Summary. This problem set contains a series of questions relating to first-order quasilinear p.d.e. and the method of characteristics. Some questions are purely theoretical while others relate to specific physical problems. Finally, when asked to write down the solution “explicitly”, I mean that you should express the solution in terms of the independent variables only. Comments and corrections should be e-mailed to: thomsons@mit.edu.

You are encouraged to collaborate with other students in this class, but you must write up your answers in your own words. You are required to list and identify clearly all sources and collaborators, including tutors. For example “Consulted: Joe Bloggs (classmate), Princeton Companion to Applied Mathematics (book)”. If no sources were consulted, then write “Consulted: none”.

Part I

P1.1) Using the method of characteristics, solve the following first-order quasilinear p.d.e.

\[ xu_x + yu_y = (x + y)u \]

subject to \( u = 1 \) on the circular arc defined by

\[ x_0(s) = 2 - \sqrt{2}\cos(s), \quad y_0(s) = \sqrt{2}\sin(s) \quad \text{for} \quad 0 \leq s < \pi/4. \]

Sketch the limiting characteristics and identify the domain of definition. Can you spot a problem if I had asked you to extend the solution for the initial data specified on \( 0 \leq s < \pi/2 \)?

P1.2) Solve the p.d.e.

\[ (1 + u)u_x + yu_y = u \]

subject to the boundary conditions

(i) \( u(x, 1) = x \) for \( 0 \leq x \leq 1 \)

(ii) \( u(x, 1) = -x \) for \( 0 \leq x \leq 1 \).

For (i), determine the domain of definition and sketch this region in \((x, y)\)-plane.

For case (ii), compute the Jacobian and find the point through which all characteristics pass. Hence find and sketch the domain of definition in the \((x, y)\)-plane. Draw the graph of \( u(x, y) \) versus \( x \) as \( y \) increases from 1 to 2.
P1.3) Using the method of characteristics, determine the (explicit) solution of
\[(x - y)u_x + (x + y)u_y = x^2 + y^2\]
subject to \(u(x, 0) = x^2/2\) for \(1 \leq x < \exp(2\pi)\). Determine the domain of definition of the solution i.e. where in the \((x, y)\)-plane is there, through each point, exactly one characteristic emanating from the initial data curve?

P1.4) Traffic flow with moving boundaries

(a) Without writing down any mathematics, explain why the quantity
\[\int_{a(t)}^{b(t)} \rho(x, t) \, dx, \quad a(t) < b(t)\]  
(1)
is constant if \(a\) and \(b\) are the position of any two cars.

(b) Assuming no cars enter or leave the road, show that (1) is independent of time. Hence deduce that
\[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0.\]

(c) Finally show that, if \(u\) is a decreasing function of \(\rho\), information propagates through the traffic at a velocity \(d(\rho u)/d\rho < u\).

P1.5) Abrupt light change in a tunnel

Consider the motion of traffic in a tunnel \(0 \leq x \leq 1\) governed by the traffic flow equation
\[\rho_t + q_x = 0\]
where \(q = \rho(2 - \rho)\). Suppose further that there are traffic lights at both ends of the tunnel (one at \(x = 0\) and one at \(x = 1\)).

(a) For this model write down (i) the wave speed, (ii) the flow velocity, (iii) the density at which the flow velocity vanishes, and (iv) the maximum flux \(q_m\) and corresponding density \(\rho_m\).

(b) At \(t = 0\), the density in the tunnel is equal to \(\rho_m\) and both lights go red. Find (explicitly) the density \(\rho\) in the tunnel for \(t \geq 0\) and find the waves that arise at both traffic lights. Physically what do these waves represent?

Part II

P2.1 Watching paint dry

Paint flowing down a vertical wall has thickness \(u(x, t)\) satisfying
\[u_t + u^2 u_x = 0 \quad \text{for} \quad t > 0.\]
A stripe of paint is applied to the wall at \(t = 0\) so that
\[u(x, 0) = \begin{cases} 
0, & x < 0 \quad \text{or} \quad x > 1, \\
1, & 0 < x < 1.
\end{cases}\]
Using the method of characteristics, show that (at least initially) the solution is given by

\[
u(x, t) = \begin{cases} 
0, & x < 0, \\
\sqrt{x/t}, & 0 < x < t, \\
1, & t < x < S(t), \\
0, & S(t) < x,
\end{cases}
\]

where there is a shock at position \( x = S(t) = 1 + t/3 \).

Show that at \( t = 3/2 \), the shock path is then given by

\[
\frac{dS}{dt} = \frac{S}{3t}
\]

and write down the solution.

P2.2 Sandpiles

In lecture, we argued that the static shape of a sandpile \( h(x, y) \) is described by the eikonal equation

\[
\left( \frac{\partial h}{\partial x} \right)^2 + \left( \frac{\partial h}{\partial y} \right)^2 = 1.
\]

What is the shape \( h(x, y) \) if the sand is piled on

(a) a square substrate with side-length \( a \)?

(b) an elliptical substrate with semiminor axis \( b \) and semimajor axis \( a \), where \( b < a \)?

In both cases, where does the Jacobian vanish?\footnote{These are ridgelines as might be observed in sand dunes in the desert.}

P2.3 Caustic in a teacup

A plane wave of the form \( \psi_I = a \exp(ikx) \), travelling in the positive \( x \) direction is incident on a reflective surface \( \Gamma \) described by the right half of the unit circle

\[
x_0(s) = \cos(s), \quad y_0(s) = \sin(s).
\]

Assuming that the reflected wave is of the form \( \psi_R = A \exp(iku_R(x, y)) \), and that \( u_R(x, y) \) satisfies the eikonal equation

\[
|\nabla u| = 1
\]

subject to the boundary condition \( u = x \) on \( \Gamma \), find and plot the resulting caustic. You may use a package such as MATLAB or Mathematica for finding and plotting the caustic, but you must show all your working.

THE END