Today Dec 15

Winding up our work on Poisson Green's function

1. Universal short distance behavior of \( G(\vec{x}, \vec{y}) \) in finite regions

2. A relation between \( G(\vec{x}, \vec{y}) \) and \( G(\vec{x}, \vec{y}, \pm) \) for both \( \pm \)

3. \( G(\vec{x}, \vec{y}) \) for interior of sphere (a ball)

4. Formulate a brief summary of all prior results for \( G(\vec{x}, \vec{y}) \)

i) For finite regions we have spectral rep.

\[
G_{D}(\vec{x}, \vec{y}) = \sum_{n=1}^{\infty} \frac{\phi_{n}(\vec{x}) \phi_{n}(\vec{y})}{\alpha_{n} N_{n}^2}
\]

True in any \( \alpha \)

\( \phi_{0} \) on \( \phi_{N} \)

ii) For open \( \mathbb{R}^{d} \) we have explicit form

\[
G_{\infty}(\vec{x}, \vec{y}) = \frac{1}{(d-2)\pi d} \frac{1}{|\vec{x} - \vec{y}|^{d-2}} + \frac{1}{\pi} \alpha_{n} \left(\frac{|\vec{x} - \vec{y}|}{\alpha_{n}}\right)
\]

iii) May change methods are powerful (and fun) when they work

1. Claim: \( G_{D} \) for any finite \( \mathbb{R}^{d} \) region has the property

\[
G_{D}(\vec{x}, \vec{y}) = G^{(d)}(\vec{x}, \vec{y}) + V(\vec{x}, \vec{y})
\]

where

\[
V(\vec{x}, \vec{y}) \text{ is regular at } \vec{x} = \vec{y} \text{ (and in all of } \mathbb{R}^{d})
\]

Significance is that short distance behavior is universal

Same for interior regions \( \mathbb{R}^{d} \) of any shape and any BC's

for fixed \( d \). Short dist. behavior is same as \( G^{(d)} \)

for infinite space \( \mathbb{R}^{d} \).

Proof of this beautiful fact for \( D \) proof on \( V \) with body \( S \) in \( d \geq 3 \)

\[
G_{D}(\vec{x}, \vec{y}) \text{ satisfying } \nabla^{2} G_{D}(\vec{x}, \vec{y}) = \delta^{(d)}(\vec{x} - \vec{y}) - G_{D}(\vec{x}, \vec{y})
\]

Then \( V(\vec{x}, \vec{y}) = G_{D} - G^{(d)} \) must satisfy

\[
\nabla^{2} V(\vec{x}, \vec{y}) = 0
\]

Then in a body value prob for Laplace with specific body data,

Thus a PDE in variable \( \vec{x} \) with \( \vec{x} \) fixed or specified \( \vec{y} \) in \( V \)
We already know quite a bit about these problems. Let's review:

(i) The boundary is unique as was proved by Green's 1st identity on p. 73. 

The body data are smooth at \( \partial \Omega \), hence over \( \Sigma = \partial \Omega \) (true since \( \Sigma \) is an inner product, so \( 1 - \gamma \) non-negative).

For smooth body data, the solution of body value prob. is also smooth. (We can thus in an example of Laplace prob. on \( \Omega \) where we assumed Poisson integral formula.)

(ii) The smooth data for \( \nabla \phi (\mathbf{r}, \omega) \) is smooth at all interior \( \mathbf{x} \) including \( \mathbf{x} = \mathbf{y} \). So our \( \phi \) is complete.

2. Anode beam. Given \( \phi = f \) for heat flow and Laplace.

(Background info for prob. 26 of Prob. 8)

(i) Recall our derivation of p. 76, 77 of Green's sol-to-problem of heat flow \( u \) with \( \Omega \) (boundary)

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \bigg|_{\partial \Omega} = A(\mathbf{r}, \omega)
\]

We solved the prob. from the case where the boundary

form had some structure for general \( d \). (Actually for \( d = 2 \))

\[
\phi (\mathbf{r}, \omega) = \int_0^\infty \lambda \phi (\mathbf{r}, \omega, \lambda) d\lambda = \int_0^\infty \lambda \phi (\mathbf{r}, \omega, \lambda) d\lambda
\]

Spectral norm \( c_0 (\mathbf{r}, \omega, \lambda) = \sum \phi_n (\mathbf{r}, \omega, \lambda) \phi_n (\mathbf{r}, \omega, \lambda) \)

(iii) on p. 76 we had the sol of Laplace prob. with body data \( f(\mathbf{x}) \).

\[
\frac{\partial^2 u}{\partial x^2} = 0 \quad x \in \Omega \quad u \bigg|_{\partial \Omega} = f(\mathbf{x})
\]

\[
\phi (\mathbf{x}) = \sum \phi _n (\mathbf{x}) G_0 (\mathbf{x}, \mathbf{y}, \lambda) d\lambda
\]

(iii) We introduced Laplace on p. 64 by physical reasoning concerning long-time limit of heat flow. Specifically we argued that

\[
\lim_{t \to \infty} u(\mathbf{x}, t) = u(\mathbf{x})
\]

will \( u \bigg|_{\partial \Omega} = f(\mathbf{x}) \) to sol of Lap with \( u \bigg|_{\partial \Omega} = f(\mathbf{x}) \)

This suggests a connection between heat flow and Laplace prob. at least for \( t \) tends body data \( A(\mathbf{x}, \omega) = f(\mathbf{x}) \).
In this case we can write a series of rates in \( \mathbf{x} \) and under
\[
U(\mathbf{x}, t) = -\sum_{\mathbf{y}} \left( \frac{\partial}{\partial t} G_{D}(\mathbf{x}, \mathbf{y}, t-t') \right) \cdot \nabla \phi(\mathbf{y})
\]

Compare with \( \mathbf{x} \mathbf{y} \), we should have
\[
\lim_{t \to \infty} U(\mathbf{x}, t) = U(\mathbf{x})
\]

for all choices of body data \( \phi(\mathbf{y}) \). Thus, suppose a certain relation between the Green's functions for heat flow and step.

\[
\lim_{t \to \infty} -\int_{0}^{t} dt' G_{D}(\mathbf{x}, \mathbf{y}, t-t') = G_{D}(\mathbf{x}, \mathbf{y})
\]

Let's prove the relation using spherical reps:

\[
G_{D}(\mathbf{x}, \mathbf{y}, t-t') = \sum_{n=1}^{\infty} \frac{4\pi}{n^{2}} e^{-n\pi(t-t')} \Phi_{n}(\mathbf{x}) \Phi_{n}(\mathbf{y})
\]

We need only the elementary integral

\[
\int_{0}^{t} dt' e^{-n\pi(t-t')} = \frac{1}{n\pi} e^{-n\pi(t-t')}
\]

Then

\[
\lim_{t \to \infty} -\int_{0}^{t} dt' G_{D}(\mathbf{x}, \mathbf{y}, t-t') = -\sum_{n=1}^{\infty} \frac{\Phi_{n}(\mathbf{x}) \Phi_{n}(\mathbf{y})}{n^{2}} = G_{D}(\mathbf{x}, \mathbf{y})
\]

In Prob 26 you are asked to verify this for Green's function full \( \mathbb{R}^{3} \).

3. Green's function a basketball in \( \mathbb{R}^{2} \), a near-to-wall applied change method. See Sec. 9.5.9 of [ref] for exact case.

Prob find soln:

\[
\nabla^{2} G(\mathbf{x}, \mathbf{y}) = 6(\mathbf{x} - \mathbf{y}) \quad |\mathbf{x}| < a
\]

\[
G_{D}(\mathbf{y}, \mathbf{y}) \bigg|_{\mathbf{y} = \mathbf{x}, |\mathbf{x}| = a} = 0
\]

It's a quick remark that there is a closed soln in unbounded full space \( G^{(2)} = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|} \) with almost certain

\[
G_{D}(\mathbf{y}, \mathbf{y}) = \frac{-1}{4\pi} \left[ \frac{1}{|\mathbf{x} - \mathbf{y}|} - \frac{1}{|\mathbf{x} - \mathbf{y}|} \right]
\]
a) The will satisfy PDE as long as \( |\overrightarrow{\partial_1}| > a \) (map outside).

b) How do we choose \( \overrightarrow{\partial_2} \) to satisfy BC \( G_0(\overrightarrow{x}, \overrightarrow{y}) | \overrightarrow{\partial_2} = a \overrightarrow{\partial_2} = 0 \).

c) Symmetry suggests \( \overrightarrow{\partial_2} \) should be in some line through center \( \overrightarrow{a} \) (in order not to introduce a new direction which would lose symmetry).

so we write \( \overrightarrow{\partial_2} = \int \frac{d\sigma}{r} \overrightarrow{\partial} = \int \frac{d\sigma}{|\overrightarrow{y}|} \).

d) Least choice is suggested by idea if inverse.

\( \overrightarrow{\partial_2} = \frac{a^2}{y^2} \frac{\partial}{\partial y} \), \( |\overrightarrow{\partial_2}| = \frac{a^2}{y} \), \( |\overrightarrow{y}| = a \).

as \( \overrightarrow{\partial_2} \to 0 \), \( \overrightarrow{\partial_2} \to \infty \).

an \( \overrightarrow{\partial} \to \overrightarrow{\partial} \), \( \overrightarrow{\partial} \to \infty \).

our choice source and image source

So we can try

\[ G_0 = -\frac{1}{4\pi} \left[ \frac{1}{|\overrightarrow{x} - \overrightarrow{y}|} - \frac{1}{|\overrightarrow{x} - a^2 \overrightarrow{y}|} \right] \]

But as \( y \to \infty \), \( \overrightarrow{\partial} \to \infty \) so \( G_0 \to \frac{1}{4\pi} \left[ \frac{1}{|\overrightarrow{x} - \overrightarrow{y}|} \right] \to 0 \).

So we cannot satisfy BC. Our ansatz needs further modification:

\[ G = -\frac{1}{4\pi} \left[ \frac{1}{|\overrightarrow{x} - \overrightarrow{y}|} - \frac{1}{|\overrightarrow{x} - a^2 \overrightarrow{y}|} \right] \] is the still

satisfies PDE in \( y \). With \( h(y) = \frac{a}{y} \) some satisfy BC.

The result can be written as

\[ G_0(\overrightarrow{x}, \overrightarrow{y}) = -\frac{1}{4\pi} \left[ \frac{1}{|\overrightarrow{x} - \overrightarrow{y}|} - \frac{a}{y} \frac{1}{|\overrightarrow{x} - a^2 \overrightarrow{y}|} \right] \]

Note that the satisifies the "universality" property. The image term is regular for \( |\overrightarrow{x}| < a \).

To show that it satisfies the BC requires some relations in \( \delta = 9.89 \)

of \( H \) (which I will not give to same time).

Using these we can show that

\[ G_0(\overrightarrow{x}, \overrightarrow{y}) = \frac{1}{4\pi} \left[ \frac{1}{|\overrightarrow{x} - \overrightarrow{y}|} - \frac{a}{y} \frac{1}{|\overrightarrow{x} - a^2 \overrightarrow{y}|} \right] \]

Ex: This satisfies our equation \( G_0(\overrightarrow{x}, \overrightarrow{y}) = G_+ (\overrightarrow{x}, \overrightarrow{y}) \)
Ex: Show that: \( G(t, x) \bigg|_{\delta = a \delta} = 0 \leq BC \)

1. The Wave Eqn: See Haberman Ch 4, 11, 12
   1) Let \( p(x, t) \) denote pressure in atmosphere at pt \( x \) and time \( t \).
      Then the PDE which describes propagating sound waves is:
      \[
      \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x, t) = \nabla^2 u(x, t)
      \]

2) Consider a string stretched between two supports, and displaced
   vertically from its rest position by \( u(x, t) \).

   The string responds to the initial stretch with motion governed by:
   \[
   \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t)
   \]

   In both cases \( c = \frac{v}{c} \). So \( c \) is a velocity, the speed
   of sound waves in the air or along string.
   In both cases, the physical mechanism is:
   \[
   \frac{\partial^2}{\partial t^2} u \quad \text{local acceleration of the medium}
   \]
   \[
   \nabla^2 u \quad \text{"stretchy" or "compression" of the medium}
   \]
   \[
   \text{due to internal forces}
   \]

   It is difficult to give a good derivation of the PDE for a
   realistic model of internal forces. (See Sec. 4.2 of H.) So we
   will assume the PDE's above and try to understand the associated
   mathematics and its physical consequences.

First let's discuss the extra cards, needed to get unique sol:

a) IC's: our PDE is 2nd order in time \( t \), so we must
   specify two IC's: initial positive \( \phi(x, 0) = \phi(x) \)
   \[
   \text{velocity } \frac{\partial \phi(x, 0)}{\partial t} = g(x)
   \]

b) If \( E \) a body, we must specify \( E = N \) BC's, which describe
   physical condition at body.

As one example, consider stretched

\[
\text{strand and over region } \Omega \text{ in plane}
\]

with \( \partial \Omega = C \)
The vertical vibrations of drumhead. \[ PDE \quad \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} (\vec{x},t) = \nabla^2 u(\vec{x},t) \] with

\[ \text{BC} \quad u(\vec{x},0) = 0 \Rightarrow \frac{\partial u(\vec{x},t)}{\partial t} \bigg|_{t=0} = 0 \]

Given the configuration of the drumhead, we define the functional

\[ E[u]_t = \frac{1}{2} \int d^2x \left[ \frac{1}{c^2} (\Delta u)^2 + \nabla u \cdot \nabla u \right] \geq 0 \]

This is called a functional because it depends on values of \( u(\vec{x},t) \) throughout \( \mathbb{R} \). Given \( u(\vec{x},t) \), \( E[u]_t \) is an ordinary function of \( t \).

**Physical Interpretation**

\( E[u]_t \) is a measure of the oscillatory motion of the drumhead. Motivation for this claim:

1) \( E[u]_t \) is a sum of kinetic energy \( (\partial_t u)^2 \) and potential energy of stretching.

2) \( E[u]_t \) is conserved \( \forall t \) if PDE + BC are satisfied.

\[
\frac{d}{dt} E[u]_t = \int d^2x \left[ \frac{1}{c^2} (\partial_t u)^2 \right] + \nabla u \cdot \nabla u \\
= \int d^2x \left[ \frac{1}{c^2} (\partial_t u)^2 - \nabla^2 \nabla u + \partial_t \mathcal{L} \right] + \nabla u \cdot \nabla u \\
\approx \frac{\partial}{\partial t} \text{PDE} - \frac{\partial}{\partial t} (\mathcal{L} + \nabla u \cdot \nabla u)
\]

By Noether:

\[ \int d^2x \frac{\partial}{\partial t} \left( \mathcal{L} + \nabla u \cdot \nabla u \right) = 0 \]

For Neumann BC this would vanish.

For either Dirichlet or Neumann BC, \( \int d^2x E[u]_t = 0 \) if PDE + BC are satisfied.

So \( E[u]_t = -E[u]_0 = \frac{1}{2} \int d^2x \left[ \frac{1}{c^2} \mathcal{L} + \nabla u \cdot \nabla u \right] \)

Energy is conserved in initial conditions.

We will use the energy functional to obtain a single proof of uniqueness. Suppose \( u_1(\vec{x},t) \) and \( u_2(\vec{x},t) \) both satisfy all our needs, namely, PDE, IC's and BC. Then

\[ u(\vec{x},t) = u_1(\vec{x},t) - u_2(\vec{x},t) \] satisfies PDE by linearity,

+ terminal BC + terminal IC's.
Then \( E[V]_{t=0} = 0 \), but conservation means that
\[
E[V]_{t} = 0 = \frac{1}{2} \int \delta x \left[ \frac{1}{c^2} (\partial_x v)^2 + \frac{\nabla \cdot \vec{V}}{V} \right] > 0
\]

But this non-negative quantity can vanish only if \( V(x,t) = \text{const.} \).
The I.C then forces \( V(x,t) = 0 \) for all \( x,t \) and we have proven uniqueness.

Summary: the wave eqn with reasonable BC's and IC's leads to a well posed problem with unique sol. The energy functional was very useful in our analysis.

General comments:

1) For \( d > 2 \), one can study the wave eqn by our tried and true methods of separation, eigenfn expansion \( \Rightarrow \) Green's fn.

2) For \( d = 1 \), these methods also work, but there are special techniques available which lead quickly to exact soln + some physical insight.

So let's start our work in 1-dim case: \( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} U(x,t) = \frac{\partial^2}{\partial x^2} U(x,t) \)

Think only about PDE, temporarily ignore BC + IC.

Observe: an ocb, \( f(x-ct) \) satisfies the PDE.

pf by chain rule: \( \partial_t f = f'(x-ct) \quad \partial_x f = f''(x-ct) \)

\( \frac{1}{c^2} \partial_x \partial_t f = \partial_x f \).

This set describes a \textbf{wave} (i.e. a disturbance of the medium)

which moves rapidly to right with velocity \( c \).

\[ f(x) \quad -ct \]

A initial disturbance with sharp leading edge at \( x_0 \) at \( t = 0 \), then
leading edge at \( x = x_0 + ct \) at time \( t \). Every detail of the
initial shape is preserved at later time \( t \).

Contrast this with one-dim heat eqn in which an initial
kink diult with sharp structure is immediately smoothed
at small \( t \).