Summary. This problem set reviews some of the basic mathematical tools that will be useful as we proceed through the course. Some, if not all, of these concepts will hopefully be familiar to you from previous courses but if not, I encourage you to revisit old lecture notes etc. to make sure you are comfortable with what is being assessed here. Comments and corrections should be e-mailed to: thomsons@mit.edu.

You are encouraged to collaborate with other students in this class, but you must write up your answers in your own words. You are required to list and identify clearly all sources and collaborators, including tutors. For example “Consulted: Joe Bloggs (classmate), Princeton Companion to Applied Mathematics (book)”. If no sources were consulted, then write “Consulted: none”.

PART I

P1.1) One- and two-variable implicit differentiation

In each of the following examples, find \( y' = \frac{dy}{dx} \) as a function of \( y \) and \( x \) given that \( y(x) \) satisfies:

(a) \( x^3 + 3xy + y^3 = 0 \) \hspace{1cm} (b) \( y = \tan \left( x + \frac{1}{2}y^2 \right) \) \hspace{1cm} (c) \( \ln(1 + y) = e^x \)

(d) \( y^2 = a^x \) \hspace{1cm} (e) \( y = \frac{1}{f(x+y)} \) \hspace{1cm} (f) \( y = f(1+xy)g(x+y) \).

Now do the same, only this time find both \( u_x = \frac{\partial u}{\partial x} \) and \( u_y = \frac{\partial u}{\partial y} \) as functions of \( u, x \) and \( y \), given that \( u(x, y) \) satisfies:

(g) \( x^3y + 3yu + yu^3 = 0 \) \hspace{1cm} (h) \( \cos(y^2u) = ye^{-x^2} \) \hspace{1cm} (i) \( \ln(1 + y) = ue^{\pi u} \).

P1.2) Differentiation within integrals (a.k.a. Leibnitz’ rule)

For each of the following evaluate \( u_x \) and \( u_y \) as functions of \( u, x, \) and \( y \) given that \( u(x, y) \) satisfies:

(a) \( u = \int_{a(x)}^{b(x)} f(x, y, s) \, ds \)

(b) \( y = \int_{x}^{u} \exp \left( y \sin(s) + xs^2 \right) \, ds \)
(c) \[ u = \int_0^x \sin(yu(s^2, s) + xs) \, ds \]

**P1.3) Single- and multi-variate Taylor expansions**

Find the Taylor expansion about \( x = 0 \) up to order \( O(x^5) \) for each of the following functions:

(a) \( f(x) = e^x \cos(x) \)
(b) \( f(x) = \sin(1 + x) \)
(c) \( f(x) = \sin(1 + x + x^3) \)

Now state the general form for the Taylor series of a function \( f \) of two variables \( x \) and \( y \) (i.e. \( f = f(x, y) \)) about the point \( (x, y) = (x_0, y_0) \) up to and including quadratic terms. Use this expression to evaluate the Taylor expansion of \( f(x, y) = y \exp(x y) \) about the point \( (x, y) = (2, 3) \).

**P1.4) Canonical forms on linear p.d.e.**

A partial differential equation which may already be familiar to you is the linear wave equation, namely

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]

(1)

where \( u = u(x, t) \) and \( c \) is the wave speed. Using the change of variables \( \xi = x - ct \) and \( \eta = x + ct \) show that the wave equation can be written in canonical form

\[
\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.
\]

Hence deduce d’Alembert’s solution

\[
u = f(\xi) + g(\eta),
\]

(2)

where \( f \) and \( g \) are arbitrary functions.

Verify by directional substitution that (2) is a solution (1).

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1This notation means, for example, \( \cos x = 1 - x^2/2 + O(x^4) \)
2In fact (2) is the general solution of (1) for the arbitrary functions \( f \) and \( g \).
Part II

P2. Conservation laws and the wave equation

In undergraduate courses on mechanics, the wave equation describing the small transverse displacements of a string is typically derived by consideration of force balances on a small elements of the string. Here we will show how the ubiquity of conservation laws may be exploited to derive the wave equation in a much more elegant (and mathematically satisfactory) fashion.

In what follows you may assume the following:

- The mass density of the string $\rho$ (mass per unit length) is constant.
- The motion of the string is restricted to the $(x, y)$-plane and the curve describing the string is given by $y = y(x, t)$. In equilibrium is described by $0 \leq x \leq L$ and $y = 0$, where $L$ is the string length.
- The transversal displacement of the string from equilibrium is small and there are no longitudinal forces.
- The tension in the string is constant throughout the motion.

Here are your tasks for this question:

a) Using conservation of transversal momentum, derive an equation for the displacement $y$.

b) For a solution of the equation in part a) to be determined, we must prescribe boundary conditions, one for each end of the string. What boundary conditions apply when

(i) the string is tied (as in a guitar or violin)  
(ii) at one end the string is free?

THE END