Some a true for radial sngn in d=3 C-space chm
in rph. coord.
Sing pt \( x=0 \) \( \Rightarrow \) origin of coord. system.
So behaviour of sngn at \( x=0 \) is physically important.

Fundamental: the sngn is both Legender + Bessel \( n \) of a mult type
and there is a simple method to find the behaviour of sngn.
Define 2nd kind ODE for \( f(x) \) to have a reg. sngn, i.e.
\( f(x) \) and \( (x-x_0)^2 \) are reg. then but \( (x-x_0)^2 \) a reg.

\[
\text{Frakasius method can be used to find behaviour if (at least one)}
\]
\[
\text{sol. near } x=x_0
\]
First apply method to Bessel at \( x=0 \)
Assume set of form \( y(x) = x^r z(x) \) where
\[
2(x) = \sum_{n=0}^{\infty} b_n x^n \quad b_0 \neq 0 \quad \text{convergent for } |x|<x_0
\]
Then \( y' = r x^{r-1} 2(x) + O(x^r) \) \( \Rightarrow \) \( x \to 0 \)
0 terms vanishing at least an fast on \( x^r \) as \( x \to 0 \)
\[
y'' = r(r-1) x^{r-2} 2(x) + O(x^{r-1})
\]
Subst in ODE (best in non-nil form)
\[
(\frac{r(r-1)}{x^2} + r-x^2) x^r 2(x) + O(x^{r+1}) = 0
\]
\( \Rightarrow \) dominant term \( x \to 0 \)
So a necessary cond for a sol of the eq. \( r = \mu^2 \Rightarrow r = \pm \mu \)
So we may let guess that \( \exists \) two lin indep sols of form
\[
y_1(x) = x^\mu 2_1(x) \quad \text{and} \quad y_2(x) = x^{-\mu} 2_2(x)
\]
Always true with \( y(x) = J_\mu(x) \) \( \not\equiv \) reg. Bessel \( n \) in Sec 7.7-7.8
of H.
Not always true.
The actual situation is governed by this, of Frakasius
set lump up:
1. Supp. that \( y'' + P(x)y' + Q(x)y = 0 \) has reg. sngn at \( x=0 \)
\[
x^2 y'' + p(x)y' + q(x)y = 0 \quad p(x) = x^2 P(x) \quad q(x) = x^2 Q(x)
\]
2. Assume \( \exists \) one sol with behav. \( y(x) = x^\mu z(x) \) as \( x \to 0 \)
3. Compute \( y', y'' \) as before. Sub in ODE and find indicial eqn. \( r(r-1) + r p_0 + q(0) = 0 \)

A necessary condition for a solution of the type assumed is that \( r \) is a root of the indicial eqn.

Suppose two real roots \( r_1, r_2 \), \( r_1 > r_2 \).

Thm: a) The ODE has one sol of the form \( y_1(x) = x^{r_1}Z_1(x) \), \( Z_1(x) \) has conv. power series and \( Z_1(0) \neq 0 \).

b) If \( r_1 - r_2 \neq n \), \( n = 0, 1, 2, \ldots \), then \( y_2(x) \) is a 2nd sol of \( y_2(x) = x^{r_2}Z_2(x) \), \( Z_2(x) \) has conv. power series.

c) If \( r_1 - r_2 = n \), the second sol has the form

\[
y_2(x) = x^{r_2}Z_2(x) + C \ln x y_1(x)
\]

Note: Info on behaviour of \( y_2(x) \) is correctly given by

Recurrence formula:

\[
y_2(x) = V(x) y_1(x)
\]

\[
V(x) = \int \frac{\psi(x) e^{-\int p(x) dx}}{y_1(x)}
\]

Refs: Birkhoff, ODE Ch 9 (Simmons, H. Ch 7, especially for Bessel fun, Sec 7.7.5)

Note: There might be two or roots \( r = r_0 \pm i\rho_0 = r_± \)

Then \( y_+(x) = x^{r_+}Z_+(x) = x^{r_0} e^{\pm \rho_0 \ln x} Z_+(x) \) and

\[
y_-(x) = y_+(x) x^{1/2}
\]

Examples: Regular Bessel fun, from Frobenius + power series method \( J_\mu(x) = \frac{(x/2)^\mu}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! n! (\mu + n + 1)} \frac{(x/2)^{2n+\mu}}{n!} \) for integer \( \mu \geq 0 \).

\( J_\mu(x) \) is ultraslow conv. series since \( c_n = \frac{1}{2^{2\mu+1}} (\mu + 1)! \)

\( c_n \sim \frac{1}{2^{2\mu+1}} (\mu + 1)! \)

Sketch

\[
\text{oscillations with decaying envelope}
\]
use $R(0)$ to find behaviour of 2nd sol near $x=0$

Since $R(x) = \frac{1}{x} \int_{0}^{x} \frac{\cos(t)}{t^2} dt = \ln t \quad \text{as} \quad x \to 0$

Then $V(x) = \int_{0}^{x} \frac{dt}{x^2 (1-t^2)}$

We take $x_0 \leq x \leq 1$ and use $J_n$ and integrate down towards $x = 0$

For $x < x_0$, we can represent

$$J_{\mu}(x) = \sum_{k=0}^{\infty} \frac{(x)^{\mu - k}}{\mu! 2^{\mu} \Gamma(k+1)}$$

as radius of convergence $\varepsilon = x_0$

(can calculate coefficients $a_n$ from series for $J_n$), but no need for this.

If $\mu > 0$, $\neq n$

$$Y(x) = \int_{0}^{x} \frac{dt}{\sqrt{z^{2} - 2zt + 1}} = \frac{Y_0}{\sqrt{z}}$$

So $Y_0 (x) = Y(x)$ $y_1 (x) = x^{-n} Z_2 (x) \quad \text{in general}$

If $\mu = 0$, $V(x) \sim \ln x + \text{reg}$

$$y_0 (x) = \ln x J_0 (x) + \mathcal{E}_0 (x) \quad \text{as} \quad \text{this requires}$$

If $\mu = n$

$$V(x) = \frac{Y_0}{\sqrt{z}} + \frac{Y_1}{x^{2n-1}} + \cdots + \frac{Y_{2n-1}}{x^{2n-1}} + \mathcal{E}_{n} (x) + \ln x$$

$$y_2 (x) = \frac{1}{\sqrt{z}} Z_2 (x) + \mathcal{E}_2 (x)$$

Conclusion $R(0)$ gives behaviour compatible with Frobenius in all cases.

It is useful and interesting to study behaviour of Bessel for an

Use method of Plüet.

Write $y(x) = x^{s} f(x)$ and choose $s$ so that

$$f(x) \quad \text{satisfies} \quad ODE \quad \text{of form} \quad f''(x) + \gamma f(x) = 0$$

$$y' = s x^{s-1} f + x^s \gamma f'$$

$$y'' = s(s+1) x^{s-2} f + 2 s x^{s-1} f' + x^s \gamma f''$$

Subsit in original $x^2 y'' + \gamma y' + (\gamma^2 - \mu^2) y = 0$
\[ x^5 \left[ x^2 f'' + 25xf' + 5(x^2 - 1)f + (x^2 - 1)f' \right] = 0 \]

Set \( s = \frac{1}{x} \) to eliminate \( f' \) term. Result

\[ y(x) = \frac{1}{n^2} f(x) \text{ with } f'' + \left(1 + \frac{1 - 4n^2}{x^2}\right)f = 0 \text{ explicitly} \]

Thus is an exact soln. \( \therefore \) which usually cannot be solved.

But it is useful be \( 1/2 \) term is \(< < 1 \) for large \( x \).

So we should expect that

\[ f(x) \rightarrow h(x) + \hat{f}(x) \]

where

\[ h(x) \text{ satisfies } h'' + h = 0 \text{ } \Rightarrow \text{ } h(x) = c_1 \cos x + c_2 \sin x \]

and \( \hat{f}(x) \rightarrow 0 \) as \( x \rightarrow \infty \)

A more detailed analysis \( \rightarrow \) \[ f(x) = c \cos(x-\theta) + r(x)/x \]

Thus in what we need to conclude that \( \hat{r}(x) \) bounded.

\[ y(x) \sim \frac{c}{x} \cos(x-\theta) + \frac{r(x)}{x^{1/2}} \text{ for } x \text{ large} \]

It is interesting that quite simple ideas have led to a good understanding of behavior of Bessel func.

1) as \( x \rightarrow 0 \) from Frob. Then \( R \rightarrow 0 \)

2) \( 1/2 \) envelope as \( x \rightarrow \infty \)

3) \# of zeros spread by \( \pi \)

A more detailed study \( \rightarrow \) reg. Bessel fun. \( J_\nu(x) \) (defined by series on p. 18) in p. 323 of H. has asymp. behavior

\[ J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{2} - \frac{\nu \pi}{2} \right) + \frac{R(x)}{x^{3/2}} \]

One makes a std. choice of second sol.: \( Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\pi}{2} - \frac{\nu \pi}{2} \right) + \frac{V(x)}{x^{3/2}} \)

Comment: 1) look at 

\[ f'' + if = 0 \text{ is an exact soln}. \]

Result \( J_{\nu/2}(x) = \sqrt{\frac{2}{x}} \sin x \text{ and } Y_{\nu/2}(x) = \sqrt{\frac{2}{x}} \cos x \)

can be expressed in terms of elem. fun.

Same is true for \( J_{\nu+1/2}(x) \) and \( Y_{\nu+1/2}(x) \). These are called spherical Bessel fun. because they occur in 3D problems.
Example: \( J_{\frac{3}{2}}(x) = \frac{\sqrt{2}}{\pi x} \left( \frac{2 \sin x}{x} - \cos x \right) \quad Y_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left( -\cos x + 5 \sin x \right) \)

For \( m \neq n \pm \frac{1}{2} \), the Bessel functions are transcendental. Defined as solutions of ODE, and all properties can be deduced from ODE.

2) One way in which Bessel functions appear is in 18.303 in text

Integral representations:
\[ J_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-itx} d \theta e^{-i\theta t} \quad x \text{ is integer} \]

Ex: Show by diff. that Bessel ODE is satisfied.
Ex: Show that \( J_n(x) \sim x^n \) as \( x \to 0 \)

**Now Topic:** Sturm-Liouville Problems. This is the setting in which ODE's occur in separable form for linear PDE's.

**General features:**
1) 2nd order linear ODE with (dy dx) at 2nd
2) \( \infty \neq \text{ anything, eigenvalue} \)
3) Any (smooth) \( f(x) \) has convergent eigenfunction expansion

**Systematic discussion in print:**
Ref H. Ch 5

- detailed summary of features of SL props
- prove some of key properties (but not all)
- illustrate by examples

**Define:** A regular SL prob on finite interval \([a,b]\) consists of:

1. Indicate the ODE in form
   \[ L \phi = \frac{d}{dx} \left( p(x) \frac{d}{dx} \phi \right) + q(x) \phi = -\lambda \] where \( p(x) \), \( q(x) \), and \( w(x) \) are real continuous, \( p(x) > 0 \) and \( p(x), w(x) > 0 \)

2. \( bc \) at \( x = a \): \( \phi(a) + k \phi'(a) = 0 \)
   at \( x = b \): \( \phi(b) + k \phi'(b) = 0 \)

**Properties:**

1. The ODE is a regular prob has unique sol for \( \infty \) eigenvalues \( \lambda_n \) \( n < \lambda < \lambda_n \) \( \cdots \lambda \to \infty \) as \( n \to \infty \) \( \lambda \) is lower bound on the eigenvalue spectrum

2. For each eigenvalue \( \lambda_n \), there is a real eigenfunc \( \phi_n(x) \)
   (unique up to \( c \phi_n(x) \))
   \( \phi_n(x) \) has exactly \( n-1 \) zeroes in \([a,b]\) internal zeroes
   \( \phi_{n+1}(x) \) has 1 zero between each pair of successive zeroes of \( \phi_n(x) \)