The distribution of lattice points in elliptic annuli

Igor Wigman
School of Mathematical Sciences
Tel Aviv University Tel Aviv 69978, Israel
e-mail: igorv@post.tau.ac.il

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Abstract

We study the distribution of the number of lattice points lying in thin elliptical annuli. It has been conjectured by Bleher and Lebowitz that, if the width of the annuli tend to zero while their area tends to infinity, then the distribution of this number, normalized to have zero mean and unit variance, is Gaussian. This has been proved by Hughes and Rudnick for circular annuli whose width shrink to zero sufficiently slowly. We prove this conjecture for ellipses whose aspect ratio is transcendental and strongly Diophantine, also assuming the width shrinks slowly to zero.

1 Introduction

Let B be an open convex domain in the plane containing the origin, with a smooth boundary, and which is strictly convex (the curvature of the boundary never vanishes). Let

$$N_B(t) := \#Z^2 \cap tB,$$

be the number of integral points in the t-dilate of B. As is well-known, as $t \to \infty$, $N_B(t)$ is approximated by the area of tB, that is

$$N_B(t) \sim At^2, \tag{1}$$

where A is the area of B.

A classical problem is to bound the size of the remainder

$$\Delta_B(t) := N_B(t) - At^2.$$

A simple geometric argument gives

$$\Delta_B(t) = O(t), \tag{2}$$

that is a bound in terms of the length of the boundary. It is known that Δ_B is much smaller than the classical bound, as Sierpinski [20] proved

$$\Delta_B(t) = O(t^{2/3}).$$

Since then the exponent $2/3$ in this estimate has been improved due to the works by many different researchers (see [14]). It is conjectured that one could replace the exponent by $1/2 + \epsilon$ for every $\epsilon > 0$.

A different problem is to study the value distribution of the normalized error term, namely, of

$$F_B(t) := \frac{\Delta_B(t)}{\sqrt{t}} = \frac{N_B(t) - At^2}{\sqrt{t}}.$$

Heath-Brown [12] treats this problem for $B = B(0,1)$, the unit circle, and shows that there exists a probability density $p(x)$, such that for every bounded continuous function $g(x)$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(F_{B(0,1)}(t))dt = \int_{-\infty}^{\infty} g(x)p(x)dx.$$

Somewhat surprisingly, the $p(t)$ is not a Gaussian: it decays as $x \to \infty$ roughly as $\exp(-x^4)$, and it can be extended to an entire function on a complex plane. Bleher [4] establishes an analogue to Heath-Brown's theorem for general ovals.

Motivated in parts by questions coming from mathematical physics, we will concentrate on counting lattice points on annuli, namely, integer points in

$$(t + \rho)B \setminus tB,$$

that is, we study the remainder term of

$$N_B(t,\, \rho) := N_B(t + \rho) - N_B(t),$$

where $\rho = \rho(t)$ is the width of annulus, depending on the inner radius t. The "expected" number of points is the area $A(2t\rho + \rho^2)$ of the annulus. Thus the corresponding *normalized* remainder term is:

$$S_B(t,\,\rho) := \frac{N_B(t+\rho) - N_B(t) - A(2t\rho + \rho^2)}{\sqrt{t}}$$

The statistics of $S_B(t,\,\rho)$ vary depending to the size of $\rho(t)$. Of particular interest to us are the following regimes:

(1) The *microscopic regime* ρt is constant. It was conjectured by Berry and Tabor [7] that the statistics of $N_B(t,\,\rho)$ are Poissonian. Eskin, Margulis and Mozes [10] proved that the pair correlation function (which is roughly equivalent to the variance of $N_B(t,\,\rho)$), is consistent with the Poisson-random model.

(2) The *"global"*, or *"macroscopic"*, *regime* $\rho(t) \to \infty$ (but $\rho = o(t)$). In such a case, Bleher and Lebowitz [5] showed that for a wide class of B's, $S_B(t,\,\rho)$ has a limiting distribution with tails which decay roughly as $\exp(-x^4)$.

(3) The *intermediate* or *"mesoscopic"*, *regime* $\rho \to 0$ (but $\rho t \to \infty$). If B is the inside of a "generic" ellipse

$$\Gamma = \left\{ (x_1,\,x_2) :\ x_1^2 + \alpha^2 x_2^2 = 1 \right\},$$

with α is Diophantine, the variance of $S_B(t,\,\rho)$ was computed in [2] to be asymptotic to

$$\sigma^2 := \frac{8\pi}{\alpha} \cdot \rho \tag{3}$$

For the circle ($\alpha = 1$), the value is $16\rho \log \frac{1}{\rho}$.

Bleher and Lebowitz [5] conjectured that $S_B(t,\,\rho)/\sigma$ has a standard Gaussian distribution. In 2004 Hughes and Rudnick [13] established the Gaussian distribution for the unit circle, provided that $\rho(t) \gg t^{-\delta}$ for every $\delta > 0$.

In this paper, we prove the Gaussian distribution for the normalized remainder term of "generic" *elliptic* annuli: We say that α is *strongly Diophantine*, if for every $n \geq 1$ there is some $K > 0$, such that for integers a_j with $\sum\limits_{j=0}^{n} a_j \alpha^j \neq 0$,

$$\left| \sum_{j=0}^{n} a_j \alpha^j \right| \gg_n \frac{1}{\left(\max\limits_{0 \leq j \leq n} |a_j| \right)^K}.$$

This holds for any algebraic α, for $\alpha = e$, and almost every real α, see section 3.2. Our principal result is:

Theorem 1.1. *Let $B = \{x^2 + \alpha^2 y^2 \leq 1\}$ with α transcendental and strongly Diophantine. Assume that $\rho = \rho(T) \to 0$, but for every $\delta > 0$, $\rho \gg T^{-\delta}$. Then for every interval $\mathcal{A}$,*

$$\lim_{T\to\infty} meas\left\{t \in [T,\, 2T] :\ \frac{S_B(t,\,\rho)}{\sigma} \in \mathcal{A}\right\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathcal{A}} e^{-\frac{x^2}{2}} dx, \tag{4}$$

where σ is given by (3)*.*

This proves the conjecture of Bleher and Lebowitz in this case.

Remarks: 1. In the formulation of theorem 1.1 we assume for technical reasons, that ρ is a function of T and independent of $t \in [T,\, 2T]$. However one may easily see that since ρ may not decay rapidly, one may refine the result for $\rho = \rho(t)$.

2. We compute statistics of the remainder term when the radius is around T. A natural choice is assuming that the radius is uniformly distributed in the interval $[T,\, 2T]$.

Our case offers some marked differences from that of standard circular annuli treated in [13]. To explain these, we note that there are two main steps in treating these distribution problems: The first step is to compute the moments of a smoothed version of S_B, defined in section 2. We will show in section 3 that the moments of the smooth counting function are Gaussian and that will suffice for establishing a normal distribution for the smooth version of our problem. The second step (section 5) is to recover the distribution of the original counting function S_B by estimating the variance of the difference between S_B and its smooth version. The proof of that invokes a truncated Poisson summation formula for the number of points of a general lattice which lie in a disk, stated and proved in section 4.

The passage from circular annuli to general elliptical annuli gives rise to new problems in both steps. The reason is that to study the counting functions one uses Poisson summation to express the counting functions as a sum over a certain lattice, that is as a sum over closed geodesics of the corresponding flat torus. Unlike the case of the circle, for a generic ellipse the sum is over a lattice where the squared lengths of vectors are no longer integers but of the form $n^2 + m^2\alpha^{-2}$, where $n, m \in \mathbb{Z}$ and α is the aspect ratio of the ellipse.

One new feature present in this case is that these lengths can *cluster* together, or, more generally, one may approximate zero too well by the means of linear combinations of lengths. This causes difficulties both in bounding the variance between the original counting function and its smoothed version, especially in the truncated summation formula of section 4, and in showing that the moments of the smooth counting function are given by "diagonal-like" contributions. This clustering can be controlled when α is strongly Diophantine.

Another problem we have to face, in evaluating moments of the smooth counting function, is the possibility of non-trivial correlations in the length spectrum. Their possible existence (e.g. in the case of algebraic aspect ratio) obscures the nature of the main term (the diagonal-like contribution) at this time. If α is transcendental this problem can be overcome, see proposition 3.8.

2 Smoothing

Rather than counting integral points inside elliptic annuli, we will count Λ-points inside $B(0,1)$-annuli, where Λ is a *lattice*. Denote the corresponding counting function N_Λ, that is,

$$N_\Lambda = \#\{\vec{n} \in \Lambda : \ |\vec{n}| \leq t\}.$$

Let $\Lambda = \langle 1, i\alpha \rangle$ be a rectangular lattice with $\alpha > 0$ *transcendental* and *strongly Diophantine* real number (almost all real α satisfy this, see section 3.2). Denote

$$S_\Lambda(t,\, \rho) = \frac{N_\Lambda(t+\rho) - N_\Lambda(t) - \frac{\pi}{d}(2t\rho + \rho^2)}{\sqrt{t}} \tag{5}$$

with $d := \det(\Lambda) = \alpha$. Thus

$$S_\Lambda(t,\, \rho) = S_B(t,\, \rho)$$

for an ellipse B as in theorem 1.1, and we will prove the result for $S_\Lambda(t,\, \rho)$.

We apply the same smoothing as in [13]: let χ be the indicator function of the unit disc and ψ a nonnegative, smooth, even function on the real line, of total mass unity, whose Fourier transform, $\hat{\psi}$ is smooth and has compact support [1]. One should notice that

$$N_\Lambda(t) = \sum_{\vec{n} \in \Lambda} \chi\left(\frac{\vec{n}}{t}\right). \tag{6}$$

[1]To construct such a function, just take a function ϕ with compact support and set $\hat{\psi} = \phi * \phi^*$ where $\phi^*(y) := \overline{\phi(-y)}$. Then $\psi = |\check{\phi}|^2$ is nonnegative.

Introduce a rotationally symmetric function Ψ on $\mathbb{R}^2$ by setting $\hat{\Psi}(\vec{y}) = \hat{\psi}(|\vec{y}|)$, where $|\cdot|$ denotes the standard Euclidian norm. For $\epsilon > 0$, set

$$\Psi_\epsilon(\vec{x}) = \frac{1}{\epsilon^2}\Psi\left(\frac{\vec{x}}{\epsilon}\right).$$

Define in analogy with (6) a *smooth* counting function

$$\tilde{N}_{\Lambda,M}(t) = \sum_{\vec{n}\in\Lambda} \chi_\epsilon(\frac{\vec{n}}{t}), \tag{7}$$

with $\epsilon = \epsilon(M)$, $\chi_\epsilon = \chi * \Psi_\epsilon$, the convolution of χ with Ψ_ϵ. In what will follow,

$$\epsilon = \frac{1}{t\sqrt{M}}, \tag{8}$$

where $M = M(T)$ is the smoothness parameter, which tends to infinity with t.

We are interested in the distribution of

$$\tilde{S}_{\Lambda,\,M,\,L}(t) = \frac{\tilde{N}_{\Lambda,\,M}(t+\frac{1}{L}) - \tilde{N}_{\Lambda,\,M}(t) - \frac{\pi}{d}(\frac{2t}{L}+\frac{1}{L^2})}{\sqrt{t}}, \tag{9}$$

which is the smooth version of $S_\Lambda(t,\,\rho)$. We assume that for every $\delta > 0$, $L = L(T) = O(T^\delta)$, which corresponds to the assumption of theorem 1.1 regarding $\rho := \frac{1}{L}$. However, we will work with a smooth probability space rather than just the Lebesgue measure. For this purpose, introduce $\omega \geq 0$, a smooth function of total mass unity, such that both ω and $\hat{\omega}$ are rapidly decaying, namely

$$|\omega(t)| \ll \frac{1}{(1+|t|)^A}, \quad |\hat{\omega}(t)| \ll \frac{1}{(1+|t|)^A},$$

for every $A > 0$.

Define the averaging operator

$$\langle f\rangle_T = \frac{1}{T}\int\limits_{-\infty}^{\infty} f(t)\omega(\frac{t}{T})dt,$$

and let $\mathbb{P}_{\omega,T}$ be the associated probability measure:

$$\mathbb{P}_{\omega,T}(f\in\mathcal{A}) = \frac{1}{T}\int\limits_{-\infty}^{\infty} 1_{\mathcal{A}}(f(t))\omega(\frac{t}{T})dt,$$

We will prove the following theorem in section 3.

Theorem 2.1. *Suppose that $M(T)$ and $L(T)$ are increasing to infinity with T, such that $M = O(T^{\delta})$ for all $\delta > 0$, and $L/\sqrt{M} \to 0$. Then if α is* transcendental *and* strongly Diophantine*, we have for $\Lambda =< 1, i\alpha >$,*

$$\lim_{T\to\infty} \mathbb{P}_{\omega,T}\left\{ \frac{\tilde{S}_{\Lambda,M,L}}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathcal{A}} e^{-\frac{x^2}{2}} d$$

for any interval $\mathcal{A}$, where $\sigma^2 := \frac{8\pi}{dL}$.

3 The distribution of $\tilde{S}_{\Lambda,M,L}$

We start from a well-known definition.

Definition: A number μ is called *Diophantine*, if $\exists K > 0$, such that for a rational p/q,

$$\left| \mu - \frac{p}{q} \right| \gg_{\mu} \frac{1}{q^K}, \tag{10}$$

where the constant involved in the "$\gg$"-notation depends only on μ. Khintchine proved that *almost all* real numbers are Diophantine (see, e.g. [17], pages 60-63).

It is obvious from the definition, that μ is Diophantine iff $\frac{1}{\mu}$ is such. For the rest of this section, we will assume that $\Lambda^* = \langle 1, i\beta \rangle$ with a Diophantine $\kappa := \beta^2$, which satisfies (10) with

$$K = K_0, \tag{11}$$

where Λ^* is the *dual* lattice, that is $\beta := \frac{1}{\alpha}$. We may assume the Diophantinity of κ, since theorem 1.1 (and theorem 2.1) assume α's being *strongly Diophantine*, which implies, in particular, Diophantinity of α, β and κ (see the definition later in this section).

We will need a generalization of lemma 3.1 in [13] to a general lattice Λ rather than $\mathbb{Z}^2$.

Lemma 3.1. *As $t \to \infty$,*

$$\tilde{N}_{\Lambda,M}(t) = \frac{\pi t^2}{d} - \frac{\sqrt{t}}{d\pi} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\cos\left(2\pi t|\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} \cdot \hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) + O\left(\frac{1}{\sqrt{t}}\right), \tag{12}$$

where, again Λ^* is the dual lattice.

Proof. The proof is essentially the same as the one which obtains the original lemma (see [13], page 642). Using *Poisson summation formula* on (7) and estimating $\hat{\chi}(t\vec{k})$ by the well-known asymptotics of the Bessel J_1 function, we get:

$$\tilde{N}_{\Lambda,M}(t) = \frac{\pi t^2}{d} - \frac{\sqrt{t}}{d\pi} \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \left\{ \frac{\cos\left(2\pi t|\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} \cdot \hat{\psi}\left(\epsilon t|\vec{k}|\right) + O\left(\frac{\hat{\psi}(\epsilon t|\vec{k}|)}{t|\vec{k}|^{\frac{5}{2}}}\right) \right\},$$

where we get the main term for $\vec{k} = 0$. Finally, we obtain (12) using (8). The contribution of the error term is obtained due to the convergence of $\sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \frac{1}{|\vec{k}|^{\frac{5}{2}}}$ as well as the fact that $\hat{\psi}(x) \ll 1$. $\square$

Unlike the standard lattice, if $\Lambda = \langle 1, i\alpha \rangle$ with an irrational α^2, then clearly there are no nontrivial multiplicities, that is

Lemma 3.2. *Let $\vec{a_i} = (n_i, m_i \cdot \alpha) \in \Lambda$, $i = 1, 2$, with an irrational α^2. If $|\vec{a_1}| = |\vec{a_2}|$, then $n_1 = \pm n_2$ and $m_1 = \pm m_2$.*

By the definition of $\tilde{S}_{\Lambda, M, L}$ in (9) and appropriately manipulating the sum in (12) we obtain the following

Corollary 3.3.

$$\tilde{S}_{\Lambda, M, L}(t) = \frac{2}{d\pi} \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}} \sin\left(2\pi\big(t + \frac{1}{2L}\big)|\vec{k}| + \frac{\pi}{4}\right) \hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) + O\left(\frac{1}{\sqrt{t}}\right), \tag{13}$$

We used

$$\sqrt{t + \frac{1}{L}} = \sqrt{t} + O\left(\frac{1}{\sqrt{t}L}\right) \tag{14}$$

in order to change $\sqrt{t + \frac{1}{L}}$ multiplying the sum in (12) for $N_\Lambda(t + \frac{1}{L})$ by $\sqrt{t}$. We use a smooth analogue of the simplest bound (2) in order to bound the cost of this change to the error term.

One should note that $\hat{\psi}$'s being compactly supported means that the sum essentially truncates at $|\vec{k}| \approx \sqrt{M}$.

Proof of theorem 2.1. We will show that the moments of $\tilde{S}_{\Lambda,\,M,\,L}$ corresponding to the smooth probability space (i.e. $\langle \tilde{S}^m_{\Lambda,\,M,\,L}\rangle_T$, see section 2) converge to the moments of the normal distribution with zero mean and variance which is given by theorem 2.1. This allows us to deduce that the distribution of $\tilde{S}_{\Lambda,\,M,\,L}$ converges to the normal distribution as T approaches infinity, precisely in the sense of theorem 2.1.

First, we show that the mean is $O(\frac{1}{\sqrt{T}})$, regardless of the Diophantine properties of α. Since ω is real,

$$\left|\left\langle \sin\left(2\pi\big(t+\frac{1}{2L}\big)|\vec{k}|+\frac{\pi}{4}\right)\right\rangle_T\right| = \left|\Im m\left\{\hat{\omega}\big(-T|\vec{k}|\big)e^{i\pi(\frac{|\vec{k}|}{L}+\frac{1}{4}}\right\}\right| \ll \frac{1}{T^A|\vec{k}|^A}$$

for any $A>0$, where we have used the rapid decay of $\hat{\omega}$. Thus

$$\left|\left\langle \tilde{S}_{\Lambda,\,M,\,L}\right\rangle_T\right| \ll \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{1}{T^A|\vec{k}|^{A+3/2}} + O\left(\frac{1}{\sqrt{T}}\right) \ll O\left(\frac{1}{\sqrt{T}}\right),$$

due to the convergence of $\sum\limits_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{1}{|\vec{k}|^{A+3/2}}$, for $A>\frac{1}{2}$

Now define

$$\mathcal{M}_{\Lambda,\,m} := \left\langle \left(\frac{2}{d\pi}\sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}}\sin\left(2\pi\big(t+\frac{1}{2L}\big)|\vec{k}|+\frac{\pi}{4}\right)\hat{\psi}\big(\frac{|\vec{k}|}{\sqrt{M}}\big)\right)^m\right\rangle_T \tag{15}$$

Then from (13), the binomial formula and the Cauchy-Schwartz inequality,

$$\left\langle \big(\tilde{S}_{\Lambda,\,M,\,L}\big)^m\right\rangle_T = \mathcal{M}_{\Lambda,\,m} + O\left(\sum_{j=1}^{m}\binom{m}{j}\frac{\sqrt{\mathcal{M}_{2m-2j}}}{T^{j/2}}\right)$$

Proposition 3.4 together with proposition 3.7 allow us to deduce the result of theorem 2.1 for a *transcendental strongly Diophantine* β^2. Clearly, α's being transcendental strongly Diophantine is sufficient. ☐

3.1 The variance

The variance was first computed by Bleher and Lebowitz [2] and we will give a version suitable for our purpose. This will help the reader to understand our computation of higher moments.

Proposition 3.4. *Let α be Diophantine and $\Lambda = \langle 1,\, i\alpha\rangle$. Then if for some fixed $\delta > 0$, $M = O\big(T^{\frac{1}{K_0+1/2+\delta}}\big)$ as $T \to \infty$, then*

$$\left\langle \big(\tilde{S}_{\Lambda,\,M,\,L}\big)^2 \right\rangle_T \sim \sigma^2 := \frac{2}{d^2\pi^2} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} r(\vec{k}) \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3} \hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right),$$

where

$$r(\vec{n}) = \begin{cases} 1, & \vec{n} = (0,\,0) \\ 2, & \vec{n} = (x,\,0) \text{ or } (0,\,y)\ , \\ 4, & \text{otherwise} \end{cases} \tag{16}$$

is the "multiplicity" of $|\vec{n}|$. Moreover, if $L \to \infty$, but $L/\sqrt{M} \to 0$, then

$$\sigma^2 \sim \frac{8\pi}{dL} \tag{17}$$

Proof. Expanding out (15), we have

$$\begin{aligned}\mathcal{M}_{\Lambda,\,2} := \frac{4}{d^2\pi^2} \sum_{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\}} & \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)\sin\left(\frac{\pi|\vec{l}|}{L}\right)\hat{\psi}\big(\frac{|\vec{k}|}{\sqrt{M}}\big)\hat{\psi}\big(\frac{|\vec{l}|}{\sqrt{M}}\big)}{|\vec{k}|^{\frac{3}{2}}|\vec{l}|^{\frac{3}{2}}} \\ & \times \left\langle \sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k}| + \frac{\pi}{4}\right)\sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{l}| + \frac{\pi}{4}\right)\right\rangle_T\end{aligned} \tag{18}$$

Now, it is easy to check that the average of the second line of the previous equation is:

$$\begin{aligned}\frac{1}{4}\Big[&\hat{\omega}\big(T(|\vec{k}|-|\vec{l}|)\big)e^{i\pi(1/L)(|\vec{l}|-|\vec{k}|)} + \\ &\hat{\omega}\big(T(|\vec{l}|-|\vec{k}|)\big)e^{i\pi(1/L)(|\vec{k}|-|\vec{l}|)} + \\ &\hat{\omega}\big(T(|\vec{k}|+|\vec{l}|)\big)e^{-i\pi(1/2+(1/L)(|\vec{k}|+|\vec{l}|))} - \\ &\hat{\omega}\big(-T(|\vec{k}|+|\vec{l}|)\big)e^{i\pi(1/2+(1/L)(|\vec{k}|+|\vec{l}|))}\Big]\end{aligned} \tag{19}$$

Recall that the support condition on $\hat{\psi}$ means that $\vec{k}$ and $\vec{l}$ are both constrained to be of length $O(\sqrt{M})$, and so the off-diagonal contribution (that is for $|\vec{k}| \neq |\vec{l}|$) of the first two lines of (19) is

$$\ll \sum_{\substack{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|,\,|\vec{k}'|\le\sqrt{M}}} \frac{M^{A(K_0+1/2)}}{T^A} \ll \frac{M^{A(K_0+1/2)+2}}{T^A} \ll T^{-B},$$

for every $B > 0$, using lemma 3.5, the fact that $|\vec{k}|, |\vec{l}| \gg 1$, $\left|\hat{\psi}\right| \ll 1$, and the assumption regarding M. We may use lemma 3.5 since we have assumed in the beginning of this section that κ is Diophantine.

Obviously, the contribution to (18) of the two last lines of (19) is negligible both in the diagonal and off-diagonal cases, and so we are to evaluate the diagonal approximation of (18), changing the second line of (18) by 1/2, since the first two lines of (19) are 2. That proves the first statement of the proposition. To find the asymptotics, we take a big parameter $Y = Y(T) > 0$ (which is to be chosen later), and write:

$$\sum_{\substack{\vec{k},\vec{k}'\in\Lambda^*\setminus\{0\}\\ |\vec{k}|=|\vec{k}'|}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}\hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right) = \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} r(\vec{k})\frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}\hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right)$$

$$= \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|^2\le Y}} + \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|^2> Y}} := I_1 + I_2,$$

Now for $Y = o(M)$, $\hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right) \sim 1$ within the constraints of I_1, and so

$$I_1 \sim \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|^2\le Y}} r(\vec{k})\frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}.$$

Here we may substitute $r(\vec{k}) = 4$, since the contribution of vectors of the form $(x,\,0)$ and $(0,\,y)$ is $O(\frac{1}{L^2})$: representing their contribution as a 1-dimensional Riemann sum.

The sum in

$$4\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|^2\le Y}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3} = \frac{4}{L}\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|^2\le Y}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{\left(\frac{|\vec{k}|}{L}\right)^3}\frac{1}{L^2}.$$

is a 2-dimensional Riemann sum of the integral

$$\iint\limits_{1/L^2\ll x^2+\kappa y^2\le Y/L^2} \frac{\sin^2\left(\pi\sqrt{x^2+\kappa y^2}\right)}{|x^2+\kappa y^2|^{3/2}}dxdy \sim \frac{2\pi}{\beta}\int\limits_{\frac{1}{L}}^{\frac{\sqrt{Y}}{L}} \frac{\sin^2(\pi r)}{r^2}dr \to d\pi^3,$$

provided that $Y/L^2 \to \infty$, since $\int\limits_0^\infty \frac{\sin^2(\pi r)}{r^2} dr = \frac{\pi^2}{2}$. We have changed the coordinates to the usual elliptic ones. And so,

$$I_1 \sim \frac{4d\pi^3}{L}$$

Next we will bound I_2. Since $\hat{\psi} \ll 1$, we may use the same change of variables to obtain:

$$I_2 \ll \frac{1}{L} \iint\limits_{x^2+\kappa y^2 \geq Y/L^2} \frac{\sin^2\left(\pi\sqrt{x^2+\kappa y^2}\right)}{|x^2+\kappa y^2|^{3/2}} dxdy \ll \frac{1}{L} \int\limits_{\sqrt{Y}/L}^{\infty} \frac{dr}{r^2} = o\left(\frac{1}{L}\right).$$

This concludes the proposition, provided we have managed to choose Y with $L^2 = o(Y)$ and $Y = o(M)$. Such a choice is possible by the assumption of the proposition regarding L.

∎

Lemma 3.5. *Suppose that $\vec{k}$, $\vec{k'} \in \Lambda^*$ with $|\vec{k}|$, $|\vec{k'}| \leq \sqrt{M}$. Then if $|\vec{k}| \neq |\vec{k'}|$,*

$$\big||\vec{k}| - |\vec{k'}|\big| \gg M^{-(K_0+1/2)}$$

Proof.

$$\big||\vec{k}| - |\vec{k'}|\big| = \frac{\big||\vec{k}|^2 - |\vec{k'}|^2\big|}{|\vec{k}| + |\vec{k'}|} \gg \frac{M^{-K_0}}{\sqrt{M}} = M^{-(K_0+1/2)},$$

by (10) and (11). ∎

3.2 The higher moments

In order to compute the higher moments we will prove that the main contribution comes from the so-called *diagonal* terms (to be explained later). In order to be able to bound the contribution of the *off-diagonal* terms, we restrain ourselves to "generic" numbers, which are given in the following definition:

Definition: We call a number η strongly Diophantine, if it satisfies the following property: for any fixed n, there exists $K_1 \in \mathbb{N}$ such that for an integral polynomial $P(x) = \sum\limits_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$, with $P(\eta) \neq 0$ we have

$$\big|P(\eta)\big| \gg_{\eta,n} h(P)^{-K_1},$$

where $h(P) = \max_{0 \le i \le n} |a_i|$ is the height of P.

The fact that the strongly Diophantine numbers are "generic" follows from various classical papers, e.g. [16].

Obviously, strong Diophantinity implies Diophantinity. Just as in the case of Diophantine numbers η is strongly Diophantine, iff $\frac{1}{\eta}$ is such. Moreover, if η is strongly Diophantine, then so is η^2. As a concrete example of a transcendental strongly Diophantine number, the inequality proven by Baker [1] implies that for any *rational* $r \neq 0$, $\eta = e^r$ satisfies the desired property.

We would like to make some brief comments concerning the number K_1, which appears in the definition of a strongly Diophantine number, although the form presented is sufficient for all our purposes.

Let η be a *real* number. One defines $\theta_k(\eta)$ to be $\frac{1}{k}$ times the supremum of the real numbers ω, such that $|P(\eta)| < h(P)^{-\omega}$ for infinitely many polynomials P of degree k. Clearly,

$$\theta_k(\eta) = \frac{1}{k} \inf\{\omega : \, |P(\eta)| \gg_{\omega, k} h^{-\omega}, \deg P = k\}.$$

It is well known [21], that $\theta_k(\eta) \ge 1$ for all *transcendental* η. In 1932, Mahler [16] proved that $\theta_k(\eta) \le 4$ for almost all real η, and that allows us to take any $K_1 > 4n$. He conjectured that

$$\theta_k(\eta) \le 1$$

which was proved in 1964 by Sprindẑuk [18], [19], making it legitimate to choose any $K_1 > n$.

Sprindẑuk's result is analogous to Khintchin's theorem which states that almost no k-tuple in $\mathbb{R}^k$ is *very well approximable* (see e.g. [17], theorem 3A), for submanifold $M \subset \mathbb{R}^k$, defined by

$$M = \{(x, \, x^2, \, \dots, \, x^k) : \, x \in \mathbb{R}\}.$$

The proof of this conjecture has eventually let to development of a new branch in approximation theory, usually referred to as "Diophantine approximation with dependent quantities" or "Diophantine approximation on manifolds". A number of quite general results were proved for a manifold M, see e.g. [15].

We prove the following simple lemma which will eventually allow us to exploit the strong Diophantinity of the aspect ratio of the ellipse.

Lemma 3.6. *If $\eta > 0$ is strongly Diophantine, then it satisfies the following property: for any fixed natural m, there exists $K \in \mathbb{N}$, such that if $z_j =$*

$a_j^2+\eta b_j^2 \ll M$, *and* $\epsilon_j=\pm1$ *for* $j=1,\ldots,m$, *with integral* a_j, b_j *and if* $\sum\limits_{j=1}^{m}\epsilon_j\sqrt{z_j}\neq 0$, *then*

$$\Big|\sum_{j=1}^{m}\epsilon_j\sqrt{z_j}\Big| \gg_{\eta,m} M^{-K}. \tag{20}$$

Proof. Let m be given. We prove that every number η that satisfies the property of the definition of a strongly Diophantine number with $n=2^{m-1}$, satisfies the inequality (20) for some K, which will depend on K_1.

Let us $\{\sqrt{z_j}\}_{j=1}^{m}$ be given. Suppose first, that there is no $\{\delta_j\}_{j=1}^{m}\in\{\pm1\}^m$ with $\sum\limits_{j=1}^{m}\delta_j\sqrt{z_j}=0$. Let us consider

$$Q=Q(z_1,\,\ldots,\,z_m):=\prod_{\{\delta_j\}_{j=1}^{m}\in\{\pm1\}^m}\sum_{j=1}^{m}\delta_j\sqrt{z_j}\neq 0.$$

Now $Q=R\big(\sqrt{z_1},\,\ldots,\,\sqrt{z_m}\big)$, where

$$R\big(x_1,\,\ldots,\,x_m\big):=\prod_{\{\delta_j\}_{j=1}^{m}\in\{\pm1\}^m}\sum_{j=1}^{m}\delta_j x_j.$$

Obviously, R is a polynomial with integral coefficients of degree 2^m such that for each vector $\underline{\delta}=(\delta_j)=(\pm1)$, $R(\delta_1x_1,\,\ldots,\,\delta_mx_m)=R(x_1,\,\ldots,\,x_m)$, and thus $Q(z_1,\,\ldots,\,z_m)$ is an integral polynomial in $z_1,\,\ldots,\,z_m$ of degree 2^{m-1}. Therefore, $Q=P(\eta)$, where P is a polynomial of degree 2^{m-1}, $P=\sum\limits_{j=0}^{2^{m-1}}c_ix^i$, with $c_i\in\mathbb{Z}$, such that $c_i=P_i(a_1,\,\ldots,\,a_m,\,b_1,\,\ldots,\,b_m)$, where P_i are polynomials. Thus there exists K_2, such that $c_i\ll M^{K_2}$, and so, by the definition of strongly Diophantine numbers, $Q\gg_{\eta,m}M^{-K_2K_1}$. We conclude the proof of lemma 3.6 in this case by

$$\Big|\sum_{j=1}^{m}\epsilon_j\sqrt{z_j}\Big|=\frac{|Q|}{\Big|\prod\limits_{\{\delta_j\}_{j=1}^{m}\neq\{\epsilon_j\}_{j=1}^{m}}\sum\limits_{j=1}^{m}\delta_j\sqrt{z_j}\Big|}\gg_{\eta,m}M^{-(K_2K_1+(2^m-1)/2)},$$

and so, setting $K:=K_2K_1+\frac{(2^m-1)}{2}$, we obtain the result of the current lemma in this case.

Next, suppose that

$$\sum_{i=1}^{m}\delta_j\sqrt{z_j}=0 \tag{21}$$

for some (given) $\{\delta_i\}_{j=1}^m \in \{\pm 1\}^m$. Denote $S := \{j : \epsilon_j = \delta_j\}$, $S' = \{1, \ldots, m\} \setminus S$. One should notice that

$$\emptyset \subsetneqq S,\, S' \subsetneqq \{1, \ldots, m\}. \tag{22}$$

Writing (21) in the new notations, we obtain:

$$\sum_{j\in S} \epsilon_j \sqrt{z_i} - \sum_{j\in S'} \epsilon_j \sqrt{z_i} = 0,$$

Finally,

$$0 \neq \Big|\sum_{j=1}^m \epsilon_j \sqrt{z_j}\Big| = 2\Big|\sum_{j\in S'} \epsilon_j \sqrt{z_j}\Big| \gg_{\eta,\, m} M^{-K}$$

for some K by induction, due to (22). $\square$

Proposition 3.7. *Let $m \in \mathbb{N}$ be given. Suppose that α^2 is* <u>*transcendental*</u> *and* <u>*strongly Diophantine*</u> *which satisfy the property of lemma 3.6 for the given m, with $K = K_m$. Denote $\Lambda = \langle 1, i\alpha\rangle$. Then if $\mathcal{M} = O\big(T^{\frac{1-\delta}{K_m}}\big)$ for some $\delta > 0$, and if $L \to \infty$ such that $L/\sqrt{M} \to 0$, the following holds:*

$$\frac{\mathcal{M}_{\Lambda,\, m}}{\sigma^m} = \begin{cases} \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!} + O\big(\frac{\log L}{L}\big), & m \text{ is even} \\ O\big(\frac{\log L}{L}\big), & m \text{ is odd} \end{cases}$$

Proof. Expanding out (15), we have

$$\mathcal{M}_{\Lambda,\, m} = \frac{2^m}{d^m \pi^m} \sum_{\vec{k_1},\ldots,\,\vec{k_m} \in \Lambda^* \setminus\{0\}} \prod_{j=1}^m \frac{\sin\left(\frac{\pi|\vec{k_j}|}{L}\right)\hat{\psi}\big(\frac{|\vec{k_j}|}{\sqrt{M}}\big)}{|\vec{k_j}|^{\frac{3}{2}}} \times \left\langle \prod_{j=1}^m \sin\left(2\pi\big(t + \frac{1}{2L}\big)|\vec{k_1}| + \frac{\pi}{4}\right)\right\rangle_T \tag{23}$$

Now,

$$\left\langle \prod_{j=1}^m \sin\left(2\pi\big(t + \frac{1}{2L}\big)|\vec{k_1}| + \frac{\pi}{4}\right)\right\rangle_T$$

$$= \sum_{\epsilon_j = \pm 1} \frac{\prod_{j=1}^m \epsilon_j}{2^m i^m} \hat{\omega}\Big(- T\sum_{j=1}^m \epsilon_j |\vec{k_j}|\Big) e^{\pi i \sum_{j=1}^m \epsilon_j \big((1/L)|\vec{k_j}| + 1/4\big)}$$

We call a term of the summation in (23) with $\sum\limits_{j=1}^{m} \epsilon_j |\vec{k_j}| = 0$ *diagonal*, and *off-diagonal* otherwise. Due to lemma 3.6, the contribution of the *off-diagonal* terms is:

$$\ll \sum_{\vec{k_1},\dots,\vec{k_m} \in \Lambda^* \setminus \{0\}} \left(\frac{T}{M^{K_m}} \right)^{-A} \ll M^m T^{-A\delta},$$

for every $A > 0$, by the rapid decay of $\hat{\omega}$ and our assumption regarding M.

Since m is constant, this allows us to reduce the sum to the *diagonal terms*. The following definition and corollary 3.9 will allow us to actually sum over the diagonal terms, making use of α's being transcendental.

Definition: We say that a term corresponding to $\{\vec{k_1}, \dots, \vec{k_m}\} \in \Big(\Lambda^* \setminus \{0\}\Big)^m$ and $\{\epsilon_j\} \in \{\pm 1\}^m$ is a *principal diagonal* term if there is a partition $\{1, \dots, m\} = \bigsqcup\limits_{i=1}^{l} S_i$, such that for each $1 \le i \le l$ there exists a primitive $\vec{n_i} \in \Lambda^* \setminus \{0\}$, with non-negative coordinates, that satisfies the following property: for every $j \in S_i$, there exist $f_j \in \mathbb{Z}$ with $|\vec{k_j}| = f_j |\vec{n_i}|$. Moreover, for each $1 \le i \le l$, $\sum\limits_{j \in S_i} \epsilon_j f_j = 0$.

Obviously, the principal diagonal is contained within the diagonal. However, if α is *transcendental*, the converse is also true. It is easily seen, given the following proposition.

Proposition 3.8. *Suppose that $\eta \in \mathbb{R}$ is a <u>transcendental</u> number. Let*

$$z_j = a_j^2 + \eta b_j^2$$

such that $(a_j, b_j) \in \mathbb{Z}_+^2$ are all different primitive vectors, for $1 \le j \le m$. Then $\{\sqrt{z_j}\}_{j=1}^{m}$ are linearly independent over $\mathbb{Q}$.

The last proposition is an analogue of a well-known theorem due to Besicovitch [6] about incommensurability of square roots of integers. A proof of a much more general statement may be found e.g. in [3] (see lemma 2.3 and the appendix).

Thus we have

Corollary 3.9. *Every <u>diagonal</u> term is a <u>principle diagonal</u> term whenether α is <u>transcendendal</u>.*

By corollary 3.9, summing over diagonal terms is the same as summing over *principal* diagonal terms. Thus:

$$\frac{\mathcal{M}_{\Lambda,m}}{\sigma^m} \sim \sum_{l=1}^{m} \sum_{\{1,\dots,m\}=\bigsqcup_{i=1}^{l} S_i} \left(\frac{1}{\sigma^{|S_1|}} \sum_{\vec{n}_1\in\Lambda^*\setminus\{0\}} {}'D_{\vec{n}_1}(S_1) \right)$$
$$\times \left(\frac{1}{\sigma^{|S_2|}} \sum_{\Lambda^*\setminus\{0\}\ni\vec{n}_2\neq\vec{n}_1} {}'D_{(\vec{n}_2)}(S_2) \right) \dots \left(\frac{1}{\sigma^{|S_l|}} \sum_{\Lambda^*\setminus\{0\}\ni\vec{n}_l\neq\vec{n}_2,\dots,\vec{n}_{l-1}} {}'D_{\vec{n}_l}(S_l) \right), \tag{24}$$

where the inner summations are over primitive 1st-quadrant vectors of $\Lambda^* \setminus \{0\}$, and

$$D_{\vec{n}}(S) = \frac{r(\vec{n})}{|\vec{n}|^{3|S|/2}} \sum_{\substack{f_j\geq 1 \\ \epsilon_j=\pm 1 \\ \sum_{j\in S}\epsilon_j f_j=0}} \prod_{j\in S} \frac{-i\epsilon_j}{d\pi f_j^{3/2}} \sin\left(\frac{\pi}{L} f_j|\vec{n}|\right) \hat{\psi}\left(\frac{|\vec{n}|}{\sqrt{M}}\right) e^{i\pi\epsilon_j/4},$$

with $r(\vec{n})$ given by (16).

Lemma 3.10 allows us to deduce that the contribution to (24) of a partition is $O(\frac{\log(L)}{L})$, unless $|S_i| = 2$ for every $i = 1,\dots,l$. In the latter case the contribution is 1 by the 2nd case of the same lemma. This is impossible for an odd m, and so, it finishes the proof of the current proposition in that case. Otherwise, suppose m is even. Then the number of partitions $\{1,\dots,m\} = \bigsqcup_{i=1}^{l} S_i$ with $|S_i| = 2$ for every $1 \leq i \leq l$ is

$$\frac{1}{(\frac{m}{2})!}\binom{m}{2}\binom{m-2}{2}\cdot\ldots\cdot\binom{2}{2} = \frac{1}{(\frac{m}{2})!}\frac{m!}{2!\,(m-2)!}\frac{(m-2)!}{2!\,(m-4)!}\cdot\ldots\cdot\frac{2!}{2!}$$
$$= \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!}$$

That concludes the proof of proposition 3.7.

□

Lemma 3.10. *If $L \to \infty$ such that $L/\sqrt{M} \to 0$, then*

$$\frac{1}{\sigma^m}\left|\sum_{\vec{n}\in\Lambda^*\setminus\{0\}} {}'D_{\vec{n}}(S)\right| = \begin{cases} 0, & |S| = 1 \\ 1, & |S| = 2 \\ O\left(\frac{\log L}{L}\right), & |S| \geq 3 \end{cases}$$

where the ' in the summation means that it is over primitive vectors (a, b).

Proof. Without loss of generality, we may assume that $S = \{1, 2, \ldots, |S|\}$, and we assume that $k := |S| \geq 3$. Now,

$$\Big| \sum_{\vec{n} \in \Lambda^* \setminus \{0\}} {}' D_{\vec{n}}(S) \Big| \ll \sum_{\vec{n} \in \Lambda^* \setminus \{0\}} \frac{1}{|\vec{n}|^{3k/2}} Q(|\vec{n}|), \tag{25}$$

where

$$Q(z) := \sum_{\{\epsilon_j\} \in \{\pm 1\}^k} \sum_{\substack{f_j \geq 1 \\ \sum_{j=1}^{k} \epsilon_j f_j = 0}} \prod_{j=1}^{k} \frac{|\sin(\frac{\pi}{L} f_j z)|}{f_j^{3/2}}.$$

Note that $Q(z) \ll 1$ for all z. We would like to establish a sharper result for $z \ll L$. In order to have $\sum_{j=1}^{k} \epsilon_j f_j = 0$, at least two of the ϵ_j must have different signs, and so, with no loss of generality, we may assume, $\epsilon_k = -1$ and $\epsilon_{k-1} = +1$. We notice that the last sum is, in fact, a Riemann sum, and so

$$Q(z) \ll \frac{L^{k-1}}{L^{3k/2}} \int_{1/L}^{\infty} \cdots \int_{1/L}^{\infty} dx_1 \cdots dx_{k-2} \sum_{\{\epsilon_j\}_{j=1}^{k-2} \in \{\pm 1\}^{k-2}} \int_{\frac{1}{L} + \max(0, -\sum_{j=1}^{k-2} \epsilon_j f_j)}^{\infty} dx_{k-1}$$

$$\times \Big(\prod_{j=1}^{k-1} \frac{|\sin(\pi x_j z)|}{x_j^{3/2}} \Big) \frac{\left| \sin\left(\pi z \cdot \left(x_{k-1} + \sum_{j=1}^{k-2} \epsilon_j x_j \right) \right) \right|}{\left(x_{k-1} + \sum_{j=1}^{k-1} \epsilon_j x_j \right)^{3/2}}$$

By changing variables $y_i = z \cdot x_i$ of the last integral, we obtain:

$$Q(z) \ll \frac{z^{k/2+1}}{L^{k/2+1}} \int_{1/L}^{\infty} \cdots \int_{1/L}^{\infty} dy_1 \cdots dy_{k-2} \sum_{\{\epsilon_j\}_{j=1}^{k-2} \in \{\pm 1\}^{k-2}} \int_{\frac{z}{L} + \max(0, -\sum_{j=1}^{k-2} \epsilon_j f_j)}^{\infty} dy_{k-1}$$

$$\times \Big(\prod_{j=1}^{k-1} \frac{|\sin(\pi y_j)|}{y_j^{3/2}} \Big) \frac{\left| \sin\left(\pi \cdot \left(y_{k-1} + \sum_{j=1}^{k-2} \epsilon_j y_j \right) \right) \right|}{\left(y_{k-1} + \sum_{j=1}^{k-1} \epsilon_j y_j \right)^{3/2}},$$

and since the last multiple integral is bounded, we may conclude that

$$Q(z) \ll \begin{cases} \frac{z^{k/2+1}}{L^{k/2+1}}, & z < L \\ 1, & z \geq L \end{cases}$$

Thus, by (25),

$$\Bigg| \sum_{\vec{n} \in \Lambda^* \setminus \{0\}} {}' D_{\vec{n}}(S) \Bigg| \ll \sum_{\substack{\vec{n} \in \Lambda^* \setminus \{0\} \\ |\vec{n}| \le L}} \frac{1}{|\vec{n}|^{3k/2}} \cdot \frac{|\vec{n}|^{k/2+1}}{L^{k/2+1}} + \sum_{\substack{\vec{n} \in \Lambda^* \setminus \{0\} \\ |\vec{n}| > L}} \frac{1}{|\vec{n}|^{3k/2}} =: S_1 + S_2.$$

Now, considering S_1 and S_2 as Riemann sums, and computing the corresponding integrals in the usual elliptic coordinates we get:

$$S_1 \ll \frac{1}{L^{k/2+1}} \sum_{\substack{\vec{n} \in \Lambda^* \setminus \{0\} \\ |\vec{n}| \le L}} \frac{1}{|\vec{n}|^{k-1}} \ll \frac{1}{L^{k/2+1}} \int_1^L \frac{dr}{r^{k-2}} \ll \frac{\log L}{L^{k/2+1}},$$

since $k \ge 3$.

Similarly,

$$S_2 \ll \int_L^\infty \frac{dr}{r^{3k/2-1}} \ll \frac{1}{L^{3k/2-2}} \ll \frac{1}{L^{k/2+(k-2)}} \ll \frac{1}{L^{k/2+1}},$$

again since $k \ge 3$.

And so, returning to the original statement of the lemma, if $k = |S| \ge 3$,

$$\frac{1}{\sigma^m} \Bigg| \sum_{\vec{n} \in \Lambda^* \setminus \{0\}} {}' D_{\vec{n}}(S) \Bigg| \ll L^{k/2} \left(\frac{\log L}{L^{k/2+1}} \right) \ll \frac{\log L}{L},$$

by (17).

In the case $|S| = 2$, by the definition of $D_{\vec{n}}$ and σ^2, we see that

$$\sum_{\vec{n} \in \Lambda^* \setminus \{0\}} {}' D_{\vec{n}}(S) = \sigma^2.$$

This completes the proof of the lemma.

$\square$

4 An asymptotical formula for N_Λ

We need an asymptotical formula for the *sharp* counting function N_Λ. Unlike the case of the standard lattice, $\mathbb{Z}^2$, in order to have a good control over the error terms we should use some Diophantine properties of the lattice we are working with. We adapt the following notations:

Let Λ be a lattice and $t > 0$ a real variable. Denote the set of squared norms of Λ by

$$SN_\Lambda = \{|\vec{n}|^2 : \; n \in \Lambda\}.$$

Suppose we have a function $\delta_\Lambda : SN_\Lambda \to \mathbb{R}$, such that given $\vec{k} \in \Lambda$, there are <u>no</u> vectors $\vec{n} \in \Lambda$ with $0 < ||\vec{n}|^2 - |\vec{k}|^2| < \delta_\Lambda(|\vec{k}|^2)$. That is,

$$\Lambda \cap \{\vec{n} \in \Lambda : \; |\vec{k}|^2 - \delta_\Lambda(|\vec{k}|^2) < |\vec{n}|^2 < |\vec{k}|^2 + \delta_\Lambda(|\vec{k}|^2)\} = A_{|\vec{k}|},$$

where

$$A_y := \{\vec{n} \in \Lambda : \; |\vec{n}| = y\}.$$

Extend δ_Λ to $\mathbb{R}$ by defining $\delta_\Lambda(x) := \delta_\Lambda(|\vec{k}|^2)$, where $\vec{k} \in \Lambda$ minimizes $|x - |\vec{k}|^2|$ (in the case there is any ambiguity, that is if $x = \frac{|\vec{n_1}|^2 + |\vec{n_2}|^2}{2}$ for vectors $\vec{n_1}, \vec{n_2} \in \Lambda$ with consecutive increasing norms, choose $\vec{k} := \vec{n_1}$). We have the following lemma:

Lemma 4.1. *For every $a > 0$, $c > 1$,*

$$\begin{aligned} N_\Lambda(t) = \frac{\pi}{d} t^2 - \frac{\sqrt{t}}{d\pi} \sum_{\substack{\vec{k} \in \Lambda^* \setminus \{0\} \\ |\vec{k}| \le \sqrt{N}}} \frac{\cos\left(2\pi t |\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} + O(N^a) \\ + O\left(\frac{t^{2c-1}}{\sqrt{N}}\right) + O\left(\frac{t}{\sqrt{N}} \cdot \left(\log t + \log(\delta_\Lambda(t^2))\right)\right) \\ + O\left(\log N + \log(\delta_{\Lambda^*}(t^2))\right) \end{aligned}$$

As a typical example of such a function, δ_Λ, for $\Lambda = \langle 1, i\alpha \rangle$, with a Diophantine $\gamma := \alpha^2$, we may choose $\delta_\Lambda(y) = \frac{c}{y^{K_0}}$, where c is a constant. In this example, if $\Lambda \ni \vec{k} = (a, b)$, then by lemma 3.2, $A_{|\vec{k}|} = (\pm a, \pm b)$, provided that γ is irrational.

Our ultimate goal in this section is to prove lemma 4.1. However, it would be more convenient to work with $x = t^2$, and by abuse of notations we will call the counting function N_Λ. Moreover, we will redefine

$$N_\Lambda(x) := \begin{cases} \#\{\vec{k} : \; |\vec{k}|^2 \le x\}, & x \ne |\vec{k}|^2 \text{ for every } \vec{k} \in \Lambda \\ \#\{\vec{k} : \; |\vec{k}|^2 < x\} + 2, & \text{otherwise} \end{cases}$$

(recall that every norm of a Λ-vector is of multiplicity 4). We are repeating the argument of Titchmarsh [22] that establishes the corresponding result for the remainder of the arithmetic function, which counts the number of different ways to write m as a multiplication of a fixed number of natural numbers.

Let $\Lambda = \langle 1, i\alpha \rangle$. For $\gamma := \alpha^2$, introduce a function $\mathcal{Z}_\gamma(s)$ (this is a special value of an Eisenstein series) where $s = \sigma + it$ is a complex variable. For $\sigma > 1$, $\mathcal{Z}_\gamma(s)$ is defined by the following converging series:

$$\mathcal{Z}_\gamma(s) := \frac{1}{4} \sum_{\vec{k} \in \Lambda \setminus 0} \frac{1}{|\vec{k}|^{2s}}. \tag{26}$$

Then $\mathcal{Z}_\gamma$ has an analytic continuation to the whole complex plane, except for a single pole at $s = 1$, defined by the formula

$$\Gamma(s)\pi^{-s}\mathcal{Z}_\gamma(s) = \int_1^\infty x^{s-1}\psi_\gamma(x)dx + \frac{1}{\sqrt{\gamma}} \int_1^\infty x^{-s}\psi_{1/\gamma}(x)dx - \frac{s - \sqrt{\gamma}(s-1)}{4\sqrt{\gamma}s(1-s)},$$

where

$$\psi_\gamma(x) := \frac{1}{4} \sum_{\vec{k} \in \Lambda \setminus 0} e^{-\pi|\vec{k}|^2 x}.$$

This enables us to compute the residue of $\mathcal{Z}_\gamma$ at $s = 1$:

$$Res(\mathcal{Z}_\gamma, 1) = \frac{\pi}{4\sqrt{\gamma}}.$$

Moreover, $\mathcal{Z}_\gamma$ satisfies the following functional equation:

$$\mathcal{Z}_\gamma(s) = \frac{1}{\sqrt{\gamma}}\chi(s)\mathcal{Z}_{1/\gamma}(1-s), \tag{27}$$

with

$$\chi(s) = \pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}. \tag{28}$$

We will adapt the notation

$$\chi_\gamma(s) := \frac{1}{\sqrt{\gamma}}\chi(s).$$

The connection between N_Λ and $\mathcal{Z}_\gamma$ is given in the following formula, which is satisfied for every $c > 1$:

$$\frac{1}{4}N_\Lambda(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{Z}_\gamma(s)\frac{x^s}{s}ds,$$

To prove it, just write $\mathcal{Z}_\gamma$ explicitly as the converging series, and use

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{y^s}{s}ds=\nu(y),$$

where

$$\nu(y):=\begin{cases}1, & y>1\\ \frac{1}{2} & y=1\\ 0; & 0<y<1\end{cases},$$

see [9], lemma on page 105, for example. One should bear in mind that the infinite integral above is not converging, and so we consider it in the symmetrical sense (that is, $\lim\limits_{T\to\infty}\int\limits_{c-iT}^{c+iT}$).

The following lemma will convert the infinite vertical integral in the last equation into a finite one, accumulating the corresponding error term. It will make use of the Diophantine properties of γ.

Lemma 4.2. *In the notations of lemma 4.1, for any constant $c>1$,*

$$\frac{1}{4}N_\Lambda(x)=\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\mathcal{Z}_\gamma(s)\frac{x^s}{s}ds+O\left(\frac{x^c}{T}\right)+O\left(\frac{x}{T}\big(\log x+\log\delta_\Lambda(x)\big)\right)\quad(29)$$

as x, $T\to\infty$.

Proof. Lemma on page 105 of [9] asserts moreover that for $y\neq 1$

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\frac{y^s}{s}ds=\nu(y)+O\left(y^c\min\big(1,\frac{1}{T|\log y|}\big)\right),\quad(30)$$

whereas for $y=1$,

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\frac{ds}{s}=\frac{1}{2}+O\left(\frac{1}{T}\right)\quad(31)$$

Suppose first that $x\neq|\vec{k}|^2$ for every $\vec{k}\in\Lambda$. Summing (30) for $y=\frac{x}{|\vec{k}|^2}$, where $\vec{k}\in\Lambda\setminus\{0\}$ gives (dividing both sides by 4):

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\mathcal{Z}_\gamma(s)\frac{x^s}{s}ds=\frac{1}{4}N_\Lambda(x)+O\left(x^c\sum_{\vec{k}\in\Lambda\setminus\{0\}}\frac{\min\left(1,\frac{1}{T\log\frac{x}{|\vec{k}|^2}}\right)}{|\vec{k}|^{2c}}\right).$$

The contribution to the error term of the right hand side of the last equality of $\vec{k} \in \Lambda$ with $|\vec{k}|^2 > 2x$ or $|\vec{k}|^2 < \frac{1}{2}x$ is

$$\ll \frac{x^c}{T} \sum_{|\vec{k}| \geq 2x \text{ or } |\vec{k}| \leq \frac{1}{2}x} \frac{1}{|\vec{k}|^{2c}} \leq \frac{x^c}{T} \mathcal{Z}_\gamma(c) \ll \frac{x^c}{T}.$$

For vectors $\vec{k_0} \in \Lambda$, which minimize $\big||\vec{k}|^2 - x\big|$ (in the case of ambiguity we choose $\vec{k_0}$ the same way we did in lemma 4.1 while extending δ_Λ), the corresponding contribution is

$$\frac{x^c}{|\vec{k}|^{2c}} \ll \frac{x^c}{x^c} = 1.$$

Finally, we bound the contribution of vectors $\vec{k} \in \Lambda \setminus \{0\}$ with $|\vec{k_0}|^2 < |\vec{k}|^2 < 2x$, and similarly, of vectors with $\frac{1}{2}x < |\vec{k}|^2 < |\vec{k_0}|^2$. Now, by the definition of δ_Λ, every such $\vec{k}$ satisfies:

$$|\vec{k}|^2 \geq |\vec{k_0}|^2 + \delta_\Lambda(x) \geq x + \frac{1}{2}\delta_\Lambda(x).$$

Moreover, $\log \frac{|\vec{k}|^2}{x} \gg \frac{|\vec{k}|^2 - |\vec{k_0}|^2}{x}$, and so the contribution is:

$$\begin{aligned}
&\ll \frac{x^c}{x^c T} x \sum_{x + \frac{1}{2}\delta_\Lambda(x) \leq |\vec{k}|^2 < 2x} \frac{1}{|\vec{k}|^2 - |\vec{k_0}|^2} \ll \frac{x}{T} \int\limits_{\sqrt{|\vec{k_0}|^2 + \delta_\Lambda(x)}}^{\sqrt{2x}} \frac{r}{r^2 - |\vec{k_0}|^2} dr \\
&= \frac{x}{2T} \int\limits_{|\vec{k_0}|^2 + \delta_\Lambda(x)}^{2x} \frac{du}{u - |\vec{k_0}|^2} \ll \frac{x}{T} \log\left(u - |\vec{k_0}|^2\right)\Big|_{|\vec{k_0}|^2 + \delta_\Lambda(x)}^{2x} \\
&\ll \frac{x}{T}\big(\log x + \log \delta_\Lambda(x)\big)
\end{aligned}$$

If $x = |\vec{k_0}|^2$ for some $\vec{k_0} \in \Lambda$, the proof is the same except that we should invoke (31) rather than (30) for $|\vec{k}| = |\vec{k_0}|$.

That concludes the proof of lemma 4.2. $\square$

Proof of lemma 4.1. We use lemma 4.2 and would like to move the contour of the integral in (29) from $\sigma = c$, $-T \leq t \leq T$ left to $\sigma = -a$ for some $a > 0$. Now, for $\sigma \geq c$,

$$\big|\mathcal{Z}_\gamma(s)\big| = O\big(1\big),$$

and by the functional equation (27) and the Stirling approximation formula,

$$\big|\mathcal{Z}_\gamma(s)\big| \ll t^{1+2a}$$

for $\sigma = -a$. Thus by the Phragmén-Lindelöf argument

$$\big|\mathcal{Z}_\gamma(s)\big| \ll t^{(1+2a)(c-\sigma)/(a+c)}$$

in the rectangle $-a - iT$, $c - iT$, $c + iT$, $-a + iT$. Using this bound, we obtain

$$\Big|\int\limits_{-a+iT}^{c+iT} Z_\gamma(s)\frac{x^s}{s}ds\Big| \ll \frac{T^{2a}}{x^a} + \frac{x^c}{T},$$

and so is $\big|\int\limits_{-a-iT}^{c-iT}\big|$. Collecting the residues at $s = 1$ with residue being the main term of the asymptotics,

$$Res\big(Z_\gamma(s)\frac{x^s}{s},\, 1\big) = \frac{\pi}{4\sqrt{\gamma}}x$$

and at $s = 0$ with

$$Res\big(Z_\gamma(s)\frac{x^s}{s},\, 0\big) = Z_\gamma(0) = O\big(1\big),$$

we get:

$$\Delta_\Lambda(x) := \frac{1}{4}N_\Lambda(x) - \frac{\pi}{4\sqrt{\gamma}}x = \frac{1}{2\pi i}\int\limits_{-a-iT}^{-a+iT}\mathcal{Z}_\gamma(s)\frac{x^s}{s}ds$$
$$+ O\left(\frac{x^c}{T}\right) + O\left(\frac{x}{T}\big(\log x + \log\delta_\Lambda(x)\big)\right) + O(1) + O\big(\frac{T^{2a}}{x^a}\big).$$

Denote the integral in the last equality by I and let $\kappa := \frac{1}{\gamma}$. Using the functional equation of $\mathcal{Z}_\gamma$ (27) again, and using the definition of $\mathcal{Z}_\kappa$ for $\sigma > 1$, (26), we get:

$$I = \frac{1}{2\pi i}\int\limits_{-a-iT}^{-a+iT}\chi_\gamma(s)\mathcal{Z}_\kappa(1-s)\frac{x^s}{s}ds = \frac{1}{2\pi i}\sum_{\vec{k}\in\Lambda^*}{}'\int\limits_{-a-iT}^{-a+iT}\frac{\chi_\gamma(s)}{|\vec{k}|^{2-2s}}\frac{x^s}{s}ds, \qquad (32)$$

where the $'$ means that the summation is over vectors in the 1st quadrant. Put

$$\frac{T^2}{\pi^2 x} := N + \frac{1}{2}\delta_{\Lambda^*}(N), \qquad (33)$$

where $N = |\vec{k_0}|^2$ for some $\vec{k_0} \in \Lambda^*$ and consider separately vectors $\vec{k} \in \Lambda^*$ with $|\vec{k}|^2 > N$ and ones with $|\vec{k}|^2 \leq N$.

First we bound the contribution of vectors $\vec{k} \in \Lambda^*$ with $|\vec{k}|^2 > N$. Write the integral in (32) as $\int\limits_{-a-iT}^{-a+iT} = \int\limits_{-a-iT}^{-a-i} + \int\limits_{-a-i}^{-a+i} + \int\limits_{-a+i}^{-a+iT}$. Then

$$\Big| \sum_{\vec{k}\in\Lambda^*}{}' \int\limits_{-a-i}^{-a+i} \frac{\chi_\gamma(s)}{|\vec{k}|^{2-2s}} \frac{x^s}{s} ds \Big| \ll x^{-a} \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2 > N}}{}' \frac{1}{|\vec{k}|^{2+2a}} \leq x^{-a} \mathcal{Z}_\kappa(1+a) \ll x^{-a}.$$

Now,

$$\begin{aligned}|J| = \Big| \int\limits_{-a+i}^{-a+iT} \frac{\chi_\gamma(s)}{|\vec{k}|^{2-2s}} \frac{x^s}{s} ds \Big| &= \frac{x^{-a}\pi^{-2a-1}}{\sqrt{\gamma}|\vec{k}|^{2+2a}} \Big| \int\limits_1^T i \frac{\Gamma(1-s)}{\Gamma(s)} \frac{\big(|\vec{k}|^2 x\big)^{ti}}{ti} \pi^{2ti} dt \Big| \\ &\ll \frac{x^{-a}}{|\vec{k}|^{2+2a}} \Big| \int\limits_1^T e^{iF(t)} \Big(t^{2a} + O\big(t^{2a-1}\big) \Big) dt \Big|,\end{aligned}$$

with

$$F(t) = 2t\big(-\log t + \log \pi + 1\big) + t \log\big(|\vec{k}|^2 x\big) = t \log \frac{\pi^2 e^2 |\vec{k}|^2 x}{t^2},$$

due to the Stirling approximation formula.

One should notice that the contribution of the error term in the last bound is

$$\ll \frac{T^{2a}}{x^a} \sum_{\vec{k}\in\Lambda^*}{}' \frac{1}{|\vec{k}|^{2+2a}} = \frac{T^{2a}}{x^a} \mathcal{Z}_\kappa(1+a) \ll N^a.$$

We would like to invoke lemma 4.3 of [22] in order to bound the integral above. For this purpose we compute the derivative:

$$F'(t) = \log\left(\frac{|\vec{k}|^2 x \pi^2}{t^2} \right) \geq \log\left(\frac{|\vec{k}|^2}{N + \frac{1}{2}\delta_{\Lambda^*}(N)} \right),$$

by the definition of N, (33). Thus in the notations of lemma 4.3 of [22],

$$\frac{F'(t)}{G(t)} = \frac{\log\left(\frac{|\vec{k}|^2 x \pi^2}{t^2} \right)}{t^{2a}} \geq \frac{\log\left(\frac{|\vec{k}|^2}{N + \frac{1}{2}\delta_{\Lambda^*}(N)} \right)}{T^{2a}}.$$

We would also like to check that $\frac{G(t)}{F'(t)}$ is monotonic. Differentiating that function and leaving only the numerator, we get:

$$-t^{2a-1}\Big(2a\log\frac{|\vec{k}|^2 x\pi^2}{t^2}+2\Big) < -2a\cdot t^{2a-1}\log\frac{|\vec{k}|^2}{N+\frac{1}{2}\delta_{\Lambda^*}(N)} < 0,$$

since $|\vec{k}^2| > N$. Thus

$$|J| \ll \frac{x^{-a}}{|\vec{k}|^{2+2a}}\frac{T^{2a}}{\log\frac{|\vec{k}|^2}{N+\frac{1}{2}\delta_{\Lambda^*}(N)}},$$

getting the same bound for $\Big|\int\limits_{-a-iT}^{-a-i}\Big|$, and therefore we are estimating

$$\sum_{\vec{k}\in\Lambda^*}{}'\frac{T^{2a}}{\log\frac{|\vec{k}|^2}{N+\frac{1}{2}\delta_{\Lambda^*}(N)}}.$$

For $|\vec{k}|^2 \geq 2N$, the contribution of the sum in (32) is:

$$\ll \frac{T^{2a}}{x^a}\sum_{\substack{\vec{k}\in\Lambda^*\\ |\vec{k}|^2\geq 2N}}\frac{1}{|\vec{k}|^{2+2a}} \leq \frac{T^{2a}}{x^a}\mathcal{Z}_\kappa(1+a) \ll N^a$$

As for vectors $\vec{k}\in\Lambda^*$ with $N+\delta_\kappa(N)\leq|\vec{k}|^2<2N$,

$$\log\frac{|\vec{k}|^2}{N+\frac{1}{2}\delta_\kappa(N)} \gg \frac{|\vec{k}|^2-N}{N},$$

which implies that the corresponding contribution to the sum in (32) is:

$$\begin{aligned}\ll \frac{T^{2a}}{x^aN^{1+a}}\sum_{\substack{\vec{k}\in\Lambda^*\\ N+\delta_\kappa(N)\leq|\vec{k}|^2<2N}}{}'\frac{N}{|\vec{k}|^2-N} &\ll \int\limits_{\sqrt{N+\delta_\kappa(N)}}^{\sqrt{2N}}\frac{r}{r^2-N-\frac{1}{2}\delta_\kappa(N)}\\ &\ll \log\big(\delta_\kappa(N)\big)+\log N\end{aligned}$$

The main term of I comes from $|\vec{k}|^2\leq N$. For such a $\vec{k}$, we write

$$\int\limits_{-a-iT}^{-a+iT} = \int\limits_{-i\infty}^{i\infty} - \Bigg(\int\limits_{iT}^{i\infty}+\int\limits_{-i\infty}^{-iT}+\int\limits_{-iT}^{-a-iT}+\int\limits_{-a+iT}^{iT}\Bigg), \tag{34}$$

that is, we are moving the contour of the integration to the imaginary axis.

Consider the first integral in the brackets. It is a constant multiple of

$$\int\limits_{T}^{\infty} e^{iF(t)} dt \ll \frac{1}{\log\left(\frac{N+\frac{1}{2}\delta_\kappa(N)}{|\vec{k}|^2}\right)},$$

and so the contribution of the corresponding sum is

$$\ll \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2\le N}} \frac{1}{|\vec{k}|^2 \log\left(\frac{N+\frac{1}{2}\delta_\kappa(N)}{|\vec{k}|^2}\right)} \ll N \int\limits_{1}^{\sqrt{N}} \frac{dr}{r\big(N+\frac{1}{2}\delta_\kappa(N)-r^2\big)}$$
$$\ll \log N + \log \delta_\kappa(N),$$

by lemma 4.2 of [22], and similarly for the second integral in the brackets in (34).

The last two give

$$\ll \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2\le N}} \frac{1}{|\vec{k}|^2} \int\limits_{-a}^{0} \left(\frac{|\vec{k}|^2 x}{T^2}\right)^{\sigma} d\sigma \ll \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2\le N}} \frac{1}{|\vec{k}|^2}\left(\frac{T^2}{|\vec{k}|^2 x}\right)^{a}$$
$$\ll \frac{T^{2a}}{x^a} \int\limits_{1}^{\sqrt{N}} \frac{dr}{r^{2a+1}} \ll N^a.$$

Altogether we have now proved:

$$\begin{aligned} \Delta_\Lambda(x) &= \frac{1}{2\pi^2 di} \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2\le N}}{}' \frac{1}{|\vec{k}|^2} \int\limits_{-i\infty}^{i\infty} \pi^{2s} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{\big(|\vec{k}|^2 x\big)^s}{s} ds + O(N^a) \\ &+ O\left(\frac{x^{c-1/2}}{\sqrt{N}}\right) + O\left(\frac{\sqrt{x}}{\sqrt{N}} \cdot \big(\log x + \log(\delta_\Lambda(x))\big)\right) \\ &+ O\big(\log N + \log(\delta_{\Lambda^*}(x))\big) \end{aligned} \tag{35}$$

Recall the integral $\int\limits_{-i\infty}^{i\infty} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{y^s}{s} ds$ is a principal value, that is $\lim\limits_{T\to\infty} \int_{-iT}^{iT}$. We have

$$\lim_{T\to\infty} \int_{-iT}^{iT} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{y^s}{s} ds = -\sqrt{y} J_{-1}(2\sqrt{y})$$

as can be seen by shifting contours. Note that the analogous Barnes-Mellin formula

$$J_\nu(x) = \frac{1}{2\pi i} \int\limits_{-i\infty}^{i\infty} \Gamma(-s)[\Gamma(\nu+s+1)]^{-1}(x/2)^{\nu+2s} ds$$

valid for $Re(\nu) > 0$ (see [11], (36), page 83), which deals with convergent integrals, is proved in this manner.

The well-known asymptotics of the bessel J-function,

$$J_{-1}(y) = \sqrt{\frac{2}{\pi y}} \cos\left(y + \frac{\pi}{4}\right) + O(y^{-3/2})$$

as $y \to \infty$, allow us to estimate the integral involved in (35) in terms of x and $\vec{k}$. Collecting all the constants and the error terms, we obtain the result of lemma 4.1. $\square$

5 Unsmoothing

Proposition 5.1. *Let a lattice $\Lambda = \langle 1, i\alpha \rangle$ with a Diophantine $\gamma := \alpha^2$ be given. Suppose that $L \to \infty$ as $T \to \infty$ and choose M, such that $L/\sqrt{M} \to 0$, but $M = O(T^\delta)$ for every $\delta > 0$ as $T \to \infty$. Suppose furthermore, that $M = O(L^{s_0})$ for some (fixed) $s_0 > 0$. Then*

$$\left\langle \left| S_\Lambda(t,\,\rho) - \tilde{S}_{\Lambda,\,M,\,L}(t) \right|^2 \right\rangle_T \ll \frac{1}{\sqrt{M}}$$

Proof. Since γ is Diophantine, we may invoke lemma 4.1 with $\delta_\Lambda(y) = \frac{c_1}{y^{K_0}}$ and $\delta_{\Lambda^*}(y) = \frac{c_2}{y^{K_0}}$, where $c_1,\, c_2$ are constants. Choosing $a = \delta'$ and $c = 1+\delta'/2$ for $\delta' > 0$ arbitrarily small and using essentially the same manipulation we used in order to obtain (13), and using (14) again, we get the following asymptotical formula:

$$S_\Lambda(t,\,\rho) = \frac{2}{d\pi} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}} \sin\left(2\pi\left(t + \frac{1}{2L}\right)|\vec{k}| + \frac{\pi}{4}\right) + R_\Lambda(N,\,t), \quad (36)$$

where

$$|R_\Lambda(N,\,t)| \ll \frac{N^{\delta'}}{\sqrt{|t|}} + \frac{|t|^{1/2+\delta'}}{\sqrt{N}} + \frac{1}{|t|^{1/2-\delta'}}.$$

Set $N = T^3$. Since M is small, the infinite sum in (13) is truncated before $n = T^3$. Thus (13) together with (36) implies:

$$
\begin{aligned}
&S_\Lambda(t,\,\rho) - \tilde{S}_{\Lambda,\,M,\,L}(t) = \\
&\frac{2}{d\pi} \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|\le T^{3/2}}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}} \sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k}| + \frac{\pi}{4}\right)\left(1-\hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right)\right) \\
&\qquad + R_\Lambda(T^3,\,t).
\end{aligned}
\tag{37}
$$

Let $P_\Lambda(N,\,t)$ denote the sum in (37). Then the Cauchy-Schwartz inequality gives:

$$
\begin{aligned}
\left\langle \left| S_\Lambda(t,\,\rho) - \tilde{S}_{\Lambda,\,M,\,L}(t) \right|^2 \right\rangle_T =& \left\langle P_\Lambda^2 \right\rangle_T + \left\langle R_\Lambda(N,\,t)^2 \right\rangle_T + \\
& O\left(\sqrt{\left\langle P_\Lambda^2 \right\rangle_T} \sqrt{\left\langle R_\Lambda(N,\,t)^2 \right\rangle_T} \right).
\end{aligned}
\tag{38}
$$

Observe that for the chosen N,

$$
\left\langle R_\Lambda(N,\,t)^2 \right\rangle_T = O\big(T^{-1+\delta'}\big)
$$

for arbitrary small $\delta' > 0$, since the above equality is satisfied pointwise.

Next we would like to bound $\left\langle P_\Lambda^2 \right\rangle_T$. Just as we did while computing the variance of the smoothed variable, $\tilde{S}_{\Lambda,\,M,\,L}$, we divide all the terms of the expanded sum into the diagonal terms and the off-diagonal ones (see section 3.1). Namely,

$$
\begin{aligned}
\left\langle P_\Lambda^2 \right\rangle_T =& \frac{2}{d^2\pi^2} \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|\le T^{3/2}}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3} \left(1-\hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right)\right)^2 \\
& + O\left(\sum_{\substack{\vec{k},\,\vec{l}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|\ne|\vec{l}|\le T^{3/2}}} \frac{1}{|\vec{k}|^{3/2}|\vec{l}|^{3/2}} \hat{\omega}\left(T\big(|\vec{k}|-|\vec{l}|\big)\right)\right)
\end{aligned}
\tag{39}
$$

We will evaluate the diagonal contribution now. For $|\vec{k}| \le \sqrt{M}$,

$$
\hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) = 1 + O\left(\frac{|\vec{k}|}{\sqrt{M}}\right),
$$

and so the diagonal contribution is:

$$\frac{1}{M}\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ 1\ll|\vec{k}|\le\sqrt{M}}}\frac{1}{|\vec{k}|}+\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ \sqrt{M}\le|\vec{k}|\le T^{3/2}}}\frac{1}{|\vec{k}|^3}\ll\frac{1}{\sqrt{M}},$$

converting the sums into corresponding integrals and evaluating these integrals in the elliptic variables.

Finally, we are evaluating the off-diagonal contribution to (39) (that is, the second sum in the right-hand side of (39)). Set $0<\delta_0<1$. With no loss of generality, we may assume that $|\vec{k}|<|\vec{l}|$. Evaluating the contribution of pairs $\vec{k}$, $\vec{l}$ with

$$|\vec{l}|^2-|\vec{k}|^2\ge\frac{|\vec{k}|}{T^{1-\delta_0}}$$

gives:

$$\ll\sum_{\substack{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|<|\vec{l}|\le T^{3/2}}}\frac{1}{|\vec{k}|^{3/2}|\vec{l}|^{3/2}}\hat{\omega}\Big(T\big(|\vec{k}|-|\vec{l}|\big)\Big)\ll T^{-A\delta_0+6}$$

for every $A>0$, since

$$T\big(|\vec{l}|-|\vec{k}|\big)=T\frac{|\vec{l}|^2-|\vec{k}|^2}{|\vec{k}|+|\vec{l}|}\gg T^{\delta_0}\frac{|\vec{k}|}{|\vec{k}|+|\vec{l}|}\ge T^{\delta_0}\frac{|\vec{k}|}{|\vec{k}|+2|\vec{k}|}\gg T^{\delta_0},$$

as otherwise,

$$T\big(|\vec{l}|-|\vec{k}|\big)\ge T\big(|\vec{k}|\big)\gg T\gg T^{\delta_0}.$$

Thus the contribution of such terms is negligible.

In order to bound the contribution of pairs of Λ^*-vectors with

$$|\vec{l}|^2-|\vec{k}|^2\le\frac{|\vec{k}|}{T^{1-\delta_0}}$$

we use the Diophantinity of β again. Recall that we chose $\delta_{\Lambda^*}(y)=\frac{c_2}{y^{K_0}}$ with a constant c_2 in the beginning of the current proof. Choose a constant $R_0>0$ and assume that $|\vec{l}|^2\le cL^{R_0}$, for a constant c. Then

$$|\vec{l}|^2-|\vec{k}|^2\ge\delta_{\Lambda^*}(L^{R_0})\gg\frac{1}{L^{K_0R_0}}\gg\frac{1}{M^{K_0R_0/2}}\gg\frac{|\vec{k}|}{T^{1-\delta_0}}.$$

Therefore, for an appropriate choice of c, there are no such pairs. Denote

$$S_n:=\left\{(\vec{k},\,\vec{l})\in(\Lambda^*)^2:\;2^n\le|\vec{k}|^2\le2^{n+1},\,|\vec{k}|^2\le|\vec{l}|^2\le|\vec{k}|^2+\frac{2^{n/2}}{T^{1-\delta_0}}\right\}$$

Thus, by dyadic partition, the contribution is:

$$\ll \sum_{n=\lfloor R_0 \log L \rfloor}^{\lceil 3 \log T \rceil} \sum_{\substack{2^n \le |\vec{k}|^2 \le 2^{n+1} \\ |\vec{k}|^2 \le |\vec{l}|^2 \le |\vec{k}|^2 + \frac{2^{n/2}}{T^{1-\delta_0}}}} \frac{1}{|\vec{k}|^{3/2} |\vec{l}|^{3/2}} \hat{\omega}\Big(T\big(|\vec{k}| - |\vec{l}|\big)\Big)$$

$$\ll \sum_{n=\lfloor R_0 \log L \rfloor}^{\lceil 3 \log T \rceil} \frac{\# S_n}{2^{3n/2}},$$

using $|\hat{\omega}| \ll 1$ everywhere. In order to bound the size of S_n, we use the following lemma, which is just a restatement of lemma 3.1 from [2]. We will prove it immediately after we finish proving proposition 5.1.

Lemma 5.2. *Let $\Lambda = \langle 1, i\eta \rangle$ be a rectangular lattice. Denote*

$$A(R, \delta) := \{(\vec{k}, \vec{l}) \in \Lambda : \; R \le |\vec{k}|^2 \le 2R, \, |\vec{k}|^2 \le |\vec{l}|^2 \le |\vec{k}|^2 + \delta\}.$$

Then if $\delta > 1$, we have for every $\epsilon > 0$,

$$\# A(R, \delta) \ll_\epsilon R^\epsilon \cdot R\delta$$

Thus, lemma 5.2 implies

$$\# S_n \ll 2^{n + \epsilon(n/2)} \max\left(1, \, \frac{2^{n/2}}{T^{1-\delta_0}}\right),$$

for every $\epsilon > 0$. Thus the contribution is:

$$\ll \sum_{n=R_0 \log L - 1}^{C \log T + 1} \frac{1}{2^{3n/2}} \cdot 2^{n + \epsilon n/2} \cdot 1 + \sum_{n=C \log T - 1}^{3 \log T + 1} \frac{1}{2^{3n/2}} \cdot 2^{n + \epsilon n/2} \cdot \frac{2^{n/2}}{T^{1-\delta_0}}$$

$$\ll L^{-R_0(1-\epsilon)/2} + \frac{\log T}{T^{1-\delta_1}} \ll L^{-R_0(1-\epsilon)/2},$$

since L is much smaller than T. Since R_0 is arbitrary, and we have assumed $M = O(L^{s_0})$, that implies

$$\langle P_\Lambda^2 \rangle_T \ll \frac{1}{\sqrt{M}}.$$

Collecting all our results, and using them on (38) we obtain

$$\left\langle \left| S_\Lambda(t, \, \rho) - \tilde{S}_{\Lambda, \, M, \, L}(t) \right|^2 \right\rangle \ll \frac{1}{\sqrt{M}} + \frac{1}{T^{1-\delta'}} + \frac{\sqrt{\log M}}{M^{1/4} T^{1/2 - \delta'/2}} \ll \frac{1}{\sqrt{M}},$$

again, since M is much smaller than T. $\square$

Proof of lemma 5.2. Let $\vec{k} = (k_1,\, i\eta k_2)$ and $\vec{l} = (l_1,\, i\eta l_2)$. Denote $\mu := \eta^2$, $n := l_1^2 - k_1^2$ and $m := k_2^2 - l_2^2$. The number of 4-tuples $(k_1,\, k_2,\, l_1,\, l_2)$ with $m \neq 0$ is

$$\#A(\delta,\, T) \ll \sum_{\substack{0 \le n - \mu m \le \delta \\ 1 \le m \le 4R}} d(n) d(m) \ll \delta \sum_{1 \le m \le 4R} d(m)^2 \ll R^{1+\epsilon}\delta$$

Next, we bound the number of 4-tuples with $m = 0$, $n \neq 0$:

$$\sum_{k_2=0}^{\sqrt{2R}} \sum_{0<n<\delta} d(n) \ll R^{1/2+\epsilon}\delta,$$

and similarly we bound the number of 4-tuples with $n = 0$, $m \neq 0$.

All in all, we have proved that

$$\#A(\delta,\, T) \ll R^{1+\epsilon}\delta$$

□

From now on we will assume that $\Lambda = \langle 1,\, i\alpha \rangle$ with a Diophantine $\gamma := \alpha^2$, and so the use of proposition 5.1 is justified.

Lemma 5.3. *Under the conditions of proposition 5.1, for all fixed $\xi > 0$,*

$$\mathbb{P}_{\omega,\, T}\left\{ \left| \frac{S_\Lambda(t,\, \rho)}{\sigma} - \frac{\tilde{S}_{\Lambda,\, M,\, L}(t)}{\sigma} \right| > \xi \right\} \to 0,$$

as $T \to \infty$, where $\sigma^2 = \frac{8\pi}{dL}$.

Proof. Use Chebychev's inequality and proposition 5.1. □

Corollary 5.4. *For a number $\alpha \in \mathbb{R}$, suppose that α^2 is $\overline{\text{strongly Diophantine}}$ and denote $\Lambda = \langle 1,\, \alpha \rangle$. Then if $L \to \infty$, but $L = O\big(\overline{T^\delta}\big)$ for all $\delta > 0$ as $T \to \infty$, then for any interval $\mathcal{A}$,*

$$\mathbb{P}_{\omega,\, T}\left\{ \frac{S_\Lambda(t,\, \rho)}{\sigma} \in \mathcal{A} \right\} \to \frac{1}{\sqrt{2\pi}} \int\limits_A e^{-\frac{x^2}{2}} dx,$$

where $\sigma^2 = \frac{8\pi}{dL}$.

Proof. Set $M = L^3$, then, obviously, L, M satisfy the conditions of lemma 5.3 and theorem 2.1. Denote $X(t) := \frac{S_\Lambda(t,\rho)}{\sigma}$ and $Y(t) := \frac{\tilde{S}_{\Lambda,M}(t)}{\sigma}$. In the new notations lemma 5.3 states that for any $\xi > 0$,

$$\mathbb{P}_{\omega,T}\Big\{|X(t) - Y(t)| > \xi\Big\} \to 0, \tag{40}$$

as $T \to \infty$. Now, for every $\epsilon > 0$,

$$\Big\{a \le X \le b\Big\} \subseteq \Big\{a - \epsilon \le Y \le b + \epsilon\Big\} \cup \Big\{|X - Y| > \epsilon\Big\},$$

and so, taking $\limsup\limits_{T\to\infty} \mathbb{P}_{\omega,T}$ of both of the sides, we obtain:

$$\begin{aligned}\limsup_{T\to\infty} \mathbb{P}_{\omega,T}\Big\{a \le X \le b\Big\} &\le \lim_{T\to\infty} \mathbb{P}_{\omega,T}\Big\{a - \epsilon \le Y \le b + \epsilon\Big\} \\ &= \frac{1}{\sqrt{2\pi}} \int\limits_{a-\epsilon}^{b+\epsilon} e^{-\frac{x^2}{2}} dx,\end{aligned}$$

due to (40) and theorem 2.1. Starting from

$$\Big\{a + \epsilon \le Y \le b - \epsilon\Big\} \subseteq \Big\{a \le X \le b\Big\} \cup \Big\{|X - Y| > \epsilon\Big\},$$

and doing the same manipulations as before, we get the converse inequality, and thus this implies the result of the present corollary.

$\square$

We are now in a position to prove our main result, namely, theorem 1.1. It states that the result of corollary 5.4 holds for $\omega = \mathbf{1}_{[1,2]}$, the indicator function. We are unable to substitute it directly because of the rapid decay assumption on $\hat{\omega}$. Nonetheless, we are able to prove the validity of the result by the means of *approximating* the indicator function with functions which will obey the rapid decay assumption. The proof is essentially the same as of theorem 1.1 in [13], pages 655-656, and we repeat it in this paper for the sake of the completeness.

Proof of theorem 1.1. Fix $\epsilon > 0$ and approximate the indicator function $\mathbf{1}_{[1,2]}$ above and below by smooth functions $\chi_\pm \ge 0$ so that $\chi_- \le \mathbf{1}_{[1,2]} \le \chi_+$, where both $\chi_\pm$ and their Fourier transforms are smooth and of rapid decay, and so that their total masses are within ϵ of unity $\left|\int \chi_\pm(x)dx - 1\right| < \epsilon$. Now, set $\omega_\pm := \chi_\pm / \int \chi_\pm$. Then $\omega_\pm$ are "admissible", and for all t,

$$(1 - \epsilon)\omega_-(t) \le \mathbf{1}_{[1,2]}(t) \le (1 + \epsilon)\omega_+(t). \tag{41}$$

Now,

$$meas\left\{t \in [T,\, 2T] :\ \frac{S_\Lambda(t,\,\rho)}{\sigma} \in \mathcal{A}\right\} = \int_{-\infty}^{\infty} \mathbf{1}_{\mathcal{A}}\left(\frac{S_\Lambda(t,\,\rho)}{\sigma}\right)\mathbf{1}_{[1,\,2]}\left(\frac{t}{T}\right)dt,$$

and since (41) holds, we find that

$$\begin{aligned}(1-\epsilon)\mathbb{P}_{\omega_-,\,T}\left\{\frac{S_{\Lambda,\,M,\,L}}{\sigma} \in \mathcal{A}\right\} &\le \frac{1}{T} meas\left\{t \in [T,\, 2T] :\ \frac{S_\Lambda(t,\,\rho)}{\sigma} \in \mathcal{A}\right\} \\ &\le (1+\epsilon)\mathbb{P}_{\omega_+,\,T}\left\{\frac{S_{\Lambda,\,M,\,L}}{\sigma} \in \mathcal{A}\right\}.\end{aligned}$$

As it was mentioned immediately after the definition of the strong Diophantinity property, α's being strongly Diophantine implies the same for α^2, making a use of corollary 5.4 legitimate. Now by corollary 5.4, the two extreme sides of the last inequality have a limit, as $T \to \infty$, of

$$(1 \pm \epsilon)\frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{x^2}{2}} dx,$$

and so we get that

$$(1-\epsilon) \int_A e^{-\frac{x^2}{2}} dx \le \liminf_{T\to\infty} \frac{1}{T} meas\left\{t \in [T,\, 2T] :\ \frac{S_\Lambda(t,\,\rho)}{\sigma} \in \mathcal{A}\right\}$$

with a similar statement for $\limsup$; since $\epsilon > 0$ is arbitrary, this shows that the limit exists and equals

$$\lim_{T\to\infty} \frac{1}{T} meas\left\{t \in [T,\, 2T] :\ \frac{S_\Lambda(t,\,\rho)}{\sigma} \in \mathcal{A}\right\} = \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{x^2}{2}} dx,$$

which is the Gaussian law. □

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