18.300 PRINCIPLES OF CONTINUOUS APPLIED MATHEMATICS
PROBLEM SET 5, DUE 13TH OF MAY AT 5:00PM

Summary. This problem set contains questions on nonlinear waves, hyperbolic systems, and shocks. Comments and corrections should be e-mailed to: thomsons@mit.edu.

You are encouraged to collaborate with other students in this class, but you must write up your answers in your own words. You are required to list and identify clearly all sources and collaborators, including tutors. For example “Consulted: Joe Bloggs (classmate), Princeton Companion to Applied Mathematics (book)”. If no sources were consulted, then write “Consulted: none”.

PART I

P1.1) (a) Starting from the Euler equations, write down the equations of two-dimensional steady gas dynamics where the pressure is a given function of the density $p = p(\rho)$.

(b) Show that the system you have written down is hyperbolic (i.e. it has two real, distinct characteristics) if the flow is supersonic where $|\mathbf{u}|^2 > c^2$ and $\mathbf{u}$ is the velocity.

(c) Finally, show that the system admits the Riemann invariants

\[
R = \frac{1}{2} |\mathbf{u}|^2 + \int_{\rho}^{p} \frac{c(\rho')^2}{\rho'} d\rho'
\]

P1.2) In class we showed that the parametric solution (in regions penetrated by positive characteristics) for the problem of a piston being withdrawn from a tube filled with a one-dimensional inviscid compressible gas is

\[
\begin{align*}
\mathbf{u} &= \dot{p}(\tau), \\
\rho &= c_0 + \frac{(\gamma - 1)}{2} \dot{p}(\tau), \\
x &= p(t) \tau + \left\{ c_0 + \frac{(\gamma + 1)}{2} \dot{p}(\tau) \right\} (t - \tau),
\end{align*}
\]

where $p(\tau)$ is the piston position.

Suppose now that the piston position takes the simple form $p = -Ut$, where $U$ is a constant. Sketch the characteristic diagram and show that the solution for the velocity $u$ takes the form

\[
u = \begin{cases}
- U & \text{in } - U < x/t < c_0 - \frac{(\gamma + 1)U}{2} \\
\frac{2}{\gamma + 1} \left( \frac{x}{t} - c_0 \right) & \text{in } c_0 - \frac{(\gamma + 1)U}{2} < x/t < c_0 \\
0 & \text{in } x/t > c_0.
\end{cases}
\]
P1.3) (a) The Rankine-Hugoniot conditions for inviscid compressible flow (under the ideal gas assumption) take the form

\[
V = \frac{[\rho u]^+}{[\rho]^+} = \frac{[p + \rho u^2]^+}{[\rho]^+} = \frac{[\rho u^3/2 + \gamma \rho u/(\gamma - 1)]^+}{[\rho u^2/2 + p/(\gamma - 1)]^+}.
\] 

(1)

Using the fact that \([V]^+ = 0\), the first of these relations can obviously be written

\[
[\rho (u - V)]^+ = 0.
\]

(2)

Now, using (2) show that the second condition may be written

\[
[p + \rho (u - V)^2]^+.
\]

(3)

(Harder) Finally, using (3) to evaluate \([p]^+\) and using the fact that \(\rho (u - V)\) is conserved and non-zero across the shock, deduce that

\[
\left[\frac{(u - V)^2}{2} + \frac{\gamma p}{\rho (\gamma - 1)}\right]^+ = 0,
\]

where \([V^2]^+ = 0\).

(b) Suppose now we have a tube filled with an inviscid compressible gas and that, instead of withdrawing a piston, we \(\text{push}\) a piston into the gas with constant speed \(U\). The gas is initially at rest with pressure \(p_0\) and density \(\rho_0\). Using (2)–(4), show that the shock speed \(V\) may be determined from

\[
V^2 - \frac{\gamma + 1}{2} UV - c_0^2 = 0
\]

where \(c_0^2 = \gamma p_0/\rho_0\). Hence determine the shock speed and sketch the characteristic diagram. How do you know that the shock solution is the correct?

Part II

In lecture, we showed that the shallow water equations are

\[
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0,
\]

(5)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -gh_x.
\]

(6)

P2.1) (a) Verify that the characteristics of (5)–(6) are \(u \pm \sqrt{gh}\) and the Riemann invariants are given by \(R = u \pm 2\sqrt{gh}\).

(b) \textbf{Dam break problem}

Suppose water of depth \(h_0\) is held in \(x > 0\) by a dam at \(x = 0\). At time \(t = 0\), the dam breaks, allowing water to flow into \(x < 0\). Sketch the characteristic diagram and
show that the solution for the height $h$ is

$$
h = \begin{cases} 
0 & \text{in } x/t < -2\sqrt{gh_0} \\
\frac{1}{9g} \left(2\sqrt{gh_0} + x/t\right)^2 & \text{in } -2\sqrt{gh_0} < x/t < \sqrt{gh_0} \\
\frac{h_0}{\sqrt{gh_0}} & \text{in } x/t > \sqrt{gh_0}.
\end{cases}
$$

Sketch a few curves of $h$ versus $x$ for increasing times $t$.

(c) **Bore moving into stationary water**

Suppose that a bore (shock in shallow water theory) moves into a stationary fluid (i.e. $u_+ = 0$). Use the Rankie-Hugoniot conditions for (5)–(6) to show that the shock speed is determined from

$$V^2 = \frac{gh_-(h_+ + h_-)}{2h_+},$$

where $h_\pm$ is the fluid height ahead of and behind the bore respectively.

By invoking the energy condition

$$Q = (\rho hu_+) \frac{g(h_- - h_+)^3}{4h_+h_-} < 0$$

select the correct sign for $V$ and hence state in which direction the bore should travel.

THE END