3. Eigenfun are orthogonal on \([a,b]\) w.r.t. weight \(w(x)\)
\[
S_a^b \int w(x) \phi_n(x) \phi_m(x) = \begin{cases} 0 & n \neq m \\ N_n^2 & n = m \end{cases} = N_n^2 \delta_{nm}
\]

\(N_n = \text{norm of eigenfun}\)

4. The set of eigenfuns \(\{\phi_n(x)\}, n = 0, 1, \ldots\) is complete on \([a,b]\)

\((\Rightarrow)\) Any well-behaved \(f(x)\) on \([a,b]\) is represented by convergent eigenfun expansion.

\[
f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)
\]


Coefficient \(a_n = \frac{1}{N_n^2} \int_a^b w(x) \phi_n(x) f(x)
\)

Convigence from truncated exp \(S_N(x) = \sum_{n=0}^{N} a_n \phi_n(x)
\)

Then \(S_N(x) \xrightarrow{N \to \infty} \begin{cases} f(x) & \text{if } f \text{ contin at } x \\ \frac{1}{2} [f(x^-) + f(x^+)] & \text{any of limit at a jump}
\end{cases}
\)

Thus is an elegant and powerful structure with many useful applics.

In many cases, once we move of the conds, assumed true will be true, e.g. \([a,1] \to [a,\infty) \text{ semi-infinite interval}\)

In those cases we have a singular \(S-L\) problem. Mathematical case are more difficult, but have a similar expan.

Comment on be's vs IC's = some ode might occur in a mechanics problem with \(x \to t\)

\[
\frac{d}{dt} (p(t) \frac{dy}{dt}) + f(t) y(t) = h(u(t) y(t)
\]

but with IC's \(y(0) = y_0, y'(0) = y_0\)

In this case \(\exists\) a unique sol for any \(\lambda\). For \(S-L\) prbs, we have BC's at 2 end pts. This changes things a lot.

3 sols only for discrete eigenvalues \(\lambda_n\).

Start discussion whose purpose is partly to prove and partly to illuminate these features.
Denote Green's formula a basic tool to establish key properties. Let \( u(x), v(x) \) be any two smooth functions on \([a, b]\).

1. Consider \( Lu = \partial_x (p \partial_x u) + q(x) u \).

2. Multiply \( v \) and \( K = \partial_x (p \partial_x u) + q(x) u \):

3. \( v \in \mathbb{R} \quad K v = \partial_x (p \partial_x v) - p \partial_x v \partial_x u + q(x) v \).

4. Subtract to get Laplace identity:

\[
K v = \partial_x (p \partial_x u - u \partial_x v) = \frac{d}{dx} \left[ p(x)(\partial_x u - u \partial_x v) \right].
\]

5. Integrate and pick up end-point contributions:

\[
\int_a^b dx \left[ v \partial_x u - u \partial_x v \right] = p(x)(\partial_x u - u \partial_x v) \Big|_a^b = p(b)(\partial_x u - u \partial_x v) - p(a)(\partial_x u - u \partial_x v) = 0.
\]

Conclusion: \( \int_a^b dx \left[ v \partial_x u - u \partial_x v \right] = 0 \) for any pair of smooth \( u(x), v(x) \) which satisfy BC's but may or may not satisfy the ODE.

We say that \( SL \) operator is self-adjoint.

Apply this to a pair of eigenfunctions: \( u = \phi_m, \quad v = \phi_n \).

We get:

\[
\phi = \int_a^b dx \left[ \phi_m \partial_x \phi_n - \phi_n \partial_x \phi_m \right] = - \int_a^b dx w(x) \left( \phi_m \partial_n \phi_n - \phi_n \partial_n \phi_m \right) = (\lambda_m - \lambda_n) \int_a^b dx w(x) \phi_m \phi_n.
\]

If eigenvalues are distinct, eigenfunctions are orthogonal.

Related result: Show that \( \forall m,n \) are real.

Use \( 1 \phi = -\lambda \phi \)
Multiply each side and integrate:
\[ \int_a^b \text{d} x \, \frac{\partial}{\partial x} \left[ \phi (\partial \phi / \partial x) + \phi \phi \right] = - \lambda \int_a^b \text{d} x \, \phi \frac{\partial^2 \phi}{\partial x^2} \]

Integrate by parts:
\[ \int_a^b \text{d} x \left[ - \phi \frac{\partial}{\partial x} \frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} \right] + \phi \frac{\partial^2 \phi}{\partial x^2} \bigg|_a^b = \int_a^b \text{d} x \, \phi \frac{\partial^2 \phi}{\partial x^2} \]

Restrict bc's to simplest types usually considered in 18.03:
- Pure Dirichlet bc: \( \phi(a) = 0 \) same at \( x = b \)
- Pure Neumann bc: \( \phi'(a) = 0 \)

Total of 2 \times 2 = 4 bc’s in all 4 cases. The end pt. (and vanish)
(Since \( \phi'^* \) changes sign, same bc)
Multiply everywhere \((\cdot) (-)\) to get
\[ \frac{1}{2} \left( \int_a^b \text{d} x \, p(x) \partial_x \phi \right) \frac{\partial \phi}{\partial x} - \int_a^b \text{d} x \, q(x) \phi \frac{\partial \phi}{\partial x} = \frac{1}{2} \int_a^b \text{d} x \, (\phi'^* \phi') \]

The only way to satisfy this is with a need.

2. Suppose \( \varphi(x) \equiv 0 \) (frequently true for PDE's in 18.303)
   Then a) \( \int_a^b \text{d} x \, p(x) \phi'^* \phi > 0 \) and b) \( \int_a^b \text{d} x \, p(x) \phi' \phi'^* > 0 \)

Conclusion: if \( \varphi(x) = 0 \) then all eigenvalues \( \lambda_n > 0 \).

3. Further \( \lambda = 0 \) is possible only \( \varphi(x) \equiv \text{const} \) is an eigenfunction.
   Not allowed with Dirichlet BC at either end.
   But we have Neumann BC at both ends \( \varphi'(a) = \varphi'(b) = 0 \)
   then \( \lambda_0 = 0 \) is an acceptable eigenfunction with \( \lambda_n > 0 \)

4. Suppose \( \varphi(x) < 0 \) then \( \lambda \Rightarrow \) all \( \lambda_n > 0 \)

Physical reason for Schrodinger eq:
\[ H \Psi = - \frac{\partial^2 \Psi}{\partial x^2} + V(x) \Psi = E \Psi \]

The caveats with our SL:
\[ L \phi = \phi'' + q(x) \phi = - \lambda \phi \]

In \( q(x) < 0 \) \( \Rightarrow - V(x) = \lambda \)

so \( q(x) < 0 \) \( \Rightarrow V(x) > 0 \) \( \Rightarrow \) energy levels \( E_n > 0 \)

Ex: Generalize 1, 2 to mixed BC’s \( \varphi(a) + k \varphi(b) = 0 \) same at \( b \).

1. Show that \( \lambda \) is always real. Hint: Green

2. If \( g(x) \equiv 0 \), find ends in \( k_a, k_b \) so that \( \lambda > 0 \)
The most important fact about $S-L$ problem is the completeness of the set $\{\phi_n(x)\}$ of eigenfunctions.

Thus gives valid expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

It is rather hard to prove validity of such an expansion (although if of Fourier series for piecewise smooth functions based on 18, 100 ideas can be given in about 1 hr of lecture time).

What we will do is to establish formula for the $a_n$, assuming that expansion valid.

1. Assume $f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$ converges.

2. Multiply $w(x) \phi_n(x)$, and integrate $\int_a^b$, assume valid to exchange order of $\sum$.

$$\int_a^b w(x) \phi_n(x) f(x) = \sum_{n=0}^{\infty} a_n \int_a^b w(x) \phi_n(x) \phi_n(x)$$

$$= \sum_{n=0}^{\infty} a_n N_n \int_a^b w(x) \phi_n(x) \phi_n(x) = a_n N_n \int_a^b w(x) \phi_n(x) \phi_n(x)$$

We have proved (non-rigorously) that

$$a_n = \frac{1}{N_n} \int_a^b w(x) \phi_n(x) f(x)$$

Normalization: since the eigenproblem is linear, $L \phi_n = -\lambda_n \phi_n$, if $\phi_n(x)$ is an eigenvector, so is $c_1 \phi_n$ for any $c_1$ and usually convenient to choose normalized eigenvector $\hat{\phi}_n(x)$:

$$\hat{\phi}_n(x) = \frac{1}{N_n^\frac{1}{2}} \phi_n(x)$$

so that $\int_a^b w(x) \hat{\phi}_n(x) \hat{\phi}_n(x) = N_n$.

Expansion formula simplifies to

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

Degression on analogy between mathematical structure of eigenvalue problem and vector spaces such as 3 dim Euclidean space $\mathbb{R}^3$.

Ref: H. Appendix to Sec 2.3 and App to Sec 5.5.
Two abstract def s: (then examples)

1. A real vector space $V$ is an abelian group whose operations are
   defined
   1) addition: if $u, v \in V$ then $u + v \in V$
   2) scalar mult.: if $c$ is a real number and $u$ is a vector in $V$,
      then $cu \in V$.

One can also define additional structures on $V$; such as a

Scalar product: a map $V \times V \to \mathbb{R}$ which obeys:
   i) $(u, v) = (v, u)$ symmetry
   ii) $(u + v, w) = (u, w) + (v, w)$ and $(cu, v) = c(u, v)$ linearity
   iii) if $(u, v) = 0 \Rightarrow u = 0$ non-degeneracy
   iv) $(u, u) \geq 0$ = 0 only if $u = 0$ positivity

These 4 properties are the exact def of a scalar product.

Examples: $\mathbb{R}^2$ realized geometrically in terms of vectors

\[ \vec{A}B \to \vec{A+B} \]

Algebraically in terms of column vectors, \( U = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} \)

and let $U + V = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$ and $cU = \begin{pmatrix} cu_1 \\ cv_1 \\ cu_2 \\ cv_2 \\ cu_3 \\ cv_3 \end{pmatrix}$

2. $C[a, b]$ vec. space of contin. func. on interval $[a, b]$
   If $f(x)$ and $g(x)$ are contin. funs, so are
   
   $(f + g)(x) = f(x) + g(x)$ pt. wise addition
   $(cf)(x) = cf(x)$ pt. wise multiplication

These statements define the linear structure of $C[a, b]$

(We know that the pointwise product $f(x)g(x)$ is also a contin. fun., but this is part of the linear structure. Rather it is part of an additional structure of a Banach algebra.)