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GLOBAL WELL-POSEDNESS FOR A PERIODIC NONLINEAR SCHRÖDINGER EQUATION IN 1D AND 2D

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Abstract. The initial value problem for the L^2 critical semilinear Schrödinger equation with periodic boundary data is considered. We show that the problem is globally well posed in $H^s(\mathbb{T}^d)$, for $s > 4/9$ and $s > 2/3$ in 1D and 2D respectively, confirming in 2D a statement of Bourgain in [3]. We use the "I-method". This method allows one to introduce a modification of the energy functional that is well defined for initial data below the $H^1(\mathbb{T}^d)$ threshold. The main ingredient in the proof is a "refinement" of the Strichartz's estimates that hold true for solutions defined on the rescaled space, $\mathbb{T}^d_\lambda = \mathbb{R}^d/\lambda\mathbb{Z}^d$, $d = 1, 2$.

1. Introduction

In this paper we study the L^2 critical Cauchy problem

$$iu_t + \Delta u - |u|^{\frac{4}{d}}u = 0, \;\; x \in \mathbb{T}^d, \; t \geq 0 \tag{1.1}$$

$$u(x, 0) = u_0(x) \in H^s(\mathbb{T}^d), \tag{1.2}$$

where $d = 1, 2$ and $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the d-dimensional torus.

We say that a Cauchy problem is locally well-posed in H^s if for any choice of initial data $u_0 \in H^s$, there exists a positive time $T = T(\|u_0\|_{H^s})$ depending only on the norm of the initial data, such that a solution to the initial value problem exists on the time interval $[0, T]$, is unique and the solution map from H^s_x to $C^0_t H^s_x$ depends continuously on the initial data on the time interval $[0, T]$. If $T = \infty$ we say that a Cauchy problem is globally well-posed.

In the case when $x \in \mathbb{R}^d$ local well-posedness for (1.1)-(1.2) in $H^s(\mathbb{R}^d)$ has been studied extensively (see, for example, [8, 16, 20]). In particular if one solves the equivalent integral equation by Picard's fixed point method and controls the nonlinearity in the iteration process by using Strichartz's type inequalities, then the problem can be shown to be locally well-posed for all $s > 0$.

Bourgain [1] adjusted this approach to the periodic case, where there are certain difficulties due mainly to a "lack of dispersion". In [1] number theoretic methods were used to show that (1.1)-(1.2) is locally well-posed in $H^s(\mathbb{T}^d)$, $d = 1, 2$ for any $s > 0$.

Assuming local existence there are many issues to be addressed about the behavior of the solution as $t \to \infty$,

- global well-posedness/blow-up behavior
- asymptotic stability
- behavior of higher order Sobolev norms of smooth solutions.

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In this note we address the question of global well-posedness for (1.1)-(1.2). We recall that solutions of (1.1) satisfy mass conservation

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

and smooth solutions also satisfy energy conservation

$$E(u)(t) = \frac{1}{2}\int |\nabla u(t)|^2 dx + \frac{d}{2(d+2)}\int |u(t)|^{2+\frac{4}{d}} dx = E(u_0).$$

For initial data in $H^s(\mathbb{T}^d)$, $s \geq 1$, the energy conservation together with local well-posedness imply global well-posedness for $s \geq 1$. However it is a more subtle problem to extend the global theory to infinite energy initial data. Bourgain [3] established global well posedness for (1.1) in $H^s(\mathbb{T})$ for any $s > 1/2-$, by combining a "normal form" reduction method (see also [5]), the "I-method", and a refined trilinear Strichartz type inequality. The normal form reduction is achieved by symplectic transformations that, in some sense, reduce the nonlinear part of the equation to its "essential part". The I-method, introduced by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao in [10, 13, 14], is based on the almost conservation of certain modified Hamiltonians. These two methods together yield global well-posedness in $H^s(\mathbb{T})$ with $s \geq 1/2$. The refined trilinear Strichartz inequality established in [3] is a qualitative estimate needed to achieve global well-posedness in $H^s(\mathbb{T})$, $s^* < s < 1/2$ for some s^*.

In this paper we continue to fill in the gap between what is known locally and what is known globally in $H^s(\mathbb{T}^d)$ for $s > 0$, when $d = 1, 2$. Our approach is based on an implementation of the I-method itself adjusted to the periodic setting via elementary number theoretic techniques.

In order to present our method, we briefly review global well-posedness results on $\mathbb{R}^d$. Using an approximation of the modified energy in the I-method, Tzirakis [25] showed that the Cauchy problem (1.1)-(1.2) is globally well posed in $H^s(\mathbb{R})$ for any $s > 4/9$. For $u_0 \in H^s(\mathbb{R}^2)$ the best known global well posedness result for (1.1)-(1.2) is in [14] where the authors proved global well posedness for any $s > 4/7$. In both results, apart from the application of the I-method, a key element in the proof is the existence of a bilinear refined Strichartz's estimate due to Bourgain, [4] (see also [9] [10]). In general dimensions $d \geq 2$ this estimate reads as follows. Let f and g be any two Schwartz functions whose Fourier transforms are supported in $|k| \sim N_1$ and $|k| \sim N_2$ respectively. Then we have

$$\|U_t f U_t g\|_{L^2_t L^2_x(\mathbb{R}\times\mathbb{R}^d)} \leq C\frac{N_2^{\frac{d-1}{2}}}{N_1^{\frac{1}{2}}}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)},$$

where U_t denotes the solution operator associated to the linear Schrödinger equation. In one dimension the estimate fails if the two frequencies are comparable but continues to hold if the frequencies are separated ($N_1 >> N_2$). Such an estimate is very useful when f is in high frequency and g is in low frequency since we can move derivatives freely to the low frequency factor.

The main difficulty in obtaining global well-posedness results in the periodic context is exactly the absence of a quantitative refinement of the bilinear Strichartz's estimate. The reader can consult the paper of Kenig, Ponce and Vega, [21], where the difference between the real and the periodic case is clearly exposed when one tries to prove bilinear estimates in different functional spaces.

In order to overcome the non-availability of a quantitative refined Strichartz's estimate, we develop a different strategy, closer in spirit to the approach in [13]. By exploiting the scaling symmetry of the equation, we analyze (1.1) on $\mathbb{T}^d_\lambda \times \mathbb{R}$, $d = 1, 2$, where $\mathbb{T}^d_\lambda = \mathbb{R}^d/\lambda\mathbb{Z}^d$. The main novelty in this analysis consists in establishing a bilinear Strichartz's inequality for λ-periodic functions which are well separated in frequency space. The constant in the inequality is quantified in terms of λ. As $\lambda \to \infty$, our estimate reduces to the refined bilinear Strichartz's inequality[1] on $\mathbb{R}^d$, $d = 1, 2$, see, for example, [4, 10, 23]. Such an estimate allows us to use the I-method machinery in an efficient way.

More precisely, when $d = 1$, beside rescaling, we follow the argument in [25] (see also [12]), where one applies the I-operator to the equation on $\mathbb{T}_\lambda$, and defines a modified second energy functional as the energy corresponding to the new "I-system". We prove that such modified second energy is "almost conserved", that is the time derivative of this new energy decays with respect to a very large parameter N. Roughly speaking, N denotes the stage at which the I-operator stops behaving like the identity and starts smoothing out the solution. We prove a local existence result for the "I-system", under a smallness assumption for the initial data, which is guaranteed by choosing λ in terms of N. The decay of the modified energy enables us to iterate the local existence preserving the same bound for $\|u^\lambda\|_{H^s}$ during the iteration process. By undoing the scaling we obtain polynomial in time bounds for u in $\mathbb{T} \times \mathbb{R}^+$, and this immediately implies global well posedness.

The precise statement of our 1D result reads as follows.

Theorem 1.1. *The initial value problem* (1.1)-(1.2) *is globally well-posed in* $H^s(\mathbb{T})$ *for* $s > \frac{4}{9}$.

For the two dimensional case Bourgain already announced in [3] that the I-method based only on the first energy would give global well-posedness for $s > 2/3$. While we were explicitly writing up the calculations to recover this claim, we noticed that in one particular case[2] of the estimate of the first energy, a better Strichartz inequality was needed to successfully conclude the argument. We then proceeded by determining a qualitative "ϵ-refined" Strichartz type estimate, see Proposition 4.6, which allowed us to implement the I-method described above to obtain indeed the following result:

Theorem 1.2. *The initial value problem* (1.1)-(1.2) *is globally well-posed in* $H^s(\mathbb{T}^2)$ *for* $s > \frac{2}{3}$.

We remark that, as we mentioned above, in 1D we introduce an approximation to the modified energy, by adding correction terms, in the spirit of [25] and [12]. In the 2D problem we did not use correction terms, since the Fourier multipliers corresponding to the approximated modified energy would be singular. This singularity is not only caused by the presence of zero frequencies, but also by orthogonality issues. This is a whole new ground that we are exploring in a different paper.

Organization of the paper. In section 2 we review the notation and some known Strichartz estimates for the periodic Schrodinger equations in 1D and 2D. In section 3 we present the proof of Theorem 1.1, while in section 4 we give a proof of Theorem 1.2.

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[1] In a way we are introducing "more dispersion" by rescaling the problem.

[2] See Case IIIb) in the proof of Proposition 4.7.

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2. Notation

In what follows we use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C. If $A \lesssim B$ and $B \lesssim A$ we say that $A \sim B$. We write $A \ll B$ to denote an estimate of the form $A \leq cB$ for some small constant $c > 0$. In addition $\langle a \rangle := 1 + |a|$ and $a\pm := a \pm \epsilon$ with $0 < \epsilon << 1$.

We recall that the equation (1.1) is L^2 invariant under the following scaling $(x,t) \to \frac{1}{\lambda^{\frac{d}{2}}}(\frac{x}{\lambda}, \frac{t}{\lambda^2})$. Thus if $u(x,t)$ solves (1.1) on $\mathbb{T}^d \times \mathbb{R}$ then

$$u^\lambda(x,t) = \frac{1}{\lambda^{\frac{d}{2}}} u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$$

is a solution of (1.1) in $\mathbb{T}^d{}_\lambda \times \mathbb{R}$ where $\mathbb{T}^d_\lambda = \mathbb{R}^d / \lambda \mathbb{Z}^d$.

Since in our argument we exploit the scaling symmetry let us recall some properties of $\lambda-$periodic functions. Define $(dk)_\lambda$ to be the normalized counting measure on $(\frac{1}{\lambda}\mathbb{Z})^d$:

$$\int a(k)(dk)_\lambda = \frac{1}{\lambda^d} \sum_{k \in (\frac{1}{\lambda}\mathbb{Z})^d} a(k).$$

We define the Fourier transform of $f(x) \in L^1_{x \in [0,\lambda]^d}$ by

$$\hat{f}(k) = \int_{[0,\lambda]^d} e^{-2\pi i k x} f(x) dx.$$

For an appropriate class of functions the following Fourier inversion formula holds:

$$f(x) = \int e^{2\pi i k x} \hat{f}(k)(dk)_\lambda.$$

Moreover we know that the following identities are true:

(1) $\|f\|_{L^2([0,\lambda]^d)} = \|\hat{f}\|_{L^2((dk)_\lambda)}$, (Plancherel)

(2) $\int_{[0,\lambda]^d} f(x)\bar{g}(x)dx = \int \hat{f}(k)\bar{\hat{g}}(k)(dk)_\lambda$, (Parseval)

(3) $\widehat{fg}(k) = \hat{f} \star_\lambda \hat{g}(k) = \int \hat{f}(k - k_1)\hat{g}(k_1)(dk_1)_\lambda$,

We define the Sobolev space $H^s = H^s([0,\lambda]^d)$ as the space equipped with the norm

$$\|f\|_{H^s} = \|\langle k \rangle^s \hat{f}(k)\|_{L^2((dk)_\lambda)}.$$

We write $U_\lambda(t)$ for the solution operator to the linear Schrödinger equation

$$iu_t - \Delta u = 0, \;\; x \in [0,\lambda]^d,$$

that is

$$U_\lambda(t)u_0(x) = \int e^{2\pi i k x - (2\pi k)^2 i t} \widehat{u_0}(k)(dk)_\lambda.$$

We denote by $X^{s,b} = X^{s,b}(\mathbb{T}^d_\lambda \times \mathbb{R})$ the completion of $\mathcal{S}(\mathbb{T}^d_\lambda \times \mathbb{R})$ with respect to the following norm, see, for example, [15]

$$\|u\|_{X^{s,b}} = \|U_\lambda(-t)u\|_{H^s_x H^b_t} = \|\langle k\rangle^s \langle \tau - 4\pi^2 k^2\rangle^b \tilde{u}(k,\tau)\|_{L^2_\tau L^2_{(dk)_\lambda}},$$

where $\tilde{u}(k,\tau)$ is the space-time Fourier Transform

$$\tilde{u}(k,\tau) = \int\int_{[0,\lambda]^d} e^{-2\pi i(k\cdot x+\tau t)} u(x,t)dxdt.$$

Furthermore for a given time interval J, we define

$$\|f\|_{X^{s,b}_J} = \inf_{g=f \text{ on } J} \|g\|_{X^{s,b}}.$$

Often we will drop the subscript J.

In what follows we will need the following known estimates. Since

$$\widehat{U_\lambda(t)u_0}(k) = e^{(2\pi k)^2 it}\widehat{u_0}(k)$$

we have that

$$\|U_\lambda(t)u_0\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2}.$$

Hence

$$\|u\|_{L^\infty_t L^2_x} = \|U_\lambda(t)U_\lambda(-t)u\|_{L^\infty_t L^2_x} \lesssim \|U_\lambda(-t)u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{X^{0,1/2+}}, \tag{2.1}$$

where in the last inequality we applied the definition of the $X^{s,b}$ spaces, the basic estimate $\|u\|_{L^\infty} \leq \|\hat{u}\|_{L^1}$, and the Cauchy-Schwartz inequality.

1D estimates. If u is on $\mathbb{T}^1$ then the estimate (2.1) combined with the Sobolev embedding theorem implies that

$$\|u\|_{L^\infty_t L^\infty_x} \lesssim \|u\|_{X^{1/2+,1/2+}}. \tag{2.2}$$

Also the following linear Strichartz's estimates was obtained in [1] in the case of the torus:

$$\|u\|_{L^4_t L^4_x} \lesssim \|u\|_{X^{0,b}}, \tag{2.3}$$

for any $b > \frac{3}{8}$, and

$$\|u\|_{L^6_t L^6_x} \lesssim \|u\|_{X^{0+,1/2+}}. \tag{2.4}$$

We note that (2.2) remains true for the λ-periodic problem. This is the case also for (2.3). The proof is essentially in [18]. In fact, it is enough to show, by the standard $X^{s,b}$ method, that

$$\frac{1}{\lambda} \sup_{(\xi,\tau)\in\frac{1}{\lambda}\mathbb{Z}\times\mathbb{R}} \sum_{k_1\in\frac{1}{\lambda}\mathbb{Z}} \langle \tau + k_1^2 + (k-k_1)^2\rangle^{1-4b} \lesssim C.$$

This is done in [18], the only difference being that there are $O(\lambda)$ numbers $k \in \frac{1}{\lambda}\mathbb{Z}$, such that $|k + x_0| < 1$ and $|k - x_0| < 1$, where $x_0^2 = |2\tau + \xi^2|$. But then summing the above series, using Cauchy-Schwartz inequality and the fact that $b > 3/8$, one gets that the left hand side of the inequality is

$$\lesssim \frac{1}{\lambda}(c_1\lambda + c_2) \lesssim C$$

for any $\lambda > 1$. So from now on we will use (2.3) for the λ-periodic solutions without any further comment. On the other hand, using scaling we can see that for the λ-periodic solutions, (2.4) takes the form

$$\|u\|_{L^6_t L^6_x} \lesssim \lambda^{0+}\|u\|_{X^{0+,1/2+}}. \tag{2.5}$$

If we interpolate equations (2.2) and (2.5) we get

$$\|u\|_{L^p_t L^p_x} \lesssim \lambda^{0+} \|u\|_{X^{\alpha_1(p),1/2+}}, \tag{2.6}$$

with $\alpha_1(p) = (\frac{1}{2} - \frac{3}{p})+$ and $6 \leq p \leq \infty$.

2D estimates. On $\mathbb{T}^2$ Bourgain proved, [1],

$$\|u\|_{L^4_t L^4_x} \lesssim \|u\|_{X^{0+,1/2+}}. \tag{2.7}$$

Again using scaling we can prove that for the λ-periodic solutions, (2.7) takes the form

$$\|u\|_{L^4_t L^4_x} \lesssim \lambda^{0+} \|u\|_{X^{0+,1/2+}}. \tag{2.8}$$

The estimate (2.1) together with Sobolev embedding gives:

$$\|u\|_{L^\infty_t L^\infty_x} \lesssim \|u\|_{X^{1+,1/2+}}. \tag{2.9}$$

Hence, by interpolation, we get

$$\|u\|_{L^p_t L^p_x} \lesssim \lambda^{0+} \|u\|_{X^{\alpha_2(p),1/2+}}, \tag{2.10}$$

with $\alpha_2(p) = (1 - \frac{4}{p})+$ and $4 \leq p \leq \infty$.

Remark 2.1. (**Decomposition remark**) Our approach to prove Theorem 1.1 and Theorem 1.2 is based on obtaining certain multilinear estimates in appropriate functional spaces which are L^2-based. Hence, whenever we perform a Littlewood-Paley decomposition of a function we shall assume that the Fourier transforms of the Littlewood-Paley pieces are positive. Moreover, we will ignore the presence of conjugates.

3. The I-method and the proof of Theorem 1.1

In this section we present the proof of Theorem 1.1. We start by recalling the definition of the operator I introduced by Colliander et al, see, for example, [10, 14]. For $s < 1$ and a parameter $N >> 1$ let $m(k)$ be the restriction to $\mathbb{Z}$ of the following smooth monotone multiplier:

$$m(k) := \begin{cases} 1 & \text{if } |k| < N \\ (\frac{|k|}{N})^{s-1} & \text{if } |k| > 2N \end{cases}$$

We define the multiplier operator $I : H^s \to H^1$ by

$$\widehat{Iu}(k) = m(k)\hat{u}(k).$$

The operator I is smoothing of order $1-s$ and we have that:

$$\|u\|_{X^{s_0,b_0}} \lesssim \|Iu\|_{X^{s_0+1-s,b_0}} \lesssim N^{1-s} \|u\|_{X^{s_0,b_0}} \tag{3.1}$$

for any $s_0, b_0 \in \mathbb{R}$.

For $u \in H^s$ we set

$$E^1(u) = E(Iu), \tag{3.2}$$

where

$$E(u)(t) = \frac{1}{2}\int |\nabla u(t)|^2 dx + \frac{1}{6}\int |u(t)|^6 dx = E(u_0).$$

We refer to $E^1(u)$ as the first modified energy. As observed by Colliander et al a hierarchy of modified energies can be formally considered for different nonlinear dispersive equations. The goal of the I-method is to prove that the modified energies are "almost conserved" i.e. they decay in time with respect to N. Since in 1D we base our approach on the analysis of a second modified energy, it is appropriate to collect some facts concerning the calculus of multilinear forms used to define the hierarchy, see, for example [25].

If $n \geq 2$ is an even integer we define a spatial multiplier of order n to be the function $M_n(k_1, k_2, \ldots, k_n)$ on $\Gamma_n = \{(k_1, k_2, \ldots, k_n) \in \frac{1}{\lambda}\mathbb{Z}^n : k_1 + k_2 + \ldots + k_n = 0\}$ which we endow with the standard measure $\delta(k_1 + k_2 + \ldots + k_n)$. If M_n is a multiplier of order n, $1 \leq j \leq n$ is an index and $l \geq 1$ is an even integer we define the elongation $X_j^l(M_n)$ of M_n to be the multiplier of order $n + l$ given by

$$X_j^l(M_n)(k_1, k_2, \ldots, k_{n+l}) = M_n(k_1, \ldots, k_{j-1}, k_j + \ldots + k_{j+l}, k_{j+l+1}, \ldots, k_{n+l}).$$

In addition if M_n is a multiplier of order n and $f_1, f_2, ..., f_n$ are functions on $\mathbb{T}_\lambda$ we define

$$\Lambda_n(M_n; f_1, f_2, ..., f_n) = \int_{\Gamma_n} M_n(k_1, k_2, \ldots, k_n) \prod_{i=1}^{n} \hat{f}_j(k_j),$$

where we adopt the notation $\Lambda_n(M_n; f) = \Lambda_n(M_n; f, \bar{f}, ..., f, \bar{f})$. Observe that $\Lambda_n(M_n; f)$ is invariant under permutations of the even k_j indices, or of the odd k_j indices.

If f is a solution of (1.1) the following differentiation law holds for the multilinear forms $\Lambda_n(M_n; f)$:

$$\partial_t \Lambda_n(M_n) = i\Lambda_n(M_n \sum_{j=1}^{n} (-1)^j k_j^2) + i\Lambda_{n+4}(\sum_{j=1}^{n} (-1)^j X_j^4(M_n)). \tag{3.3}$$

Note that in this notation the first modified energy (3.2) reads as:

$$E^1(u) = \frac{1}{2}\int |\partial_x Iu|^2 dx + \frac{1}{6}\int |Iu|^6 dx = -\frac{1}{2}\Lambda_2(m_1 k_1 m_2 k_2) + \frac{1}{6}\Lambda_6(m_1 ... m_6)$$

where $m_j = m(k_j)$.

We define the second modified energy

$$E^2(u) = -\frac{1}{2}\Lambda_2(m_1 k_1 m_2 k_2) + \frac{1}{6}\Lambda_6(M_6(k_1, k_2, ..., k_6)),$$

where $M_6(k_1, k_2, ..., k_6)$ is the following multiplier:

$$M_6(k_1, k_2, ..., k_6) = \frac{m_1^2 k_1^2 - m_2^2 k_2^2 + m_3^2 k_3^2 - m_4^2 k_4^2 + m_5^2 k_5^2 - m_6^2 k_6^2}{k_1^2 - k_2^2 + k_3^2 - k_4^2 + k_5^2 - k_6^2}. \tag{3.4}$$

Notice that the zero set of the denominator corresponds to the resonant set for a six-waves interaction. We remark that M_6 contains more "cancellations" than the multiplier $m_1 ... m_6$ that appears in E^1.

The differentiation rule (3.3) together with the fundamental theorem of calculus implies the following Lemma, which will be used to prove that E^2 is almost conserved.

Lemma 3.1. *Let u be an H^1 solution to (1.1). Then for any $T \in \mathbf{R}$ and $\delta > 0$ we have*

$$E^2(u(T + \delta)) - E^2(u(T)) = \int_T^{T+\delta} \Lambda_{10}(M_{10}; u(t))dt, \tag{3.5}$$

with $M_{10} = c \sum \{M_6(k_{abcde}, k_f, k_g, k_h, k_i, k_j) - M_6(k_a, k_{bcdef}, k_g, k_h, k_i, k_j) +$

$M_6(k_a, k_b, k_{cdefg}, k_h, k_i, k_j) - M_6(k_a, k_b, k_c, k_{defgh}, k_i, k_j) +$

$M_6(k_a, k_b, k_c, k_d, k_{efghj}, k_j) - M_6(k_a, k_b, k_c, k_d, k_e, k_{fghij})\},$

where the summation runs over all permutations $\{a, c, e, g, i\} = \{1, 3, 5, 7, 9\}$ *and* $\{b, d, f, h, j\} = \{2, 4, 6, 8, 10\}$. *Furthermore if* $|k_j| \ll N$ *for all* j *then the multiplier* M_{10} *vanishes.*

As it was observed in [25] where the equation (1.1) was considered on $\mathbb{R}$ one has

Proposition 3.2. *The multiplier* M_6 *defined in (3.4) is bounded on its domain of definition.*

For the proof see [25].

Now we proceed to present the steps leading to the proof of Theorem 1.1.

Step 1: Local well-posedness for the I-system. The first step towards the proof of Theorem 1.1 is to apply the I-operator to (1.1) and prove a local well-posedness result for the I-initial value problem

$$\begin{aligned} & iIu_t + Iu_{xx} - I(|u|^4 u) = 0 \\ & Iu(x,0) = Iu_0(x) \in H^1(\mathbb{T}_\lambda), t \in \mathbb{R}. \end{aligned} \tag{3.6}$$

In order to obtain the local well-posedness we need the following technical lemma:

Lemma 3.3. *Let* $\eta \in C_0^\infty$ *be a bump function that is supported on* $[-2, 2]$ *and equals 1 on* $[-1, 1]$ *and set* $\eta_\delta(t) = \eta(\frac{t}{\delta})$. *For* $b, b' \in \mathbb{R}$ *with* $-1/2 < b' \leq 0 \leq b \leq b' + 1$ *and* $\delta \leq 1$ *we have:*

(1) $\|\eta_1(t)U_\lambda(t)u_0\|_{X^{s,b}} \lesssim \|u_0\|_{H^s}$,
(2) $\|\eta_\delta(t)\int_0^\delta U_\lambda(t-\tau)F(\tau)d\tau\|_{X^{s,b}} \lesssim \delta^{1+b'-b}\|F\|_{X^{s,b'}}$.

Proof. (1) We compute (see, for example, [15])

$$\begin{aligned} \|\eta_1(t)U_\lambda(t)u_0\|_{X^{s,b}} &= \|U_\lambda(-t)\eta_1(t)U_\lambda(t)u_0\|_{H_t^b H_x^s} \\ &= \|\eta_1(t)u_0\|_{H_t^b H_x^s} \\ &= \|u_0\|_{H^s}\|\eta_1(t)\|_{H_t^b} \\ &\sim \|u_0\|_{H^s}. \end{aligned}$$

(2) The following inequality is true and its proof can be found in [15], [21],

$$\|\eta_\delta(t)\int_0^\delta G(\tau)d\tau\|_{H_t^b} \lesssim \delta^{1+b'-b}\|G\|_{H_t^{b'}}.$$

But then (2) follows if we apply the above inequality for fixed ξ, multiplying by $< \xi >^{2s}$ and taking the L^2 norm in ξ. □

Remark. It is shown in [11] that if

$$\|uv\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}},$$

then

$$\|I(uv)\|_{X^{1,b-1}} \lesssim \|Iu\|_{X^{1,b}}\|Iv\|_{X^{1,b}},$$

where the constants in the inequality above are independent of N. From now on we use this fact and refer to it as the "invariant lemma". For details see Lemma 12.1 in [11].

Now we present the local well-posedness result for (3.6). Because of the loss in the periodic Strichartz estimates (2.5) represented by the λ^{0+} factor, the local well-posedness is not as straightforward as in the real case. (There one can essentially use the Leibnitz rule for the fractional derivatives and the Cauchy-Schwartz inequality along with interpolated Strichartz's estimates to bound the nonlinearity). In 1D we can still follow the same approach thanks to (2.3), which is valid for the λ-periodic problem. In 2D for the λ-periodic problem we need a more complicated argument that is due to Bourgain, [2], and uses a local variant of the periodic Strichartz inequality that is proved there. For details see [2]. As a final comment, note that our local well-posedness results are not optimal but they suffice for the purposes of the global theory that we establish. Thus without aiming at sharpness, we prove local well-posedness for $s > 5/16$ in 1D (see Proposition 3.4 and Corollary 3.5) and $s > 7/20$ in 2D (see Proposition 4.1 and Corollary 4.2).

Proposition 3.4. *Consider the I-initial value problem* (3.6), $\lambda = 1$. *If* $\|Iu_0\|_{H^1} \lesssim 1$, *then* (3.6) *is locally well-posed for any* $s > \frac{5}{16}$ *in* $[0, \delta] \sim [0, 1]$.

Proof. By Duhamel's formula and Lemma 3.3 we have that

$$\|Iu\|_{X^{1,1/2+}} \lesssim \|Iu_0\|_{H^1} + \delta^{\frac{1}{2}-\epsilon}\|I(|u|^4u)\|_{X^{1,-1/2+2\epsilon}}. \tag{3.7}$$

We shall prove that

$$\|I(|u|^4u)\|_{X^{1,-1/2+2\epsilon}} \lesssim \|Iu\|^5_{X^{1,1/2+}}.$$

By the invariant lemma we know that to prove such an estimate it suffices to prove

$$\||u|^4u\|_{X^{s,-1/2+2\epsilon}} \lesssim \|u\|^5_{X^{s,1/2+}}.$$

By Hölder's inequality combined with the Leibnitz rule ([22]) we have

$$\||u|^4u\|_{X^{s,-1/2+2\epsilon}} \leq \||u|^4u\|_{X^{s,0}} \lesssim \|J^s u\|_{L^4_t L^4_x}\|u\|^4_{L^{16}_t L^{16}_x},$$

where J^s is the Bessel potential of order s. Using (2.6) with $\lambda = 1$ we obtain

$$\|u\|_{L^{16}_t L^{16}_x} \lesssim \|u\|_{X^{5/16+,1/2+}} \lesssim \|u\|_{X^{s,1/2+}},$$

for $s > \frac{5}{16}$. Note that we also use (2.3) in order to bound $\|J^s u\|_{L^4_t L^4_x}$. Thus

$$\||u|^4u\|_{X^{s,-1/2+2\epsilon}} \lesssim \|u\|^5_{X^{s,1/2+}}$$

which combined with (3.7) gives

$$\|Iu\|_{X^{1,1/2+}} \lesssim \|Iu_0\|_{H^1} + \delta^{\frac{1}{2}-\epsilon}\|Iu\|^5_{X^{1,1/2+}}.$$

Now by standard nonlinear techniques, see, for example, [12] we have that for $\delta \sim 1$

$$\|Iu\|_{X^{1,1/2+}} \lesssim \|Iu_0\|_{H^1}.$$

□

Corollary 3.5. *Consider the I-initial value problem* (3.6). *If* $\|Iu_0\|_{H^1} \lesssim 1$, *then* (3.6) *is locally well-posed for any* $s > \frac{5}{16}$ *in* $[0, \delta] \sim [0, \frac{1}{\lambda^{a\epsilon}}]$, *where a is a fixed positive integer ($a \sim 20$ will do it).*

Proof. The proof is line by line the same as in the previous proposition. The important step is that in the iteration process where $x \in [0, \lambda]$, δ has a fixed power and this fact gives us a local well-posedness result on the time interval $[0, \frac{1}{\lambda^{a\epsilon}}]$, where $\frac{1}{\lambda^{a\epsilon}} \sim \frac{1}{\lambda^{0+}}$ for all practical purposes. □

Step 2: $\mathbf{E^2(u)}$ is a small perturbation of $\mathbf{E^1(u)}$. Now we shall prove that the second energy $E^2(u)$ is a small perturbation of the first energy $E^1(u)$. This is the content of the following proposition.

Proposition 3.6. *Assume that u solves (3.6) with $s \geq 2/5$. Then*

$$E^2(u) = E^1(u) + O(1/N)\|Iu\|^6_{H^1}.$$

Moreover if $\|Iu\|_{H^1} = O(1)$ then

$$\|\partial_x Iu\|^2_{L^2} \lesssim E^2(u) + O(\frac{1}{N}). \tag{3.8}$$

Proof. By the definition of first and second modified energy we have

$$E^2(u) = -\frac{1}{2}\Lambda_2(m_1k_1m_2k_2) + \frac{1}{6}\Lambda_6(M_6) = E^1(u) + \frac{1}{6}\Lambda_6(M_6 - \prod_{i=1}^{6} m_i). \tag{3.9}$$

Therefore it suffices to prove the following pointwise in time estimate

$$|\Lambda_6(M_6 - \prod_{i=1}^{6} m_i)|(t) \lesssim O(1/N)\|Iu(\cdot,t)\|^6_{H^1_x}. \tag{3.10}$$

Combining the **Decomposition Remark** 2.1 with the fact that M_6 is bounded (by Proposition 3.2) and that m is bounded (by its definition) it is enough to show that

$$\int_{\Gamma_6} \prod_{j=1}^{6} \widehat{u}(k_j) \lesssim \frac{1}{N}\|Iu(\cdot,t)\|^6_{H^1_x}.$$

Towards this aim, we break all the functions into a sum of dyadic constituents u_j, each with frequency support $\langle k\rangle \sim 2^j, j = 0, ...$ We will be able to sum over all frequency pieces u_j of u since our estimate will decay geometrically in these frequencies. Hence we need to show the following:

$$\int_{\Gamma_6} \prod_{j=1}^{6} \widehat{u_j}(k_j) \lesssim \frac{1}{N}(N_1...N_6)^{0-} \prod_{j=1}^{6} \|Iu_j(\cdot,t)\|^6_{H^1_x} \tag{3.11}$$

where u_j is supported around $\langle k\rangle \sim N_j = 2^{h_j}$ for some h_j's.

Denote by $N_1^\star \geq N_2^\star \geq N_3^\star \geq N_4^\star \geq N_5^\star \geq N_6^\star$ the decreasing rearrangement of $N_1, ..., N_6$, and by $u_j^\star$ the function u_j supported in Fourier space around $N_j^\star$, $j = 1, ..., 6$.

Since we are integrating over Γ_6, we have that $N_1^\star \sim N_2^\star$. Moreover, we may assume that $N_1^\star \sim N_2^\star \gtrsim N$, otherwise $M_6 - \prod_{i=1}^6 m_i \equiv 0$ and (3.10) follows trivially.

Since $\frac{1}{m(N_1^\star)(N_1^\star)^{1-}} \lesssim \frac{1}{N^{1-}}$ we have that

$$\int_{\Gamma_6} \prod_{j=1}^{6} \widehat{u_j}(k_j) \lesssim \frac{(N_1^\star)^{0-}}{N^{1-}} \int (\widehat{JIu_1^\star}) \prod_{j=2}^{6} \widehat{u_j^\star} \lesssim \frac{(N_1^\star)^{0-}}{N^{1-}} \|Iu_1^\star\|_{H^1_x} \prod_{j=2}^{6} \|u_j^\star\|_{L^{10}_x}$$

by reversing Plancherel and applying Hölder's inequality. Moreover by Sobolev embedding

$$\|u_j^\star\|_{L^{10}_x} \lesssim \|u_j^\star\|_{H^{2/5}_x}$$

and for $s \geq 2/5$ we have

$$\|u_j^\star\|_{H^{2/5}_x} \lesssim \|Iu_j^\star\|_{H_x^{1+\frac{2}{5}-s}} \lesssim \|Iu_j^\star\|_{H^1_x}.$$

Thus (3.11) follows. ∎

Step 3: Decay of $\mathbf{E^2(u)}$. In order to estimate the decay of $E^2(u)$ in time we need to bound the right hand side of (3.5). Towards this aim we prove a bilinear estimate using an elementary number theory argument. Let $\eta \in C_0^\infty$ be a bump function that is supported on $[-2,2]$ and equals 1 on $[-1,1]$.

Proposition 3.7. *Let ϕ_1 and ϕ_2 be $\lambda-$periodic functions whose Fourier transforms are supported on $\{k : |k| \sim N_1\}$ and $\{k : |k| \sim N_2\}$ respectively with $N_1 >> N_2$. Then*

$$\|\eta(t)(U_\lambda(t)\phi_1)\ \eta(t)(U_\lambda(t)\phi_2)\|_{L^2_t L^2_x} \lesssim C(\lambda, N_1)\|\phi_1\|_{L^2}\|\phi_2\|_{L^2}$$

where

$$C(\lambda, N_1) = \left\{ \begin{array}{ll} 1, & \textit{if } N_1 \le 1 \\ (\frac{1}{N_1} + \frac{1}{\lambda})^{\frac{1}{2}}, & \textit{if } N_1 > 1 \end{array} \right\}.$$

Moreover,

$$\|\phi_1\ \phi_2\|_{L^2_t L^2_x} \lesssim C(\lambda, N_1)\|\phi_1\|_{X^{0,1/2+}}\|\phi_2\|_{X^{0,1/2+}}.$$

Remark. We observe that as $\lambda \to \infty$ we recover the refined bilinear estimate in $\mathbb{R}$ with constant $C(N_1) = (\frac{1}{N_1})^{1/2}$, see, for example, [4, 10, 23].

Proof. Let ψ be a positive even Schwartz function such that $\psi = \hat{\eta}$. Then we have

$$\begin{aligned} B &= \|\eta(t)(U_\lambda(t)\phi_1)\ \eta(t)(U_\lambda(t)\phi_2)\|_{L^2_t L^2_x} \\ &= \left\| \int_{k=k_1+k_2,\, \tau=\tau_1+\tau_2} \widehat{\phi_1}(k_1)\widehat{\phi_2}(k_2)\psi(\tau_1 - k_1^2)\ \psi(\tau_2 - k_2^2)\ (dk_1)_\lambda\ (dk_2)_\lambda\ d\tau_1\ d\tau_2 \right\|_{L^2_\tau L^2_k} \\ &\lesssim \left\| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ (dk_1)_\lambda\ (dk_2)_\lambda \right)^{1/2} \times \right. \end{aligned}$$

$$\left. \times \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ (dk_1)_\lambda\ (dk_2)_\lambda \right)^{1/2} \right\|_{L^2_\tau L^2_k}, \tag{3.12}$$

where to obtain (3.12) we used Cauchy-Schwartz and the following definition of $\widetilde{\psi} \in \mathcal{S}$

$$\int_{\tau=\tau_1+\tau_2} \psi(\tau_1 - k_1^2)\ \psi(\tau_2 - k_2^2)\ d\tau_1 d\tau_2 = \widetilde{\psi}(\tau - k_1^2 - k_2^2).$$

An application of Hölder gives us the following upper bound on (3.12)

$$M \left\| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ (dk_1)_\lambda\ (dk_2)_\lambda \right)^{1/2} \right\|_{L^2_\tau L^2_k}, \tag{3.13}$$

where

$$M = \left\| \int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ (dk_1)_\lambda\ (dk_2)_\lambda \right\|^{1/2}_{L^\infty_\tau L^\infty_k}.$$

Now by integration in τ followed by Fubini in k_1, k_2 and two applications of Plancharel we have

$$\left\| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ (dk_1)_\lambda\ (dk_2)_\lambda \right)^{1/2} \right\|_{L^2_{k,\tau}} \lesssim \|\phi_1\|_{L^2_x}\|\phi_2\|_{L^2_x},$$

which combined with (3.12), (3.13) gives

$$B \lesssim M \|\phi_1\|_{L^2_x} \|\phi_2\|_{L^2_x}. \tag{3.14}$$

We find an upper bound on M as follows:

$$M \lesssim \left(\frac{1}{\lambda} \sup_{\tau,k} \#S \right)^{1/2}, \tag{3.15}$$

where

$$S = \{k_1 \in \frac{1}{\lambda}\mathbb{Z} \mid |k_1| \sim N_1,\ |k-k_1| \ \sim N_2,\ k^2 - 2k_1(k-k_1) = \tau + O(1)\},$$

and $\#S$ denotes the number of elements of S. When $N_1 \leq 1$ then $\#S \lesssim O(\lambda)$, which implies $C(\lambda, N_1) \lesssim 1$. When $N_1 > 1$ rename $k_1 = z$. Then

$$S = \{z \in \frac{1}{\lambda}\mathbb{Z} \mid |z| \sim N_1,\ |k-z| \ \sim N_2,\ k^2 - 2z(k-z) = \tau + O(1)\}.$$

Let z_0 be an element of S i.e.

$$|z_0| \sim N_1, \ \ |k - z_0| \sim N_2, \tag{3.16}$$

and

$$k^2 + 2z_0^2 - 2kz_0 = \tau + O(1). \tag{3.17}$$

In order to obtain an upper bound on $\#S$, we shall count the number of $\bar{z}$'s $\in \frac{1}{\lambda}\mathbb{Z}$, such that $z_0 + \bar{z} \in S$. Thus

$$|z_0 + \bar{z}| \sim N_1, \ \ |z_0 + \bar{z} - k| \sim N_2, \tag{3.18}$$

and

$$k^2 + 2\left(z_0 + \bar{z}\right)^2 - 2k\left(z_0 + \bar{z}\right) = \tau + O(1). \tag{3.19}$$

However by (3.17) we can rewrite the left hand side of (3.19) as follows

$$\begin{aligned} k^2 + 2\left(z_0 + \bar{z}\right)^2 - 2k\left(z_0 + \bar{z}\right) &= k^2 + 2z_0^2 + 2\left(\bar{z}\right)^2 + 4z_0\bar{z} - 2kz_0 - 2k\bar{z} \\ &= \tau + O(1) + 2\bar{z}^2 + 4z_0\bar{z} - 2k\bar{z}. \end{aligned}$$

Hence it suffices to count $\bar{z}$'s $\in \frac{1}{\lambda}\mathbb{Z}$ satisfying (3.18) and such that

$$\bar{z}^2 + 2\bar{z}(z_0 - \frac{k}{2}) = O(1), \tag{3.20}$$

where z_0 satisfies (3.16) - (3.17).

By (3.16) and (3.18) we have that

$$|\bar{z}| = |(\bar{z} + z_0 - k) - z_0 + k| \lesssim N_2 + N_2.$$

Hence

$$|\bar{z}| \lesssim N_2 << N_1. \tag{3.21}$$

On the other hand, by (3.16) we have that

$$|z_0 - \frac{k}{2}| \sim N_1. \tag{3.22}$$

Now rewrite (3.20) as follows

$$\bar{z}\left(\bar{z} + 2(z_0 - \frac{k}{2})\right) = O(1). \tag{3.23}$$

Therefore, using (3.21)-(3.22), the estimate (3.23) implies that

$$|\bar{z}|\, N_1 = O(1).$$

Since $\bar{z} \in \frac{1}{\lambda}\mathbb{Z}$ this implies that the number of $\bar{z}$'s of size $1/N_1$ is λ/N_1. Hence

$$\#S \lesssim 1 + \frac{\lambda}{N_1},$$

which combined with (3.15) gives

$$M \le \left(\frac{1}{N_1} + \frac{1}{\lambda}\right)^{1/2},$$

and therefore

$$C(\lambda, N_1) \le \left(\frac{1}{N_1} + \frac{1}{\lambda}\right)^{\frac{1}{2}}.$$

□

Now we are ready to prove desired decay of $E^2(u)$ which follows from the following proposition:

Proposition 3.8. *For any λ-periodic Schwartz function u, with period $\lambda \ge N$, and any δ given by the local theory, we have that*

$$|\int_0^\delta \Lambda_{10}(M_{10}; u(t))| \lesssim \lambda^{0+} N^{-5/2+} \|Iu\|^{10}_{X^{1,1/2+}},$$

for $s > 11/28$.

Proof. We perform a dyadic decomposition as in Proposition 3.6, and we borrow the same notation. From now on we do not keep track of all the different $\lambda^{k\epsilon}$, $k \in \mathbb{R}$, and we just write λ^{0+}

We observe that M_{10} is bounded as an elongation of the bounded multiplier M_6. Since the multiplier M_{10} vanishes on Γ_{10} when all frequencies are smaller than N, we can assume that there are $N_1^*, N_2^* \gtrsim N$. Now we divide the proof in two cases.

Case 1. $N_1^\star \sim N_2^\star \sim N_3^\star \gtrsim N$

Since $\frac{1}{m(N_1^\star)m(N_2^\star)m(N_3^\star)(N_1^\star N_2^\star N_3^\star)^{1-}} \lesssim \frac{1}{N^{3-}}$ we have

$$|\int_0^\delta \int M_{10} \prod_{j=1}^{10} \hat{u}_j| \lesssim \frac{(N_1^\star)^{0-}}{N^{3-}} \|JIu_1^\star\|_{L^6_t L^6_x} \|JIu_2^\star\|_{L^6_t L^6_x} \|JIu_3^\star\|_{L^6_t L^6_x} \prod_{j=4}^{10} \|u_j^\star\|_{L^{14}_t L^{14}_x}$$

$$\lesssim \lambda^{0+} \frac{(N_1^\star)^{0-}}{N^{3-}} \|Iu\|^3_{X^{1,1/2+}} \|u\|^7_{X^{(\frac{1}{2}-\frac{3}{14})+,1/2+}} \tag{3.24}$$

$$\lesssim \lambda^{0+} (N_1^\star)^{0-} N^{-3+} \|Iu\|^{10}_{X^{1,1/2+}} \tag{3.25}$$

where in order to obtain (3.24) we use (2.6) and to obtain (3.25) we use the fact that for $s > 2/7$ the following inequality holds

$$\|u\|_{X^{(\frac{1}{2}-\frac{3}{14})+,1/2+}} = \|u\|_{X^{2/7+,1/2+}} \lesssim \|Iu\|_{X^{1,1/2+}}.$$

Case 2. $N_1^\star \sim N_2^\star \gg N_3^\star$

Since $\delta < 1$ we can insert $\eta(t)$ where η is a bump function supported in $[-1/2, 3/2]$ and equals 1 in $[0,1]$. Also notice that $\frac{1}{m(N_1^\star)m(N_2^\star)(N_1^\star N_2^\star)^{1-}} \lesssim \frac{1}{N^{2-}}$. Hence using the Cauchy-Schwartz inequality we get:

$$|\int_0^\delta \eta(t) \int M_{10} \prod_{j=1}^{10} \hat{u}_j| \lesssim \frac{(N_1^\star)^{0-}}{N^{2-}} \|\eta J I u_1^\star u_3^\star\|_{L_t^2 L_x^2} \|J I u_2^\star \prod_4^{10} u_i^\star\|_{L_t^2 L_x^2}$$

$$\lesssim \frac{(N_1^\star)^{0-}}{N^{2-}} (\frac{1}{\lambda} + \frac{1}{N_1^*})^{\frac{1}{2}} \|J I u_1^\star\|_{X^{0,1/2+}} \|u_3^\star\|_{X^{0,1/2+}} \|J I u_2^\star \prod_4^{10} u_i^\star\|_{L_t^2 L_x^2} \tag{3.26}$$

$$\lesssim \frac{(N_1^\star)^{0-}}{N^{5/2-}} \|I u_1^\star\|_{X^{1,1/2+}} \|u_3^\star\|_{X^{0,1/2+}} \|J I u_2^\star\|_{L_t^4 L_x^4} \prod_{j=4}^{10} \|u_j^\star\|_{L_t^{28} L_x^{28}} \tag{3.27}$$

$$\lesssim \lambda^{0+} \frac{(N_1^\star)^{0-}}{N^{5/2-}} \|I u\|^3_{X^{1,1/2+}} \prod_{j=4}^{10} \|u_j\|_{X^{1/2-3/28,1/2+}} \tag{3.28}$$

$$\lesssim \lambda^{0+} (N_1^\star)^{0-} N^{-5/2+} \|I u\|^{10}_{X^{1,1/2+}}$$

where to obtain (3.26) we use Proposition 3.7, to obtain (3.27) we use Hölder's inequality and the facts that $\lambda \geq N$ and $N_1^* \gtrsim N$, to obtain (3.28) we use (2.6), and in the last line we use that $s > 11/28$. □

Now we are ready to prove our main theorem on $\mathbb{T}^1$.

Step 4: Proof of Theorem 1.

Proof. Let $u_0 \in H^s$, where $4/9 < s < 1/2$. By definition, $\frac{|m(k)|}{|k|^{s-1}} \lesssim N^{1-s}$, hence

$$\|\partial_x I u_0^\lambda\|_2 = \|m(k)|k|\widehat{u_0^\lambda}\|_2 = \|\frac{|m(k)|}{|k|^{s-1}} |k|^s \widehat{u_0^\lambda}\|_2 \leq N^{1-s} \||k|^s \widehat{u_0^\lambda}\|_2 \lesssim \frac{N^{1-s}}{\lambda^s} \|u_0\|_{\dot{H}^s}.$$

By the Gagliardo-Nirenberg inequality we have:

$$\|I u_0^\lambda\|_6^6 \lesssim \|\partial_x I u_0^\lambda\|_2^2 \|I u_0^\lambda\|_2^4 \lesssim \frac{N^{2-2s}}{\lambda^{2s}} \|u_0\|^2_{\dot{H}^s} \|u_0\|_2^4.$$

Combining the two inequalities above with the definition of the first modified energy we obtain

$$E^1(u_0^\lambda) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x I u_0^\lambda|^2 dx + \frac{1}{6} \int_{\mathbb{R}} |I u_0^\lambda|^6 dx \lesssim \frac{N^{2-2s}}{\lambda^{2s}} \|u_0\|^2_{\dot{H}^s}.$$

We choose $\lambda \sim N^{\frac{1-s}{s}}$, which for $s < 1/2$ implies $\lambda \gtrsim N$. Then we have that $E^1(u_0^\lambda) \lesssim 1$. Therefore

$$\|I u_0^\lambda\|^2_{H^1} \lesssim 1, \tag{3.29}$$

which allows us to apply Propositions 3.4, 3.6. The main idea of the proof is that if in each step of the iteration we have the same bound for $\|I u_0^\lambda\|_{H^1}$ then we can iterate again with the same timestep. Because we iterate the u^λ solutions, the interval of local existence is of order $\frac{1}{\lambda^{a\epsilon}}$. But this creates no additional problem since λ is given in terms of N. Moreover N is a fixed number and thus the interval of local existence does not shrink. This is the reason why we don't write down the λ^{0+} dependence in the following string of inequalities. Now, by Lemma 3.1, Proposition 3.6, Proposition 3.8 and (3.29) we have that

$$E^2(u^\lambda(\delta)) \lesssim E^2(u^\lambda(0)) + C N^{-5/2+} \lesssim E^1(u^\lambda(0)) + \frac{1}{N} \|I u_0^\lambda\|^6_{H^1} + C N^{-5/2+} \lesssim 1,$$

for N large enough. Combining the inequality above with Proposition 3.6 we have,

$$\|\partial_x Iu(\delta)^\lambda\|_2^2 \lesssim E^2(u^\lambda(\delta)) + O(\frac{1}{N}) \lesssim 1.$$

Thus

$$\|Iu^\lambda(\delta)\|_{H^1} \lesssim 1$$

and we can continue the solution in $[0, M\delta] = [0, T]$ as long as $T \ll N^{5/2-}$. Hence

$$\|Iu^\lambda(T)\|_{H^1} \lesssim 1$$

for all $T \ll N^{5/2-}$. From the definition of I this implies that

$$\|u^\lambda(T)\|_{H^s} \lesssim 1$$

for all $T \ll N^{5/2-}$. Undoing the scaling we have that

$$\|u(T)\|_{H^s} \lesssim C_{N,\lambda}$$

for all $T \ll \frac{N^{5/2-}}{\lambda^2}$. But $\frac{N^{5/2-}}{\lambda^2} \sim N^{\frac{9s-4}{2s}}$ goes to infinity as $N \to \infty$ since $s > 4/9$. □

4. The I-method and the proof of Theorem 1.2.

In this section we present the proof of Theorem 1.2. We will focus on the analysis of the λ-periodic problem:

$$iu_t + \Delta u - |u|^2 u = 0 \tag{4.1}$$

$$u(x, 0) = u_0(x) \in H^s(\mathbb{T}^2_\lambda), t \in \mathbb{R}, \tag{4.2}$$

for $0 < s < 1$. As for the one dimensional case, we will use the I-method. The definitions of the multiplier m and of the operator I are the same as in the 1D context. Precisely, given a parameter $N >> 1$ to be chosen later, we define $m(k)$ to be the following multiplier:

$$m(k) = \begin{cases} 1 & \text{if } |k| < N \\ \left(\frac{|k|}{N}\right)^{s-1} & \text{if } |k| > 2N \end{cases}$$

Then $I : H^s \to H^1$ will be defined as the following multiplier operator:

$$(\widehat{Iu})(k) = m(k)\hat{u}(k).$$

This operator is smoothing of order $1 - s$, that is:

$$\|u\|_{s_0,b_0} \lesssim \|Iu\|_{s_0+1-s,b_0} \lesssim N^{1-s}\|u\|_{s_0,b_0} \tag{4.3}$$

for any $s_0, b_0 \in \mathbb{R}$.

The first modified energy of a function $u \in H^s$ is defined by

$$E^1(u) = E(Iu),$$

where we recall that

$$E(u)(t) = \frac{1}{2}\int |\nabla u(t)|^2 dx + \frac{1}{4}\int |u(t)|^4 dx.$$

Our main objective will be to prove that the modified energy of a solution to the λ-periodic problem (4.1)-(4.2) is almost conserved in time.

We now proceed to present the steps leading to the proof of Theorem 1.2.

Step 1: Local well-posedness for the I-system. The first step towards the proof of Theorem 1.2 is to apply the I-operator to (1.1) and prove a local well-posedness result for the I-initial value problem

$$\begin{aligned} &iIu_t + I\Delta u - I(|u|^2 u) = 0 \\ &Iu(x,0) = Iu_0(x) \in H^s(\mathbb{T}^2_\lambda), t \in \mathbb{R}. \end{aligned} \tag{4.4}$$

This is the content of the next proposition.

Proposition 4.1. *Consider the I-initial value problem* (4.4), *λ=1. If $\|Iu_0\|_{H^1} \lesssim 1$, then* (4.4) *is locally well-posed for any $s > \frac{7}{20}$ in $[0,\delta] \sim [0,1]$.*

Proof. By Duhamel's formula as in Proposition 3.4 we have that

$$\|Iu\|_{X^{1,1/2+}} \lesssim \|Iu_0\|_{H^1} + \delta^{\frac{1}{20}-\epsilon} \|I(|u|^2 u)\|_{X^{1,-\frac{9}{20}}}$$

By the "invariant lemma" in [11] we know that the estimate

$$\|I(|u|^2 u)\|_{X^{1,-\frac{9}{20}}} \lesssim \|Iu\|^3_{X^{1,1/2+}}$$

is implied by

$$\||u|^2 u\|_{X^{s,-\frac{9}{20}}} \lesssim \|u\|^3_{X^{s,1/2+}}. \tag{4.5}$$

Since the rest of the proof is identical to Proposition 3.4 we briefly recall how (4.5) can be obtained. Note that this estimate was first proved by Bourgain [2] and his proof is based upon a local variant of the well known periodic Strichartz estimate

$$\|u\|_{L^4_t L^4_x} \lesssim N^{s_1} \left(\sum_{k \in Q} \int d\tau (1 + |\tau - |k|^2|)^{2b_1} |\hat{u}(k,\tau)|^2 \right)^{\frac{1}{2}} \tag{4.6}$$

where $b_1 > \frac{1-\min(1/2,s_1)}{2}$. Here we omit the proof and refer the reader to [2] for details. □

Corollary 4.2. *Consider the I-initial value problem* (4.4). *If $\|Iu_0\|_{H^1} \lesssim 1$, then* (4.4) *is locally well-posed for any $s > \frac{7}{20}$ in $[0,\delta] \sim [0, \frac{1}{\lambda^{a\epsilon}}]$, where a is a fixed positive integer ($a \sim 100$ will do it).*

Proof. The proof is identical to the argument of Bourgain in [2], the only difference being the fact that (4.6) holds true for the λ-periodic solutions, but now with a factor of order λ^{0+} on the right hand side. Again the fixed power of δ in the local theory, enables us to prove the local well-posedness result for the λ-periodic problem in the interval $[0, \frac{1}{\lambda^{a\epsilon}}]$. □

Step 2: Decay of the first energy. In order to prove that the first energy is almost conserved, we will use the bilinear estimate for λ-periodic functions stated in Proposition 4.6. Its proof is based on some number theoretic facts that we recall in the following three lemmas; see also related estimates in the work of Bourgain [6].

The following lemma is known as **Pick's Lemma** [24]:

Lemma 4.3. *Let Ar be the area of a simply connected lattice polygon. Let E denote the number of lattice points on the polygon edges and I the number of lattice points in the interior of the polygon. Then*

$$Ar = I + \frac{1}{2}E - 1.$$

Lemma 4.4. *Let $\mathcal{C}$ be a circle of radius R. If γ is an arc on $\mathcal{C}$ of length $|\gamma| < \left(\frac{3}{4}R\right)^{1/3}$, then γ contains at most 2 lattice points.*

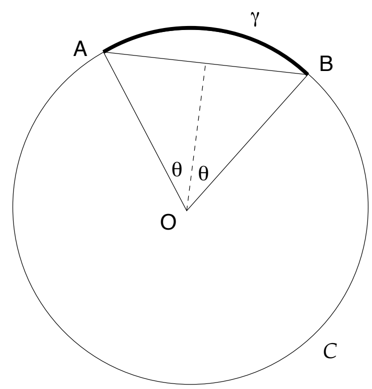

FIGURE 1. Triangle area.

Proof. We prove the lemma by contradiction. Assume that there are 3 lattice points P_1, P_2 and P_3 on an arc $\gamma = AB$ of $\mathcal{C}$, and denote by $T(P_1, P_2, P_3)$ the triangle with vertices P_1, P_2 and P_3. Then, by Lemma 4.3 we have

$$\text{Area of } T(P_1, P_2, P_3) = I + \frac{1}{2}E - 1 \geq I + \frac{3}{2} - 1 = I + \frac{1}{2} \geq \frac{1}{2}.$$

We shall prove that under the assumption that $|\gamma| < \left(\frac{3}{4}R\right)^{1/3}$, then

$$\text{Area of } T(P_1, P_2, P_3) < \frac{1}{2}, \tag{4.7}$$

hence γ must contain at most two lattice points.

We observe that (see Figure 1)

$$\text{Area of the sector } ABO = R^2\theta,$$
$$\text{Area of the triangle } ABO = R^2 \sin\theta\cos\theta.$$

Hence, for any P_1, P_2, P_3 on γ we have

$$\text{Area of } T(P_1, P_2, P_3) \leq R^2\theta - R^2\sin\theta\cos\theta = R^2(\theta - \frac{1}{2}\sin(2\theta)). \tag{4.8}$$

One can easily check that

$$\theta - \frac{1}{2}\sin(2\theta) \leq \frac{2}{3}\theta^3. \tag{4.9}$$

Thus (4.8), (4.9) and the fact that $|\gamma| = R\theta$ imply that

$$\text{Area of } T(P_1, P_2, P_3) \leq \frac{2}{3}R^2\theta^3 = \frac{2}{3}R^2(|\gamma|R^{-1})^3 < \frac{1}{2},$$

where to obtain the last inequality we used the assumption that $|\gamma| < \left(\frac{3}{4}R\right)^{1/3}$. Therefore (4.7) is proved.

□

Also we recall the following result of Gauss, see, for example [19]

Lemma 4.5. *Let K be a convex domain in $\mathbb{R}^2$. If*

$$N(\lambda) = \#\{\mathbb{Z}^2 \cap \lambda K\},$$

then, for $\lambda >> 1$

$$N(\lambda) = \lambda^2 |K| + O(\lambda),$$

where $|K|$ denotes the area of K and $\#S$ denotes the number of points of a set S.

Now we are ready to present the bilinear estimate for λ-periodic functions. Let $\eta \in C_0^\infty$ be a bump function that is supported on $[-2,2]$ and equals 1 on $[-1,1]$.

Proposition 4.6. *Let ϕ_1 and ϕ_2 be $\lambda-$periodic functions, $\lambda \geq 1$, whose Fourier transforms are supported on $\{k : |k| \sim N_1\}$ and $\{k : |k| \sim N_2\}$ respectively, with $N_1 >> N_2 > 1$.*

(a) *Then*

$$\|\eta(t)(U_\lambda(t)\phi_1)\ \eta(t)(U_\lambda(t)\phi_2)\|_{L^2_t L^2_x} \lesssim (\lambda N_2)^\epsilon \|\phi_1\|_{L^2} \|\phi_2\|_{L^2}, \tag{4.10}$$

for any $\epsilon > 0$. Hence

$$\|\phi_1\ \phi_2\|_{L^2_t L^2_x} \lesssim (\lambda N_2)^\epsilon \|\phi_1\|_{X^{0,1/2+}} \|\phi_2\|_{X^{0,1/2+}}.$$

(b) *Moreover, if $\lambda >> 1$ then*

$$\|\eta(t)(U_\lambda(t)\phi_1)\ \eta(t)(U_\lambda(t)\phi_2)\|_{L^2_t L^2_x} \lesssim \left(\frac{1}{\lambda} + \frac{N_2}{N_1}\right)^{1/2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2}. \tag{4.11}$$

Remark. Note that as $\lambda \to \infty$ we recover the improved bilinear Strichartz with constant $C(N_1, N_2) = (\frac{N_2}{N_1})^{1/2}$.

Proof. (a) Let ψ be a positive even Schwartz function such that $\psi = \hat{\eta}$. Then we have

$$\begin{aligned} B &= \|\eta(t)(U_\lambda(t)\phi_1)\ \eta(t)(U_\lambda(t)\phi_2)\|_{L^2_t L^2_x} \\ &= \| \int_{k=k_1+k_2,\ \tau=\tau_1+\tau_2} \widehat{\phi_1}(k_1)\widehat{\phi_2}(k_2)\psi(\tau_1 - k_1^2)\ \psi(\tau_2 - k_2^2)\ (dk_1)_\lambda\ (dk_2)_\lambda\ d\tau_1\ d\tau_2 \|_{L^2_\tau L^2_k} \\ &\lesssim \| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ (dk_1)_\lambda\ (dk_2)_\lambda \right)^{1/2} \times \end{aligned}$$

$$\times \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ (dk_1)_\lambda\ (dk_2)_\lambda \right)^{1/2} \|_{L^2_\tau L^2_k} \tag{4.12}$$

where to obtain (4.12) we used Cauchy-Schwartz and the following definition of $\widetilde{\psi} \in \mathcal{S}$

$$\int_{\tau=\tau_1+\tau_2} \psi(\tau_1 - k_1^2)\ \psi(\tau_2 - k_2^2)\ d\tau_1\ d\tau_2 = \widetilde{\psi}(\tau - k_1^2 - k_2^2).$$

An application of Hölder gives us the following upper bound on (4.12)

$$M \| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ (dk_1)_\lambda\ (dk_2)_\lambda \right)^{1/2} \|_{L^2_\tau L^2_k}, \tag{4.13}$$

where

$$M = \| \int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ (dk_1)_\lambda\ (dk_2)_\lambda \|^{1/2}_{L^\infty_\tau L^\infty_k}.$$

Now by integration in τ followed by Fubini in k_1, k_2 and two applications of Plancharel we have

$$\| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ (dk_1)_\lambda\ (dk_2)_\lambda \right)^{1/2} \|_{L^2_\tau L^2_k} \lesssim \|\phi_1\|_{L^2_x} \|\phi_2\|_{L^2_x},$$

which combined with (4.12) and (4.13) gives

$$B \lesssim M\|\phi_1\|_{L^2_x}\|\phi_2\|_{L^2_x}. \tag{4.14}$$

We find an upper bound on M as follows:

$$M \lesssim \left(\frac{1}{\lambda^2}\sup_{\tau,k} \#S\right)^{\frac{1}{2}}, \tag{4.15}$$

where

$$S = \{k_1 \in \mathbb{Z}^2/\lambda \mid |k_1| \sim N_1,\ |k-k_1|\ \sim N_2,\ |k|^2 - 2k_1\cdot(k-k_1) = \tau + O(1)\},$$

and $\#A$ denotes the number of lattice points of a set A.

For notational purposes, let us rename $k_1 = z$, that is

$$S = \{z \in \mathbb{Z}^2/\lambda \mid |z| \sim N_1,\ |k-z|\ \sim N_2,\ |k|^2 + 2|z|^2 - 2k\cdot z = \tau + O(1)\}.$$

Let z_0 be an element of S i.e.

$$|z_0| \sim N_1,\ \ |k-z_0| \sim N_2, \tag{4.16}$$

and

$$|k|^2 + 2|z_0|^2 - 2k\cdot z_0 = \tau + O(1). \tag{4.17}$$

In order to obtain an upper bound on $\#S$, we shall count the number of l's $\in \mathbb{Z}^2$ such that $z_0 + \frac{l}{\lambda} \in S$ where z_0 satisfies (4.16) - (4.17). Thus such l's must satisfy

$$\left|z_0 + \frac{l}{\lambda}\right| \sim N_1,\ \ \left|z_0 + \frac{l}{\lambda} - k\right| \sim N_2, \tag{4.18}$$

and

$$|k|^2 + 2\left|z_0 + \frac{l}{\lambda}\right|^2 - 2k\cdot(z_0 + \frac{l}{\lambda}) = \tau + O(1). \tag{4.19}$$

However by (4.17) we can rewrite the left hand side of (4.19) as follows

$$\begin{aligned}|k|^2 + 2\left|z_0 + \frac{l}{\lambda}\right|^2 - 2k\cdot(z_0 + \frac{l}{\lambda}) &= |k|^2 + 2|z_0|^2 + 2\left|\frac{l}{\lambda}\right|^2 + 4z_0\cdot\frac{l}{\lambda} - 2k\cdot z_0 - 2k\cdot\frac{l}{\lambda}\\ &= \tau + O(1) + 2\left|\frac{l}{\lambda}\right|^2 + 4z_0\cdot\frac{l}{\lambda} - 2k\cdot\frac{l}{\lambda}.\end{aligned}$$

Therefore (4.19) holds if

$$\left|\frac{l}{\lambda}\right|^2 + 2\frac{l}{\lambda}\cdot(z_0 - \frac{k}{2}) = O(1). \tag{4.20}$$

Moreover, (4.16) and (4.18) yield

$$\left|\frac{l}{\lambda}\right| = \left|\frac{l}{\lambda} + z_0 - k - z_0 + k\right| \lesssim N_2 + N_2,$$

that is

$$|l| \lesssim \lambda N_2. \tag{4.21}$$

Finally we observe that (4.16) together with the assumption that $N_1 >> N_2$ implies that

$$N_1 \sim N_1 - N_2 \sim \left|\left|\frac{z_0}{2} - \frac{k}{2}\right| - \left|\frac{z_0}{2}\right|\right| \leq \left|z_0 - \frac{k}{2}\right| \leq \left|\frac{z_0}{2} - \frac{k}{2}\right| + \left|\frac{z_0}{2}\right| \sim N_2 + N_1 \lesssim N_1,$$

i.e.

$$\left| z_0 - \frac{k}{2} \right| \sim N_1. \tag{4.22}$$

Hence, it suffices to count the $l's \in \mathbb{Z}^2$ satisfying (4.20) and (4.21) where z_0 is such that (4.22) holds.

Let $w = (a, b)$ denote the vector $z_0 - \frac{k}{2}$. Thus we need to count the number of points in the set A^λ

$$A^\lambda = \{ l \in \mathbb{Z}^2 : \big| \, |l|^2 + 2\lambda l \cdot w \, \big| = O(\lambda^2), \ |l| \lesssim \lambda N_2, \ |w| \sim N_1 \}, \tag{4.23}$$

or equivalently,

$$A^\lambda = \{ (x, y) \in \mathbb{Z}^2 : \left| x^2 + y^2 + 2\lambda(ax + by) \right| \le c\lambda^2, \ x^2 + y^2 \le (k_2 \lambda N_2)^2, \ a^2 + b^2 \sim N_1^2 \}, \tag{4.24}$$

for some $c, k_2 > 0$. Let $\mathcal{C}_-^\lambda$, $\mathcal{C}_+^\lambda$ be the following circles,

$$\begin{aligned} \mathcal{C}_-^\lambda &: \ (x + \lambda a)^2 + (y + \lambda b)^2 = -c\lambda^2 + \lambda^2(a^2 + b^2) \\ \mathcal{C}_+^\lambda &: \ (x + \lambda a)^2 + (y + \lambda b)^2 = c\lambda^2 + \lambda^2(a^2 + b^2) \end{aligned}$$

and for any integer n, let $\mathcal{C}_n^\lambda$ be the circle

$$\mathcal{C}_n^\lambda : \ (x + \lambda a)^2 + (y + \lambda b)^2 = n + \lambda^2(a^2 + b^2).$$

Finally, let $\mathcal{D}^\lambda$ denote the disk

$$\mathcal{D}^\lambda : \ x^2 + y^2 \le (k_2 \lambda N_2)^2.$$

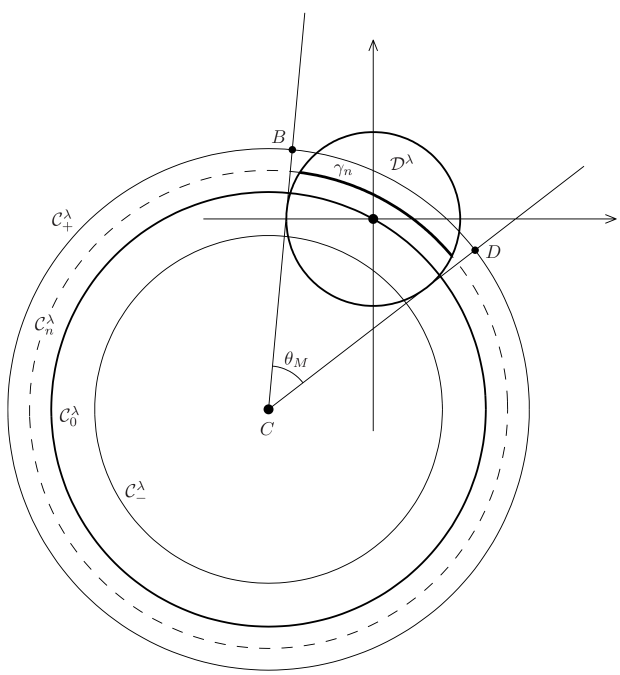

Figure 2. Circular sector

We need to count the number of lattice points inside $\mathcal{D}^\lambda$ that are on arcs of circles $\mathcal{C}_n^\lambda$, with

$$-c\lambda^2 \leq n \leq c\lambda^2.$$

Precisely, the total number of lattice point in A^λ can be bounded from above by

$$2c\lambda^2 \times \#(\mathcal{C}_n^\lambda \cap \mathcal{D}^\lambda). \tag{4.25}$$

Denote by γ_n the arc of circle $\mathcal{C}_n^\lambda$ which is contained in $\mathcal{D}^\lambda$. Notice that (see Figure 2)

$$|\gamma_n| \leq R_M \theta_M \tag{4.26}$$

where $R_M = \sqrt{c\lambda^2 + k_1\lambda^2 N_1^2}$ for some constant $k_1 > 0$, and θ_M is the angle between the line segment CB and CD, which lie along the tangent lines from $C = (-\lambda a, -\lambda b)$ to the circle $x^2 + y^2 = (k_2\lambda N_2)^2$. Hence,

$$\sin\theta_M \leq k\frac{N_2}{N_1},$$

for some constant $k > 0$. Since $N_1 >> N_2$, we can assume that $\sin\theta_M > \frac{1}{2}\theta_M$. Hence,

$$\theta_M < 2k\frac{N_2}{N_1}. \tag{4.27}$$

In order to count efficiently the number of lattice points on each γ_n, we distinguish two cases based on the application of Lemma 4.4.

Case 1: $2k\frac{N_2}{N_1} < \left(\frac{3}{4}\right)^{\frac{1}{3}} R_M^{-\frac{2}{3}}$**.**
In this case (4.26)-(4.27) guarantee that the hypothesis of Lemma 4.4 is satisfied by each arc of circle γ_n. Hence, on each γ_n there are at most two lattice points.

Case 2: $2k\frac{N_2}{N_1} \geq \left(\frac{3}{4}\right)^{\frac{1}{3}} R_M^{-\frac{2}{3}}$**.**
In this case we approximate the number of lattice points on γ_n by the number of lattice points on $\mathcal{C}_n^\lambda$ (see for example [1] , [7]):

$$\#\mathcal{C}_n^\lambda \lesssim R_M^\epsilon \sim (\lambda N_1)^\epsilon \lesssim (\lambda N_2)^{3\epsilon} \tag{4.28}$$

for any $\epsilon > 0$.

Combining the estimate in (4.25), **Case 1** and **Case 2** we conclude that

$$\#S \lesssim \lambda^2 + \lambda^2\,(\lambda N_2)^\epsilon,$$

for any $\epsilon > 0$. Since $\lambda, N_2 \geq 1$, together with (4.15), this implies that

$$M \lesssim (\lambda N_2)^\epsilon,$$

for all positive ϵ's. Hence (4.10) follows.

(b) As in part (a) we will count the number of points in the set A^λ given by (4.23). According to (4.24) we have that

$$A^\lambda = \mathbb{Z}^2 \cap \lambda B_1, \tag{4.29}$$

with

$$B_1 = \{(x,y) \in \mathbb{R}^2 :\ \left|x^2 + y^2 + 2(ax + by)\right| \leq c,\ x^2 + y^2 \leq (k_2N_2)^2,\ a^2 + b^2 \sim N_1^2\}. \tag{4.30}$$

In the rest of the proof for arbitrary two points P_1, P_2 on a circle with center C we denote by $S(C, P_1, P_2)$ the solid sector contained between the lines CP_1,CP_2 and the circle arc between P_1 and P_2.

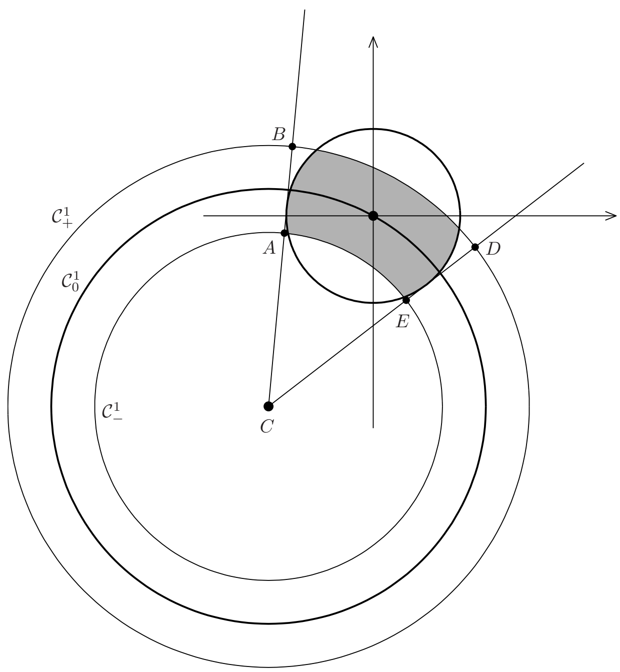

FIGURE 3. Circular sectors when $\lambda = 1$

We observe that (see Figure 3)

$$B_1 \subset S(C, B, D) \setminus S(C, A, E),$$

where CB and CD lie along the tangent lines from $C = (-a, -b)$ to the circle $x^2 + y^2 = (k_2 N_2)^2$. Hence, setting $S_1 = S(C, B, D), S_2 = S(C, A, E)$, and $H = S_1 \setminus S_2$ we have

$$\begin{aligned}
\# A^\lambda &= \#\{\mathbb{Z}^2 \cap \lambda B_1\} \\
&\leq \#\{\mathbb{Z}^2 \cap \lambda S_1\} - \#\{\mathbb{Z}^2 \cap \lambda S_2\} \\
&= \lambda^2 |S_1| - \lambda^2 |S_2| + O(\lambda) && (4.31) \\
&= \lambda^2 |H| + O(\lambda), && (4.32)
\end{aligned}$$

where to obtain (4.31) we used Lemma 4.5 twice. In order to compute the area of H notice that since $a^2 + b^2 \sim N_1^2$, we have (see Figure 3)

$$\sin\theta \sim \frac{N_2}{N_1}.$$

Hence, by the assumption $N_1 >> N_2$, we get $\theta \sim \frac{N_2}{N_1}$. Therefore

$$|H| = (r_1^2 - r_2^2)\theta \sim \frac{N_2}{N_1},$$

with

$$r_1^2 = c + (a^2 + b^2), \quad r_2^2 = -c + (a^2 + b^2).$$

This combined with (4.32) implies that

$$\#A^\lambda \lesssim \lambda + \lambda^2 \frac{N_2}{N_1},$$

that together with (4.15) gives

$$M \lesssim \left(\frac{1}{\lambda} + \frac{N_2}{N_1}\right)^{1/2}.$$

□

Now we are ready to prove the desired decay of the first modified energy, which is the content of the following proposition:

Proposition 4.7. *Let $s > 1/2$, $\lambda \lesssim N$, $t > 0$, and $u_0 \in H^s(\mathbb{T}^2_\lambda)$ be given. If u is a solution to (4.1) $-$ (4.2) then*

$$|E^1(u)(t) - E^1(u)(0)| \lesssim \frac{1}{N^{1-}} \|Iu\|_{X^{1,1/2+}}.$$

Proof. The definition of E together with equation (4.1) give that:

$$\begin{aligned}\partial_t E(Iu)(t) &= \mathrm{Re} \int_{\mathbb{T}^2_\lambda} \overline{I(u)_t}(|Iu|^2 Iu - \Delta Iu - iIu_t)\\ &= \mathrm{Re} \int_{\mathbb{T}^2_\lambda} \overline{I(u)_t}(|Iu|^2 Iu - I(|u|^2 u))\end{aligned}$$

Therefore, integrating in time, and using the Parseval's formula we get:

$$\begin{aligned}E^1(u)(t) - E^1(u)(0) &= \\ &= \int_0^t \int_{\Gamma_4} \left(1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right) \widehat{\overline{I\partial_t u}}(k_1)\widehat{Iu}(k_2)\widehat{\overline{Iu}}(k_3)\widehat{Iu}(k_4)\end{aligned} \tag{4.33}$$

Using the equation (4.1), we then get:

$$E^1(u)(t) - E^1(u)(0) = \mathrm{Tr}_1 + \mathrm{Tr}_2,$$

where

$$\begin{aligned}\mathrm{Tr}_1 &= \\ &= \int_0^t \int_{\Gamma_4} \left(1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right) \widehat{\Delta \overline{Iu}}(k_1)\widehat{Iu}(k_2)\widehat{\overline{Iu}}(k_3)\widehat{Iu}(k_4)\end{aligned} \tag{4.34}$$

$$\begin{aligned}\mathrm{Tr}_2 &= \\ &= \int_0^t \int_{\Gamma_4} \left(1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right) \widehat{\overline{I(|u|^2 u)}}(k_1)\widehat{Iu}(k_2)\widehat{\overline{Iu}}(k_3)\widehat{Iu}(k_4)\end{aligned} \tag{4.35}$$

We wish to prove that:

$$|\mathrm{Tr}_1| + |\mathrm{Tr}_2| \lesssim \frac{1}{N^{1-}}, \tag{4.36}$$

for a constant $C = C(\|Iu\|_{X^{1,1/2+}})$. According to the **Decomposition Remark** in our estimates we may ignore the presence of the complex conjugates.
In order to obtain the estimate (4.36), we break u into a sum of dyadic pieces u_j, each with frequency support $\langle k \rangle \sim 2^j$, $j = 0, \ldots$. We will then be able to sum over all the frequency

pieces u_j of u, since our estimates will decay geometrically in these frequencies.
We start by analyzing Tr_1. First notice that,

$$\|\Delta(Iu)\|_{X^{-1,1/2+}} \leq \|Iu\|_{X^{1,1/2+}},$$

therefore, the estimate on Tr_1 will follow, if we prove that:

$$\left|\int_0^t \int_{\Gamma_4} \left(1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right) \widehat{\overline{\phi_1}}(k_1)\widehat{\phi_2}(k_2)\widehat{\overline{\phi_3}}(k_3)\widehat{\phi_4}(k_4)\right| \lesssim \frac{1}{N^{1-}}(N_1N_2N_3N_4)^{0-}\|\phi_1\|_{X^{-1,1/2+}} \prod_{i=2}^{i=4} \|\phi_i\|_{X^{1,1/2+}} \tag{4.37}$$

for any λ-periodic function $\phi_i, i = 1, ..., 4$ with positive spatial Fourier transforms supported on

$$\langle k\rangle \sim 2^{l_i} \equiv N_i, \tag{4.38}$$

for some $l_i \in \{0, 1, ...\}$.
By the symmetry of the multiplier, we can assume that

$$N_2 \geq N_3 \geq N_4. \tag{4.39}$$

Moreover, since we are integrating over Γ_4, we have that $N_1 \lesssim N_2$. Therefore, we only need to get the factor N_2^{0-} in the estimate (4.37), in order to be able to sum over all dyadic pieces.
Now we consider three different cases, by comparing N to the size of the N_i's.

Case I. $N >> N_2$. In this case, the symbol in Tr_1 is identically zero, and the desired bound follows trivially.

Case II. $N_2 \gtrsim N >> N_3 \geq N_4$. Since we are on Γ_4, we also have that $N_1 \sim N_2$. By the mean value theorem, we have

$$\left|1 - \frac{m(k_2+k_3+k_4)}{m(k_2)}\right| = \left|\frac{m(k_2) - m(k_2+k_3+k_4)}{m(k_2)}\right| \lesssim \left|\frac{m'(k_2)}{m(k_2)}\right| N_3 \sim \frac{N_3}{N_2}.$$

Using the pointwise bound above, the Cauchy-Schwartz inequality, and Plancharel's theorem we get:

$$|\mathrm{Tr}_1| \lesssim \frac{N_3}{N_2}\|\phi_1\phi_3\|_{L_t^2L_x^2}\|\phi_2\phi_4\|_{L_t^2L_x^2} \tag{4.40}$$

By two applications of Hölder's inequality, followed by four applications of the Strichartz estimate (2.8) we obtain:

$$|\mathrm{Tr}_1| \lesssim \frac{N_3}{N_2}\lambda^{0+}\frac{(N_1)^{1+}}{(N_2N_3N_4)^{1-}}\|\phi_1\|_{X^{-1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}}.$$

Since $N_1 \sim N_2 \gtrsim N$, and $\lambda \lesssim N$, it follows immediately that in this case

$$|\mathrm{Tr}_1| \lesssim \frac{1}{N^{1-}}N_2^{0-}\|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}}.$$

Case III. $N_2 \geq N_3 \gtrsim N$. We will use the following crude bound on the multiplier:

$$\left|1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right| \lesssim \frac{m(k_1)}{m(k_2)m(k_3)m(k_4)}, \tag{4.41}$$

which follows from the fact that we are integrating on Γ_4. We also need the following estimate:

$$\frac{1}{m(k)|k|^{\alpha}} \lesssim N^{-\alpha}, \tag{4.42}$$

for $\alpha \geq 1/2-$, $|k| \gtrsim N$.
We distinguish two subcases:

IIIa)$N_2 \sim N_3 \gtrsim N$. Using Plancharel's, followed by Holder's inequality, and then by four applications of the Strichartz estimate (2.8), we get:

$$\begin{aligned} |\mathrm{Tr}_1| &\lesssim \frac{m(k_1)}{m(k_2)m(k_3)m(k_4)}\lambda^{0+}\frac{(N_1)^{1+}}{(N_2N_3N_4)^{1-}}\times \\ &\times \|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}} \\ &\lesssim \lambda^{0+}\frac{1}{N^{1-}m(k_4)N_4^{1-}}\|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}}, \end{aligned} \tag{4.43}$$

where to obtain (4.43) we use that $N_1 \leq N_2$ together with (4.42) with $\alpha = 1-$ ($N_3 \gtrsim N$). Now if $N_4 \gtrsim N$, we conclude with an application of the monotonicity property (4.42). If $N_4 << N$, then we use that $m(k_4) = 1$ and $N_4 \geq 1$. Hence, since $\lambda \lesssim N$, we get

$$|\mathrm{Tr}_1| \lesssim \frac{1}{N^{1-}}N_2^{0-}\|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}}$$

IIIb)$N_2 >> N_3 \gtrsim N$. Since we are integrating on Γ_4, we get $N_1 \sim N_2$. Let us emphasize that in the case **IIIb)** we could not use the same strategy as in the case **IIIa)**. More precisely, the approach of **IIIa)** consisting of Plancharel's, followed by Holder's inequality, and then by four applications of the Strichartz estimate (2.8) would produce a bound of the type

$$\begin{aligned} &\frac{m(k_1)}{m(k_2)m(k_3)m(k_4)}\lambda^{0+}\frac{(N_1)^{1+}}{(N_2N_3N_4)^{1-}}\times \|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}} \\ &\lesssim \frac{1}{N^{1-}}\lambda^{0+}N_1^{0+}\|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}} \end{aligned}$$

which we cannot sum in N_1 (here we cannot cancel N_1^{0+} by N_3^{0+} since $N_3 << N_1$). Thus to handle the case **IIIb)** we do need an improved Strichartz estimate given in Proposition 4.6.

In order to obtain the desired estimate, we apply the Cauchy-Schwartz inequality, together with Proposition 4.6 for the pair ϕ_1, ϕ_3 and for the pair ϕ_2, ϕ_4. Thus

$$\begin{aligned} |\mathrm{Tr}_1| &\lesssim \frac{m(k_1)}{m(k_2)m(k_3)m(k_4)}(\lambda N_3)^{\epsilon}(\lambda N_4)^{\epsilon}\frac{N_1}{N_2N_3N_4}\times \\ &\times \|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}} \lesssim \\ &\lesssim \frac{1}{m(k_3)m(k_4)}\ \lambda^{0+}\ \frac{1}{N_3^{1-}N_4^{1-}}\times \qquad (4.44) \\ &\times \|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}} \lesssim \\ &\lesssim \frac{1}{N^{1-}}\lambda^{0+}\frac{1}{m(k_4)N_4^{1-}}\|\phi_1\|_{X^{-1,1/2+}}\|\phi_2\|_{X^{1,1/2+}}\|\phi_3\|_{X^{1,1/2+}}\|\phi_4\|_{X^{1,1/2+}} \qquad (4.45) \end{aligned}$$

where in order to obtain (4.44) we have used that $N_1 \sim N_2$ and in order to obtain (4.45) we have used (4.42) once with $\alpha = 1-$ ($N_3 \gtrsim N$). Finally as observed in the case **IIIa)**, $m(k_4)N_4^{1-} \geq 1$. Therefore

$$|\mathrm{Tr}_1| \lesssim \frac{1}{N^{1-}} N_2^{0-} \|\phi_1\|_{X^{-1,1/2+}} \|\phi_2\|_{X^{1,1/2+}} \|\phi_3\|_{X^{1,1/2+}} \|\phi_4\|_{X^{1,1/2+}}.$$

Now, we proceed to analyze Tr_2. We claim that the following stronger estimate holds:

$$|\mathrm{Tr}_2| \lesssim \frac{1}{N^{2-}} \|Iu\|^6_{X^{1,1/2+}}.$$

As for Tr_1 it suffices to prove the following bound:

$$\mathrm{LHS} = \int_0^t \int_{\Gamma_6} m_{123}(m_{456} - m_4 m_5 m_6) \prod_{j=1}^6 \widehat{\phi_j} \lesssim \frac{N_{max}^{0-}}{N^{2-}} \prod_{i=1}^6 \|I\phi_i\|_{X^{1,1/2+}},$$

with the ϕ_i's having positive spatial Fourier transform, supported on

$$\langle k \rangle \sim 2^{l_i} \equiv N_i,$$

for some $l_i \in \{0, 1, ...\}$, and with N_{max}, N_{med} denoting respectively the biggest and the second biggest frequency among the N_i's.

Since we are integrating over Γ_6, we can assume that $N_{max} \sim N_{med}$. Also, we only need to consider the case when $N_{max} \gtrsim N$, otherwise the symbol in (LHS) is zero, and the estimate above holds trivially.

Now, using Holder's inequality, together with the estimate (4.42) twice, and the fact that the multiplier m is bounded, we then obtain:

$$\begin{aligned}
\mathrm{LHS} &\leq \frac{N_{max}^{1-} N_{med}^{1-} m_{max} m_{med}}{N_{max}^{1-} N_{med}^{1-} m_{max} m_{med}} \int_0^t \int_{\Gamma_6} \prod_{j=1}^6 \widehat{\phi_j}(k,t) \\
&\lesssim \frac{N_{max}^{0-}}{N^{2-}} \|J^{1-} I\phi_{max}\|_{L_t^4 L_x^4} \|J^{1-} I\phi_{med}\|_{L_t^4 L_x^4} \prod_{k=1}^4 \|\phi_k\|_{L_t^8 L_x^8}.
\end{aligned}$$

The Strichartz estimate (2.8), and the interpolation estimate (2.10) then imply:

$$\mathrm{LHS} \lesssim \frac{N_{max}^{0-}}{N^{2-}} \|I\phi_{max}\|_{X^{1,1/2+}} \|I\phi_{med}\|_{X^{1,1/2+}} \prod_{k=1}^4 \|\phi_k\|_{X^{1/2+,1/2+}} \lesssim \frac{N_{max}^{0-}}{N^{2-}} \prod_{j=1}^6 \|I\phi_j\|_{X^{1,1/2+}},$$

where in the last inequality we have used (4.3). □

Step 3: Proof of Theorem 1.2. We are now ready to exhibit the proof of Theorem 1.2. As we explained in Theorem 1.1 we do not keep track of the λ^{0+} factors.

Proof. Let u_0 be an initial data, and let $N >> 1$ be given. Recall that, for any $\lambda \in \mathbb{R}$, we set $u_0^\lambda(x,t) = \frac{1}{\lambda} u_0(\frac{x}{\lambda}, \frac{t}{\lambda^2})$. Now, let us choose a rescaling parameter λ so that $E^1(u_0^\lambda) \lesssim 1$, that is

$$\lambda \sim N^{\frac{1-s}{s}}.$$

Then, by Corollary 4.2 and Proposition 4.7, there exists a δ, and a solution u_λ to (4.1), with initial condition u_0^λ, such that

$$E^1(u_\lambda)(\delta) \lesssim E^1(u_\lambda)(0) + O\left(\frac{1}{N^{1-}}\right).$$

Therefore, we can continue our solution u_λ until the size of $E^1(u_\lambda)(t)$ reaches 1, that is at least $C \cdot N^{1-}$ times. Hence

$$E^1(u_\lambda)(C \cdot N^{1-}\delta) \sim 1.$$

Given $T >> 1$, we choose our parameter $N >> 1$, so that

$$T \sim \frac{C \cdot N^{1-}\delta}{\lambda^2} \sim N^{\frac{3s-2}{s}-}.$$

We observe that the exponent of N is positive as long as $s > 2/3$, hence N is well-defined for all times T. Now, undoing the scaling, we get that:

$$E^1(u)(T) \lesssim T^{\frac{4(1-s)}{3s-2}+},$$

with u solution to (1.1)-(1.2), $d = 2$. This bound implies the desired conclusion. □

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