Thus, we need to solve for $G_p$ because it has some paradoxes.

1) $(\partial_t - \partial^2_x) G_p(x,y,t) = 0$

2) PBC \hspace{1cm} G_p(x,y,t) = G_p(-\pi - y, t) \hspace{1cm} \forall t$

3) $\partial_x G_p(x,y,t) = \partial_x G_p(-\pi - y, t)$

4) future evol of pt source at $x=y$ (plus periodically spaced images)

$v)$ symmetry $G_p(x - y, t) = G_p(y - x, t)$

\[ G_p(x - y, t) \sim \sum_{m = -\infty}^{\infty} e^{-\left(\frac{(y + (2m + 1)\pi)^2}{4t}\right)} \text{ term with } m \leq 0, 1, 2, \ldots \]

\[ G_p(-\pi - y, t) \sim \sum_{m = -\infty}^{\infty} e^{-\left(\frac{(y + (2m + 1)\pi)^2}{4t}\right)} \text{ term with } m \geq 0, 1, 2, \ldots \]

Rate of convergence: \{ best at small $t$ when only a few images term have appreciable size \}

Still reasonable at large $t$

\underline{Summary:} image change expansion of $G_p$ is correct, and has complementary convergence properties to spectral nexp.

Heat flow on $[0, \pi]^2$: BC: \hspace{1cm} 0 at $x=0$ and $x=\pi$

\[ \left\{ \begin{array}{l}
G_{p/n}(x,y,t) = G_p(x,y,t) + G_p(x,y,t) \\
\end{array} \right. \text{ at } x=0 \text{ and } y=\pi
\]

It satisfies \((\partial_t - \partial^2_x) G = 0\)

It satisfies \(\partial_x G = 0\) \hspace{1cm} \text{at } x=0 \hspace{1cm} \text{by construction}

(\text{It is even in } y \hspace{1cm} \text{by inspection. By symmetry})

\[ G_{0/n}(x,y,t) = G_p(y,x,t) + G_p(y+x,t) \text{ new odd even in } x \]

What about $x = \pi$

\[ G_{p/n}(\pi,y,t) = G_p(\pi - y, t) = G_p(\pi + y, t) \]

\[ = G_p(y - \pi, t) + G_p(y + \pi, t) \]

so \(\text{odd} = 0\) \hspace{1cm} \text{by periodicity of } G_p

So we do get a new image change rep for $G_p(x,y,t)$.

Ex: \text{ does } \frac{\partial}{\partial_x} G_{0/n}(x,y,t) \bigg|_{x = \pi} = 0?
Last appic of heat eq: Go back to our
result for whom. Dirichlet problem

\[ u(x, t) = - \int_0^\infty d\xi \left[ \frac{\partial}{\partial y} G(x, y, t; \xi) \right]_{y=\pi} = B(t) \]

\[ - \frac{\partial}{\partial y} G(x, y, t) \bigg|_{y=0} = A(t) \]

Can we simply write a formula of same structure
to solve the corresponding problem in 2 dims.

See p. 96b.

General flow chart for our work on heat flow
in all situations:

- Basic tool
- Separation of variables
- Eigenparameter expansions

**Boundary value**
- Dirichlet: \( u(\partial, t) = T \)
- Neumann: \( \partial u/\partial n = 0 \)

**Initial value problem**
- \( u(x, 0) = f(x) \)

**Source**
- \( Q(x, t) \)

**Green's function**
- \( G(x, y, t) \)

**Spectral representation**
Change of topic: Laplace / Poisson eqtn \( \nabla^2 u = \rho(x) \)

1. Review our early work on pp 72 - 79 of notes

2. Green's fn for Laplace / Neumann problem

3. Laplace / Poisson on \( \mathbb{R}^d \) and its Green's fn.

**Review:**

1. Uniqueness for \( \nabla^2 u = \rho \) on \( \mathbb{R} \)

   - with BC: \( \mathcal{D} \), \( u(x) \big|_{\partial \mathcal{D}} = f(x) \)

   \( \text{on } N \), \( n \cdot \nabla u \big|_{\partial \mathcal{D}} = -\Phi(x) \)

   a. Set of \( \mathcal{D} \), prob in unique (with or w/o. source)

   b. \( N \) prob (w/o. source) has a sol only if

   \[ \int_{\mathcal{D}} d\mathcal{S} \Phi(x) = 0 \]

   - sol \( u \) unique up to additive const: \( u_1(x) = u(x) + c \)

   \( N \) prob with source and non-zero BC's

   \( \nabla^2 u = \rho(x) \)

   \( n \cdot \nabla u = -\Phi(x) \)

   \[ \int_{\mathcal{D}} d\mathcal{S} \nabla^2 u = \int_{\mathcal{D}} d\mathcal{S} \rho(x) \]

   \[ L_0 = \int_{\mathcal{D}} d\mathcal{S} n \cdot \nabla u = -\int_{\mathcal{D}} d\mathcal{S} \Phi(x) \]

   A sol can exist only if the body data + source

   satisfy

   \( \int_{\mathcal{D}} d\mathcal{S} \rho(x) + \int_{\mathcal{D}} d\mathcal{S} \Phi(x) = 0 \)

2. Connection between Laplace eqtn + Laplace eigenvalue prob

\( \nabla^2 \varphi_n = -\lambda_n \varphi_n \)

Green's fn: \( \nabla^2 G_D(x, \tilde{x}) = \delta(x - \tilde{x}) \)

\( G_D(x, \tilde{x}) \big|_{\tilde{x} \in \mathcal{D}} = 0 \)

We showed that \( G_D \) has universal spectral map

\[ G_D(x, \tilde{x}) = \sum_{n=1}^{\infty} \frac{\varphi_n(x) \varphi_n(\tilde{x})}{-\lambda_n N_n^2} \quad \text{for } \lambda_n > 0 \]

\( \ast \)
Then \( \nabla_x^2 G_D (\vec{x}, \vec{y}) = \sum_{n} \left( \frac{\phi_n(x) \phi_n(y)}{N_n^2} \right) = \delta (\vec{x} - \vec{y}) \),

Using \( G_D \), we can write the unique solution of Dirichlet problem with both source and body data

\[
u (\vec{x}) = \int d^2 y \, G_D (\vec{x}, \vec{y}) \rho (\vec{y}) + \int d y \, \hat{n} \cdot \nabla \, G_D (\vec{x}, \vec{y}) f (\vec{y})
\]

We would like to find \( G_N \) for Neumann problem, but there is trouble due to zero mode: \( \rho_0 (\vec{x}) \equiv 1/A \).

We might try \( G_N (\vec{x}, \vec{y}) = \sum_{n=0}^{\infty} \frac{\phi_n(x) \phi_n(y)}{-\lambda_n N_n^2} \),

where \( \phi_n(x) \) are Neumann eigenfunctions, \( \nabla^2 \phi_n = -\lambda_n \phi_n \),

\[
\frac{\partial}{\partial \vec{n}} \phi_n |_{\vec{n} \cdot \nabla \phi_n = 0} = 0
\]

BUT zero mode term \( n = \infty \) is excluded so sum is well defined.

There are several ways to define a modified Green's function to avoid this problem. One way is to define the Neumann for \( G_N \).

\[
G_N (\vec{x}, \vec{y}) = \sum_{n=1}^{\infty} \frac{\phi_n(x) \phi_n(y)}{-\lambda_n N_n^2}
\]

\[
\nabla_x^2 G_N = \sum_{n=1}^{\infty} \frac{\phi_n(x) \phi_n(y)}{N_n^2} = \delta (\vec{x} - \vec{y}) - \frac{1}{A}
\]

Connection to rhs, etc. completeness sum includes \( \phi_0 = \frac{1}{\sqrt{A}} \).

We now show that it still \( G_N \) allows us to solve Neumann/Poisson problem \( \nabla^2 \nu = \rho (x) \times \hat{n} \), \( \nabla \cdot \nu |_{\partial \Omega} = 0 \).

We would try:

\[
u (\vec{x}) = \int d^2 y \, G_N (\vec{x}, \vec{y}) \rho (\vec{y})
\]

\[
\nabla^2 \nu = \int d^2 y \left[ \delta (\vec{x} - \vec{y}) - \frac{1}{A} \right] \rho (\vec{y}) = \rho (x) - \frac{1}{A} \int d^2 y \, \rho (\vec{y}) = 0
\]

and term vanishes because of solvability condition, on previous page so are sol with modified Green's fn works!
Poisson's Eqtn in \( \mathbb{R}^3 \): \( \nabla^2 u(x) = \rho(x) \)

(The body canis in finite \( \mathbb{R} \) replaced by a point cond.

we won't be explicit about this.)

We expect Green soln of form \( u(x) = \int d^3y \, G(x-y) \rho(y) \)

For \( G \) we will write a spherical rep and also go to \( x \rightarrow 0 \), but using complete set of plane waves:

\[
G(x-y) = -\frac{1}{(2\pi)^3} \int d^3k \frac{e^{ik \cdot x} - e^{ik \cdot y}}{k^2} \]

\( c) \) \( G \) n a fn of \( x-y \) because of trans sym\( s \) of \( \mathbb{R}^2 \)

\( d) \) we will see that integral will well defined despite \( \frac{1}{k^2} \rightarrow \infty \) as \( k \rightarrow 0 \)

\( e) \) formal check of PDE

\[
\nabla^2 G(x-y) = -\frac{1}{(2\pi)^3} \int d^3k \frac{-k^2 e^{ik \cdot x} - e^{ik \cdot y}}{k^2} = \delta(x-y)
\]

Calculate \( \delta \) explicitly -- no loss of generality if we take \( y = 0 \)

\( G(x) = \frac{1}{(2\pi)^3} \int d^3k \frac{e^{ik \cdot x}}{k^2} \)

The integral is best done in sph. coords in \( k \)-space:

\[
k_3 = k \cos \theta \\
k_1 = k \sin \theta \cos \phi \\
k_2 = k \sin \theta \sin \phi \\
\frac{k \cdot x}{r} = \frac{k_1 x_1 + k_2 x_2 + k_3 x_3}{r}
\]

Choose coords so that geom. angle \( u \) \( k \cdot x \) is equal to \( \cos \theta \) of sph. coords.

We then get:

\[
G(x) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \frac{d\theta}{k} \sin \theta \int_0^{\pi} d\phi \int_0^{2\pi} d\alpha \, e^{i k \cdot x} \cos \theta
\]

\[ a) \] \( \int_0^{2\pi} d\phi = 2\pi \)

\[ b) \] \( \frac{1}{2\pi} \int_0^{2\pi} d\alpha \, e^{i k_1 x_1 + i k_2 x_2 + i k_3 x_3} = \frac{1}{2\pi} (e^{ik_1 x_1} - e^{-ik_1 x_1}) = \cos k_1 x_1 \]