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Approximation of Solutions of the Cubic Nonlinear Schrödinger Equations by Finite-Dimensional Equations and Nonsqueezing Properties

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We prove a nonsqueezing result for certain nonlinear Schrödinger equations, extending to the NLSE some recent work of S. Kuksin [Kuk]. The main feature of this situation is that the flow map is not a compact perturbation of a linear map in the symplectic Hilbert space. The method consists of a direct reduction of the problem to a finite-dimensional model where the symplectic capacity preservation holds. This is achieved by elaborating some of the techniques of [Bo1], [Bo2]. The precise form of the nonlinearity is of importance in this argument.

Section 1

We will consider for simplicity the cubic NLSE

$$i\, u_t + u_{xx} + u|u|^2 = 0 \tag{1}$$

$$u = u(x,t) \text{ periodic in } x$$

with initial data

$$u(x,0) = \phi(x). \tag{2}$$

The same argument applies equally well to equations of the form $i\, u_t + u_{xx} + A(x,t)u + B(x,t)\, u|u|^2 = 0$ with A, B real smooth functions in x, t, both periodic in x. (These equations are Hamiltonian but not integrable.)

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We assume $\phi \in L^2(\mathbf{T})$. There is the equivalent integral equation

$$u(t) = U(t)\ \phi + i \int_0^t U(t-\tau)\ w(\tau)\ d\tau \tag{3}$$

where $U(t)$ is the group $e^{i\partial_x^2}$ and w stands for $u|u|^2$.

In [Bo], we studied the wellposedness problem of (1), (2) and we summarize some facts of relevance in this discussion. Assume $t \in [0, T]$ and define a quasi norm for functions u

$$|||u||| = \inf \left\{ \sum_k \int d\lambda\ (1 + |\lambda - k^2|)\ |\widehat{u}(k, \lambda)|^2 \right\}^{1/2} \tag{4}$$

where the infimum is taken over all representations

$$u(x, t) = \sum_k \int d\lambda\ \widehat{u}(k, \lambda)\ e^{i(kx + \lambda t)} \tag{5}$$

the equality (5) being valid on $\mathbf{T} \times [0, T]$.

Using the norm $|||\ |||$, it is shown in [Bo] that the local Cauchy problem (1), (2) is well posed for $\phi \in L^2(\mathbf{T})$. This follows from an application of Picard's fixpoint theorem applied to (3). Using a Fourier analysis, one establishes the inequalities

$$|||u||| \le \|\phi\|_2 + \gamma(T)\ |||u|||^3 \tag{6}$$

and also, for $t \in [0, T]$,

$$\|u(t)\|_2 \le \|\phi\|_2 + \gamma(T)\ |||u|||^3 \tag{7}$$

where $\gamma(T) \to 0$ for $T \to 0$.

Similarly, if u, v are the solutions corresponding to L^2-data ϕ, ψ, one has

$$|||u - v||| \le \|\phi - \psi\|_2 + \gamma(T)\ (|||u|||^2 + |||v|||^2)\ |||u - v||| \tag{8}$$

$$\|u(t) - v(t)\|_2 \le \|\phi - \psi\|_2 + \gamma(T)\ (|||u|||^2 + |||v|||^2)\ |||u - v|||. \tag{9}$$

Thus, letting T be sufficiently small (depending on $\|\phi\|_2$, $\|\psi\|_2$ size), (8), (9) yield for $t \in [0, T]$

$$|||u - v||| \le 2\ \|\phi - \psi\|_2 \qquad \text{and } \|u(t) - v(t)\|_2 \le 2\ \|\phi - \psi\|_2. \tag{10}$$

Recall also the conservation of the L^2-norm

$$\int_{\mathbf{T}} |u(t)|^2 \, dx. \tag{11}$$

This fact permits an iteration of the local result to get global wellposedness. In particular one gets from (10), for all time t,

$$\|u(t) - v(t)\|_2 \le c \, (\|\phi\|_2, \|\psi\|_2)^{|t|} \, \|\phi - \psi\|_2. \tag{12}$$

The key analytical fact used in obtaining inequalities (6)–(9) is the following L^2-L^4-bound:

$$\left\| \sum_k \int d\lambda \, \widehat{u}(k,\lambda) \, e^{i(kx+\lambda t)} \right\|_{L^4(\mathbf{T}\times[0,1])} \le c \left(\sum_k \int d\lambda \, (1 + |\lambda - k^2|)^{3/4} \, |\widehat{u}(k,\lambda)|^2 \right)^{1/2} \tag{13}$$

(see [Bo], Proposition 2.6).

In what follows, we will repeat most of this analysis, making certain refinements of it. Here the specific algebraic structure of the nonlinear term $u|u|^2$ will matter.

Section 2

For a given positive integer N, consider the modified equation

$$i \, v_t + v_{xx} + P_N(v|v|^2) = 0 \tag{14}$$

with data

$$v(x,0) = \phi(x), \qquad \phi = P_N \, \phi \tag{15}$$

where P_N is the Dirichlet projection with respect to the x-variable, i.e.,

$$P_N \, \phi = \sum_{|n|\le N} \widehat{\phi}(n) \, e^{inx}.$$

Thus $v = \sum_{|n|\le N} v_n(t) \, e^{inx}$, and considering the Hamiltonian formulation

$$\frac{dv}{dt} = i \, \frac{\partial H}{\partial \overline{v}} \tag{16}$$

$$H(\phi) = \frac{1}{2} \int_{\mathbf{T}} |\phi'|^2 - \frac{1}{4} \int_{\mathbf{T}} |\phi|^4 \tag{17}$$

one gets a finite-dimensional phase space $(\mathrm{Re}\, \phi, \mathrm{Im}\, \phi)$.

The purpose of this article is to prove following result.

Proposition 1. Consider the solutions u, v to the Cauchy problems

$$\begin{cases} i\, u_t + u_{xx} + u|u|^2 = 0 \\ \qquad u(x,0) = \phi(x) \end{cases} \tag{18}$$

$$\begin{cases} i\, v_t + v_{xx} + P_N(v|v|^2) = 0 \\ \qquad v(x,0) = \phi(x) \end{cases} \tag{19}$$

where $\phi = P_N\, \phi$. Fix a positive integer N' and a time t. Then one has an approximation

$$\|P_{N'}(u(t) - v(t))\|_2 < \varepsilon \tag{20}$$

provided $N > N(N', |t|, \varepsilon, \|\phi\|_2)$.

The main application is to extend the nonsqueezing results discussed in [Kuk] to the flow of equation (1) (which has a noncompact nonlinearity). Using the notations from [Kuk], denote by (p, q) the canonical coordinates and

$$B(r) = \{(p,q) \;\;|\;\; |p|^2 + |q|^2 < r^2\} \qquad \text{(ball)}$$
$$\mathbf{T}(r) = \{(p,q) \;\;|\;\; p_1^2 + q_1^2 < r^2\} \qquad \text{(cylinder).}$$

Gromov's (finite-dimensional) squeezing theorem asserts that there is no symplectic embedding of $B(r)$ into $\mathbf{T}(R)$ unless $R \geq r$. In the context of equation (14), this fact has following consequence. Let B_r be some ball in $L^2(\mathbf{T})$ of radius r and $\mathbf{T}_r^{(k)}$ some cylinder in $L^2(\mathbf{T})$ defined with respect to the kth coordinate $(|k| \leq N)$[1]. If $S_N(t)$ is the flow map associated to (14), then

$$S_N(t)\;(B_r) \subset \mathbf{T}_R^{(k)} \tag{21}$$

implies $R \geq r$.

Denote by $S(t)$ the flow map corresponding to the cubic NLSE (1). Proposition 1 yields an estimate

$$\|P_k\; S(t)\; P_N - P_k\; S_N(t)\; P_N\| < \varepsilon \tag{22}$$

[1] not necessarily centered at the origin

provided $N > N(k, |t|, \varepsilon, B)$, where $\| \ \|$ refers to the sup L^2-norm on the ball B. From (22), one has

$$P_k\ S_N(t)\ (P_N\ B_r) \xrightarrow{N\to\infty} P_k\ S(t)\ (P_N\ B_r) \tag{23}$$

and hence, by (21), the following proposition.

Proposition 2. Denote by $S(t)$ the flow map of the NLSE (1). Then (with previous notations) $S(t)\ (B_r) \subset \mathbf{T}_R^{(k)}$ implies $R \geq r$.

Remark. If balls and cylinders are centered at the origin, the previous statement is obvious from the L^2-conservation. Otherwise the result seems nontrivial.

Going back to Proposition 1, we show the following lemmas.

Lemma 3. Consider the solutions u, v to the Cauchy problems

$$\begin{cases} i\ u_t + u_{xx} + u|u|^2 = 0 \\ \quad u(x,0) = \phi(x) \end{cases} \tag{24}$$

$$\begin{cases} i\ v_t + v_{xx} + v|v|^2 = 0 \\ \quad v(x,0) = \psi(x) \end{cases} \tag{25}$$

and assume $\|\phi\|_2 = \|\psi\|_2$. Then for $|t| < T(\|\phi\|_2)$ one has

$$\|P_{N_0}(u(t) - v(t))\|_2 \leq \|P_{N_1}(\phi - \psi)\|_2 + \varepsilon \tag{26}$$

provided $N_1 - N_0 > C_1\ \varepsilon^{-C_1}$ (C_1 numerical).

Lemma 4. Consider the solutions u, v to the Cauchy problems

$$\begin{cases} i\ u_t + u_{xx} + u|u|^2 = 0 \\ \quad u(x,0) = \phi(x) \end{cases} \tag{27}$$

$$\begin{cases} i\ v_t + v_{xx} + P_N\ (v|v|^2) = 0 \\ \quad v(x,0) = \phi(x) \end{cases} \tag{28}$$

where $\phi = P_N\, \phi$. Then for $|t| < T(\|\phi\|_2)$ one has

$$\|P_{N_0}(u(t) - v(t))\|_2 \le \varepsilon \tag{29}$$

provided $N - N_0 > C_1\, \varepsilon^{-C_1}$.

To deduce Proposition 1, one breaks up $[0, t]$ in time intervals $[t_i, t_{i+1}]$ of size $T(\|\phi\|_2)$. For fixed i, compare on $[t_i, t_{i+1}]$ the solutions to the initial value problems

$$\begin{cases} i\, u_t + u_{xx} + u|u|^2 = 0 \\ \quad u(x, t_i) = u(t_i)\ (x) \end{cases} \tag{30}$$

$$\begin{cases} i\, \tilde{u}_t + \tilde{u}_{xx} + \tilde{u}|\tilde{u}|^2 = 0 \\ \quad \tilde{u}(x, t_i) = v(t_i)\ (x) \end{cases} \tag{31}$$

$$\begin{cases} i\, v_t + v_{xx} + P_N\ (v|v|^2) = 0 \\ \qquad v(x, t_i) = v(t_i)\ (x). \end{cases} \tag{32}$$

Observe that $\|u(t_i)\|_2 = \|v(t_i)\|_2 = \|\phi\|_2$. Denoting by $\{N_i\}$ a decreasing sequence of positive integers $< N$, Lemma 3 implies that $\left\|P_{N_{i+1}}(u(t_{i+1}) - \tilde{u}(t_{i+1}))\right\|_2 \le \left\|P_{N_i}(u(t_i) - v(t_i))\right\|_2 + (N_i - N_{i+1})^{-c_2}$ for some $c_2 > 0$ and Lemma 4 yields $\left\|P_{N_{i+1}}(\tilde{u}(t_{i+1}) - v(t_{i+1}))\right\|_2 \le (N - N_{i+1})^{-c_2}$. Hence

$$\left\|P_{N_{i+1}}(u(t_{i+1}) - v(t_{i+1}))\right\|_2 \le \left\|P_{N_i}(u(t_i) - v(t_i))\right\|_2 + (N_i - N_{i+1})^{-c_2} \tag{33}$$

and (33) implies $\|P_{N'}(u(t) - v(t))\|_2 \le \sum (N_i - N_{i+1})^{-c_2}$. Since the number of steps is controlled by $\|\phi\|_2$, Proposition 1 follows.

Section 3

We now come to the main analysis needed in Lemmas 3 and 4. Consider the nonlinear expression $w = u\,|u|^2 = u\,\bar{u}\,u \equiv w(u, u, u)$ appearing in (3) and rewrite it using (5). We get

$$w(u, u, u) = \sum_{k = k_1 - k_2 + k_3} \int_{\lambda = \lambda_1 - \lambda_2 + \lambda_3} e^{i(kx + \lambda t)}\ \hat{u}(k_1, \lambda_1)\ \overline{\hat{u}(k_2, \lambda_2)}\ \hat{u}(k_3, \lambda_3), \tag{34}$$

and splitting the $\sum_{k = k_1 - k_2 + k_3}$ summation as

$$\sum_{\substack{k = k_1 - k_2 + k_3 \\ k_2 \neq k_1, k_3}} - \sum_{\substack{k = k_1 - k_2 + k_3 \\ k_1 = k_2 = k_3}} + \sum_{\substack{k = k_1 - k_2 + k_3 \\ k_1 = k_2}} + \sum_{\substack{k = k_1 - k_2 + k_3 \\ k_3 = k_2}}$$

(34) clearly yields, since $\int |u(t)|^2 = \int |\phi|^2$,

$$\sum_{\substack{k=k_1-k_2+k_3\\ k_2\neq k_1,k_3}} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i(kx+\lambda t)}\, \widehat{u}(k_1,\lambda_1)\, \overline{\widehat{u}(k_2,\lambda_2)}\, \widehat{u}(k_3,\lambda_3) \tag{35}$$

$$-\sum_{k} e^{ikx} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i\lambda t}\, \widehat{u}(k,\lambda_1)\, \overline{\widehat{u}(k,\lambda_2)}\, \widehat{u}(k,\lambda_3) \tag{36}$$

$$+\,2\left(\int |\phi|^2\right)\cdot \sum_{k} \int d\lambda\; e^{i(kx+\lambda t)}\, \widehat{u}(k,\lambda). \tag{37}$$

The corresponding contributions to the integral term in (3) are

$$\sum_{\substack{k=k_1-k_2+k_3\\ k_2\neq k_1,k_3}} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{ikx}\, \frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\, \widehat{u}(k_1,\lambda_1)\, \overline{\widehat{u}(k_2,\lambda_2)}\, \widehat{u}(k_3,\lambda_3) \tag{38}$$

$$-\sum_{k} e^{ikx} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} \frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\, \widehat{u}(k,\lambda_1)\, \overline{\widehat{u}(k,\lambda_2)}\, \widehat{u}(k,\lambda_3) \tag{39}$$

$$+\int |\phi|^2 \cdot \sum_{k} e^{ikx} \int d\lambda\; \frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\, \widehat{u}(k,\lambda). \tag{40}$$

Fix positive integers K, Δ and denote $P_L\, u$ by u_L. It is clear from (38)–(40) that

$$\begin{aligned} &\int_0^t U(t-\tau)\,(P_K\, w)\,(\tau)\, d\tau \\ &-\int_0^t U(t-\tau)\left[P_K\, w(u_{K+\Delta}, u_{K+\Delta}, u_{K+\Delta}) + 2\left(\int (|\phi|^2 - |u_{K+\Delta}|^2)\, dx\right)\, u_K\right](\tau)\, d\tau \end{aligned} \tag{41}$$

is obtained by considering the following subsum of (38):

$$\sum_{\substack{k=k_1-k_2+k_3;\, k_2\neq k_1,k_3\\ |k|\le K;\, \max|k_i|>K+\Delta}} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{ikx}\, \frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\, \widehat{u}(k_1,\lambda_1)\, \overline{\widehat{u}(k_2,\lambda_2)}\, \widehat{u}(k_3,\lambda_3). \tag{42}$$

Define

$$c(k,\lambda) = (1+|\lambda-k^2|^{1/2})\, |\widehat{u}(k,\lambda)| \tag{43}$$

so that

$$|||u||| = \|c\|_{\ell^2_k L^2_\lambda}. \tag{44}$$

One may then estimate $|||(42)|||$ considering the expressions

$$\sum_{\substack{k=k_1-k_2+k_3;k_2\neq k_1,k_3\\ |k|\leq K;\max|k_i|>K+\Delta}} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} \frac{c(k_1,\lambda_1)}{|\lambda_1-k_1^2|^{1/2}} \frac{c(k_2,\lambda_2)}{|\lambda_2-k_2^2|^{1/2}} \frac{c(k_3,\lambda_3)}{|\lambda_3-k_3^2|^{1/2}} \frac{a(k,\lambda)}{|\lambda-k^2|^{1/2}} \tag{45}$$

and

$$\sum_{\substack{k=k_1-k_2+k_3;k_2\neq k_1,k_3\\ |k|\leq K;\max|k_i|>K+\Delta}} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} \frac{c(k_1,\lambda_1)}{|\lambda_1-k_1^2|^{1/2}} \frac{c(k_2,\lambda_2)}{|\lambda_2-k_2^2|^{1/2}} \frac{c(k_3,\lambda_3)}{|\lambda_3-k_3^2|^{1/2}} \frac{b(k)}{|\lambda-k^2|} \tag{46}$$

where $\sum_k \int d\lambda\, |a(k,\lambda)|^2 \leq 1$, $\sum_k |b(k)|^2 \leq 1$, and $|\cdot|$ stands for $|\cdot|+1$ in the denominators. Here, we distinguish in (42) the cases $|\lambda-k^2| > 1$ and $|\lambda-k^2| < 1$. The technicalities involved here may be found in [Bo] (see also [Bo2]) and are inessential in this discussion. Also the estimates performed next on (45) will apply to (46), defining $a(k,\lambda) = (b(k))/(|\lambda - k^2|^{1/2+\varepsilon})$.

The main point is the observation

$$-k^2 + k_1^2 - k_2^2 + k_3^2 = -2k_2^2 + 2k_1k_2 + 2k_3k_2 - 2k_1k_3 = 2(k_1-k_2)(k_2-k_3)$$

and hence

$$\max\left(|\lambda_1 - k_1^2|\,,\ |\lambda_2-k_2^2|\,,\ |\lambda_3-k_3^2|\,,\ |\lambda-k^2|\right) \geq 2|k_2-k_1|\,|k_2-k_3|. \tag{47}$$

None of the factors in the right side of (47) vanishes, since we assume $k_2 \neq k_1, k_3$. Also for $i = 1,2,3$ one has

$$|k-k_i| \leq |k_1-k_2| + |k_3-k_2| \tag{48}$$

so that, by (47), (48) and the assumption $|k|\leq K$, $\max|k_i| > K+\Delta$, one finds that

$$\max\left(|\lambda_1-k_1^2|, |\lambda_2-k_2^2|, |\lambda_3-k_3^2|, |\lambda-k^2|\right) \geq \Delta. \tag{49}$$

Now, essentially speaking (cf. [Bo]), the estimate on (45) is obtained by considering $\int\int dx\ dt\ F_1^3 F_2$ where $\widehat{F}_1(k,\lambda) = (c(k,\lambda))/(1+|\lambda-k^2|^{3/8})$, $\widehat{F}_2(k,\lambda) = (a(k,\lambda))/(1+|\lambda-k^2|^{3/8})$, bounding the integral by $\|F_1\|_4^3 \; \|F_2\|_4 \leq c \cdot \|c\|^3_{\ell^2_k L^2_\lambda} = c|||u|||^3$, invoking inequality (13). Taking an exponent $3/8 + \varepsilon$ instead of $3/8$, there is an extra gain of a factor $|T|^{\varepsilon'}$, $\varepsilon' = \varepsilon'(\varepsilon)$, considering a small time interval $[0, T]$. The preceding discussion and the presence of the $(1/2)$-exponents for the different denominator factors in (45), (49) permit one to estimate (45), (46) by ($c_3 > 0 =$ some constant)

$$(45), (46) \leq T^{c_3} \; \Delta^{-c_3} \; |||u|||^3, \tag{50}$$

the main point being the saving of a Δ^{-c_3}-factor for this restricted summation in k_1, k_2, k_3. Thus also

$$|||(42)||| \leq T^{c_3} \; \Delta^{-c_3} \; |||u|||^3 \tag{51}$$

and since the second term in (41) is bounded in $|||\;\;|||$-norm by $T^{c_3} \; |||u_{K+\Delta}||| \; \left(|||u_{K+\Delta}|||^2 + \|\phi\|_2^2\right)$ it follows from the integral equation (3) that

$$u_K(t) = U(t) \; \phi_K + i \int_0^t U(t - \tau) \; (P_K \; w) \; (\tau) \; d\tau$$

$$|||u_K||| \leq \|\phi_K\|_2 + T^{c_3} \; \left(|||u|||^2 + \|\phi\|_2^2\right) \; |||u_{K+\Delta}||| + T^{c_3} \; \Delta^{-c_3} \; |||u|||^3. \tag{52}$$

Choosing T sufficiently small, depending on $|||u|||$, $\|\phi\|_2$, and hence $\|\phi\|_2$, we get

$$|||u_K||| \leq c_4 \; \|\phi_K\|_2 + \delta \; |||u_{K+\Delta}||| + \Delta^{-c_3} \tag{53}$$

where $\delta > 0$ is a sufficiently small constant. A straightforward iteration of (53) r times yields

$$|||u_K||| \leq \|\phi_{K+r\Delta}\|_2 + \delta^r \; \|\phi\|_2 + \Delta^{-c_3} \tag{54}$$

and hence, for an appropriate choice of r, Δ, assuming $N_1 > N_0$,

$$|||u_{N_0}||| \leq \|\phi_{N_1}\|_2 + (N_1 - N_0)^{-c_5} \tag{55}$$

and also

$$\|u_{N_0} \; (t)\|_2 \leq \|\phi_{N_1}\|_2 + (N_1 - N_0)^{-c_5}. \tag{56}$$

Section 4 Proofs of the lemmas

The proofs are a variant on previous estimates in Section III. Consider first Lemma 3. From the integral equation (3), one gets

$$(u - v)(t) = U(t)\ (\phi - \psi) + i \int_0^t U(t-\tau)\ \left[w(u,u,u) - w(v,v,v)\right]\ (\tau)\ d\tau. \tag{57}$$

The difference expressions (41) written for u and v are both bounded by $T^{c_3}\ \Delta^{-c_3}\ (|||u|||^3 + |||v|||^3)$. Consider the second term in (41) and subtract these expressions for u and v. Since

$$\int |\phi|^2\ dx = \int |\psi|^2\ dx \tag{58}$$

we obtain

$$\begin{aligned}&\int_0^t U(t-\tau) \Bigg[P_K\ w(u_{K+\Delta}, u_{K+\Delta}, u_{K+\Delta}) - P_K\ w(v_{K+\Delta}, v_{K+\Delta}, v_{K+\Delta})\\ &+ 2\left(\int |\phi|^2\right)(u_K - v_K) - \left(\int |u_{K+\Delta}|^2\ dx\right)\ u_K + \left(\int |v_{K+\Delta}|^2\ dx\right)\ v_K\Bigg]\ (\tau)\ d\tau.\end{aligned} \tag{59}$$

The $|||\ |||$-norm of this expression is bounded by

$$T^{c_3}\ \left(|||u_{K+\Delta}|||^2 + |||v_{K+\Delta}|||^2 + \|\phi\|_2^2\right)\ |||u_{K+\Delta} - v_{K+\Delta}||| \tag{60}$$

because the differences of the trilinear expressions yield a factor $u_K - v_K$ or $u_{K+\Delta} - v_{K+\Delta}$. Thus for sufficiently small T one gets instead of (53)

$$|||u_K - v_K||| \leq \|\phi_K - \psi_K\|_2 + \delta\ |||u_{K+\Delta} - v_{K+\Delta}||| + \Delta^{-c_3} \tag{61}$$

and also

$$\|u_K(t) - v_K(t)\|_2 \leq \|\phi_K - \psi_K\|_2 + \delta\ |||u_{K+\Delta} - v_{K+\Delta}||| + \Delta^{-c_3}. \tag{62}$$

One again iterates (61) and gets for $N_1 > N_0$ the estimates

$$|||u_{N_0} - v_{N_0}||| \leq \|\phi_{N_1} - \psi_{N_1}\|_2 + (N_1 - N_0)^{-c_5} \tag{63}$$

and

$$\|u_{N_0}(t) - v_{N_0}(t)\|_2 \leq \|\phi_{N_1} - \psi_{N_1}\|_2 + (N_1 - N_0)^{-c_5}. \tag{64}$$

This proves Lemma 3.

Next consider Lemma 4. The integral equation (3) gives now

$$(u - v)(t) = i \int_0^t U(t-\tau) \left[w(u,u,u) - P_N \, w(v,v,v) \right] (\tau) \, d\tau. \tag{65}$$

Hence

$$(u_K - v_K)(t) = i \int_0^t U(t-\tau) \left[P_K \, w(u,u,u) - P_K \, w(v,v,v) \right] (\tau) \, d\tau \tag{66}$$

provided K is kept less than N. As above in the proof of Lemma 3, one obtains

$$|||u_K - v_K||| \le \delta \; |||u_{K+\Delta} - v_{K+\Delta}||| + \Delta^{-c_3} \tag{67}$$

$$\|u_K(t) - v_K(t)\|_2 \le \delta \; |||u_{K+\Delta} - v_{K+\Delta}||| + \Delta^{-c_3}. \tag{68}$$

Iterating (67) and keeping $K + r\Delta < N$ yields

$$|||u_K - v_K||| \le \delta^r \left(|||u||| + |||v||| \right) + \Delta^{-c_3}. \tag{69}$$

Hence for $N_0 < N_1$

$$|||u_{N_0} - v_{N_0}||| \le (N - N_0)^{-c_5}; \qquad \|u_{N_0}(t) - v_{N_0}(t)\|_2 \le (N - N_0)^{-c_5}.$$

This proves Lemma 4.

Section 5 Further comments

Due to some specific structures used in the argument (see (47)–(49)) of an arithmetical nature, it is not clear how to replace $u|u|^2$ in (1) by other nonlinearities (with comparable growth properties). The previous method applies however to NLSE of the form

$$i \, u_t + u_{xx} + A(x,t) \, u + B(x,t) \, u|u|^2 = 0 \tag{70}$$

where A, B are real sufficiently smooth functions of x, t, both periodic in x. (We do not intend to work out refinements here under weaker assumptions.) It is indeed clear that for smooth A, B there are no significant problems to carry out the harmonic analysis above when replacing the term $u|u|^2$ by $A \; u + B \; u|u|^2$. Moreover, the L^2-norm is a conserved quantity.

Considering a more general Hamiltonian NLS with nonlinearity $(\partial/\partial\bar{u})G(u, \bar{u}; t, x)$, G of degree ≤ 4 in u, $\bar{u}$, it may be proved that bounded solutions are not uniformly asymptotically stable as $t \to \infty$ (see [Bo3]).

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