Before launching into the math, ask physical question: What do we expect for \( u(x,t) \) at large \( t \)?

Heat will flow in and out of ends until: \( u(x,t) \to u_s(x) \) steady state (equal) temp dist \( \to \infty \) since \( \frac{\partial u(x,t)}{\partial t} \to 0 \). \( u_s(x) \) must satisfy \( \frac{\partial^2 u_s(x)}{\partial x^2} = 0 \)

Given \( u(x,t) = u_s(x) + v(x,t) \)

BC's are satisfied by unique choice

\[
T_i + \frac{T_2 - T_1}{L} x
\]

How to solve full problem: write \( u(x,t) = u_s(x) + v(x,t) \). Then \( v \) satisfies \( \frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \)

BC's: \( v(0,t) = 0 \quad v(L,t) = 0 \) pure Dirichlet (homogenous)

IC \( v(x,0) = f(x) - u_s(x) \) modified IC, \( \checkmark \)

Essent. same prob. studied on pp 3-6 of notes.

Recall sol'n: \( v(x,t) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{L} \right) e^{-\frac{n^2 \pi^2 k t}{L^2}} \)

\( A_n \) is chosen from initial data.

Summarize physical picture:

- use only fact that \( v(x,t) \to 0 \) as \( t \to \infty \)
- At large \( t \), sol of full prob \( u(x,t) \to u_s(x) \)
- all memory of initial \( f(x) \) is lost at large \( t \)
- The \( v(x,t) \) part dies away, \( u_s(x) \) is left.

Consider long-time limit of heat flow with insulation BC's

At large \( t \), we expect \( u(x,t) = ax + b \) where before

BC's \( \Rightarrow \) slope \( a = 0 \)

Use conservation of total thermal energy: \( b \) for \( b \).

\[
E = \int_0^L dx \, u(x,t)
\]

\[
\frac{dE}{dt} = \frac{\partial}{\partial t} \int_0^L dx \, u(x,t) = c k \int_0^L dx \frac{\partial^2 u}{\partial x^2} = c k \left[ \frac{\partial u}{\partial x} \right]_0^L = 0
\]

For insulated BC's (but not in general), total energy is indep.

Energy at \( t = 0 \) = Energy at \( t = \infty \)

\[
\int_0^L dx \, u(x,0) \quad b = \frac{1}{L} \int_0^L dx \, u(x,t)
\]
Summary: For heat flow with insulated BC's
\[ u(x,t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \]
\[ b = \frac{1}{2} \int_0^L dx f(x) \quad \text{mean value of initial} \]
\[ \phi(x,t) \]

Tricky question: (for next PSet): What is long time limit of
heat flow with non-homog. flux BC's
\[ \partial_t u(0,t) = -\frac{1}{k} \phi_0 \]
\[ \partial_x u(L,t) = -\frac{1}{k} \phi_L \]
Hint: use your head. The sol. take 3-4 guidelines to write,
return to D'alembert heat flow (with no wall BC's) from pp. 3-6:

Sol: from Sep. of var., and eigenfun expansion:
\[ u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2 \pi^2 k t}{L^2}} \sin \left( \frac{n \pi x}{L} \right) \]

Common
\[ b_n = \frac{2}{L} \int_0^L dx \sin \left( \frac{n \pi x}{L} \right) f(x) \]

1. Convergence at \( t = 0 \) determined by how smooth \( f(x) \) is.

Example: square spike of width 2 \( \varepsilon \)
centered at \( x = x_0 \), height \( u_0 \)

Ex: Calculate
\[ b_n = \frac{4 u_0}{n \pi} \sin \left( \frac{n \pi x_0}{L} \right) \sin \left( \frac{n \pi x}{L} \right) \]
\[ \frac{1}{n} \quad \text{as} \quad n \rightarrow \infty \]
Slow convergence at \( t = 0 \), but at any
small \( t > 0 \) the effective \( \phi \) envelope \( B_n \rightarrow b \)

So the series converges to super-smooth

Temp. dist. No matter how 'jampy' the initial data, the
Temp. dist. is super-smooth in \( 1 \) MS later.

Physical Principle: diffusion is a smoothing process:

2) For Dirichlet heat flow: \( u(x,t) \rightarrow 0 \) at large \( t \), and
down to shape at large \( t \) in least eigenfun:
\[ \phi_1(x) = \sin \left( \frac{\pi x}{L} \right) \]
True because least mode has slowest fall-off in \( t \).
To bring this out more clearly, define a time scale $t_0 = \frac{L^2}{\kappa_1}$

Thus, $T_1(t) = e^{-\frac{\kappa_1}{t_0} t}$

For $t > 0$, the higher modes are damped by

1) $1/n$ fall-off of $b_n$  
2) $e^{-\sqrt{n}/t_0}$ of $T_n(t)$ most important

Rough measure of importance of higher modes:

\[
\frac{T_2(t)}{T_1(t)} = e^{-3\sqrt{2}/t_0} = \begin{cases} 
2.22 & \text{at } t = \frac{1}{2} t_0 \\
1.05 & \text{at } t = t_0 \\
1.0025 & \text{at } t = 2 t_0 
\end{cases} \quad \text{with } b_0 = 0
\]

It is a universal feature of boundary heat flow in finite regions in any dim.

$u(x, t) \to 0$ with shape of lowest eigenmode.

True for regions $R$ of any shape and even for inhomogeneous materials.

Note that eigenfunctions satisfy $\nabla^2 \phi_n(x) = -\kappa_1 \phi_n(x)$ $\phi_n(\partial R) = 0$

Slightly different situation for those Newton's b.c.'s (insulating)

Then $b_0 = 0$ in an eigenvalue, and

$\phi_0 = \kappa_1 b_0 = \frac{1}{\kappa_1} \int_0^L \phi(x) dx$ is fixed by energy cons.

Then $u(x, t) \to b_0 + b_1 e^{-\kappa_1/L} t \phi_1(x)$

For uniform rod $\phi_n(x) = \sin\left(\frac{n \pi x}{L}\right)$, see Sec 2.4 of H.

Another universal feature of heat flow (both D + N)

Time scale $t_0 = \frac{L^2}{\kappa_1}$ of diffusion process.

Spatial scale $L = \text{length of rod}$.

Qualitative relation: Time scale $\propto (\text{length scale})^2$

Length scale $\propto (\text{time scale})^{1/2}$

This relation of the scales is quite universal, and we will encounter it in several ways in this course.

A bit more detailed example where:

$\kappa_{1n} = \frac{n^2 \pi^2}{L^2}$

Def: Spatial scale of $n$th mode $\sin\left(\frac{n \pi x}{L}\right)$ in wavelength

$\lambda_n = \frac{2\pi}{\kappa_{1n}} = \frac{2L}{n\pi}$

Time scale of $n$th mode $\frac{T_n(t)}{T_1(t)} = e^{-\frac{\kappa_{1n}}{t_0} t}$ in $T_n = \frac{L^2}{\kappa_{1n}} n^2$.

Again, we see $t_n = \frac{L^2}{\kappa_{1n}} n^2 \lambda_n^2$.
General remark about modeling physical processes;
Important to identify the relevant scales in the problem.
Much can be learned by combining the units with dimensional analysis.

Ex: Given heat eq $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$, we can change units of time by $\tau' = k \tau$,
By chain rule $\frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial \tau'} \frac{\partial \tau'}{\partial \tau}$

Or can change units of length by $x' = \frac{1}{k} x$ and also

$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x'^2}$.

Conclusion: The diffusion constant $k$ can be eliminated by choice of units. So $k$ is unessential for the mathematical analysis. No loss of generality to take $k=1$.

A main theoretical issue: Uniqueness Thm. for Heat Flow:
Let $u_1(x,t)$ and $u_2(x,t)$ be sols. of following prob.
1) Both satisfy uniform heat eq $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad$ \text{temp.}$
2) \quad \text{same IC} : u(x,0) = f(x)
3) \quad \text{same BC if four prescribed } T: u(0,t) = T(0)$
\text{for prescribed flux } \frac{\partial u}{\partial x}(0,t) = \Phi(t)$, with same choice at $x=L$.
\text{(total of } 2+2 = 4 \text{ distinct BC's)}

Case 1: $V(x,t) = u_1(x,t) - u_2(x,t)$. We want to show that $V(x,t) \equiv 0$, $V$ solution
1) $\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2}$ some PDE by linearity.
2) $V(x,0) = 0 \quad$ \text{from IC.} \quad \text{homog BC's}
3) either $V(0,t) = 0$ or $V(L,t) = 0$, same at $x=L$.

Step 1: Multiply PDE by $V$:
$V \frac{\partial V}{\partial \tau} = V \frac{\partial^2 V}{\partial x^2}$
$\int \frac{1}{2} \frac{\partial V}{\partial \tau} \, dx \, dx = \int \frac{1}{2} \frac{\partial V}{\partial \tau} \, dx \, dx \, dx$

Step 2: Integrate w.r.t. spatial variable
$\frac{1}{2} \int \frac{\partial V}{\partial \tau} \, dx \, dx = \frac{1}{2} \int \frac{\partial V}{\partial \tau} \, dx \, dx \, dx$

Int. \quad \text{by}\ \partial x = 0$
$\int_0^L \frac{\partial V}{\partial \tau} \, dx \, dx = \frac{1}{2} \int_0^L \frac{\partial V}{\partial \tau} \, dx \, dx \, dx$

But $\text{RHS} = -\int_0^L \frac{\partial V}{\partial \tau} \, dx \, dx \, dx$