

9/29/2011

Lecture #7

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$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u = \lambda |u|^{p-2} u & x \in \mathbb{R}, \mathbb{T} \\ u|_{t=0} = u_0 \in H^s \quad s \geq 1 \end{cases}$$

Then one can show that it has as many conservation laws and combining them one gets that

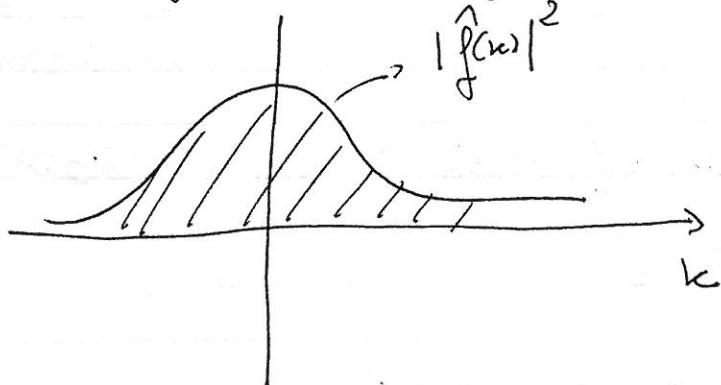
$$\|u(t)\|_{H^s} \leq C \quad \forall t \in \mathbb{R}$$

$\downarrow$   
depends on norm of initial state.

Is this true in general?

Why do we care about these bounds?

Consider a function  $f(x)$  (smooth)



The subgraph = ~~area under~~  $\|f\|_{L^2}^2$

If  $u(x, t)$  is solution of

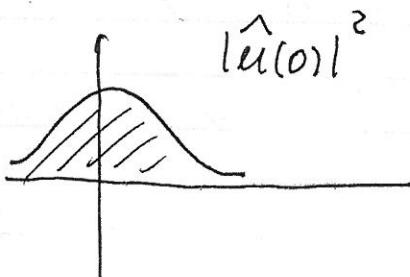
$$(NLS)_P \quad \begin{cases} i\partial_t u + \Delta u = \lambda |u|^{p-2} u & x \in \mathbb{R}^n, \mathbb{T}^n \\ u|_{t=0} = u_0 \end{cases}$$

Then Are subgraph of  $|\hat{u}(t, \kappa)|^2 = \text{mass conserved}$

recall do the energy is conserved.

Question:

If at  $t=0$ :



1) What can we say about the support of  $|\hat{u}(t)|^2$  of time  $t$ ? The one of the subgraph is constant but not the profile.

2) Is there a "migration" to high frequencies?

(~~if~~ if ~~the~~ NLS is dispersing we also have the subgraph  $|\kappa|^2 |\hat{u}(t)|^2$  is bounded, so if  $\kappa$  is growth it cannot be too large). This migration is called "forward cascade".

4) Note that this is not a blow up result!

In the cases we ~~can~~ ask this question. We

know that solutions  $f!$  and smooth!

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The con  $\mathbb{R}^n$  compared to  $T^*$

For the (NLS) above, if we know  $J!$  of global smooth solutions we also know scattering.

This means that we can prove: given a solution  $u(x, t)$  to NLS s.t.  $u \in C(\mathbb{R}, H^s)$  s.t.

$\exists u_{\pm} \in H^s$  s.t.

$$\lim_{t \rightarrow \pm\infty} \|u(t, \cdot) - S(t)u_{\pm}\|_{H^s} = 0$$

when  $S(t)u_{\pm}$  are linear solutions!

This is typical of  $\mathbb{R}^n$ , that is when boundaries conditions are not involved. This is an effect due to strong dispersion. In the periodic case (or NLS in other "fanning" manifolds)

the situation is more complex and not well understood yet.

Q: How is scattering connected to forward cascade?  
To understand if at time  $t$  the

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Support of  $(\hat{u}(t))^\sharp(k)$  has moved to large  $|k|$ ,

one can look at  $t \rightarrow +\infty$  of

$$E^S(u(t)) = \int |\hat{u}(t)|^2(k) \underset{k \rightarrow \infty}{\sim} dk \quad \text{for } S \gg 1$$

$\hookrightarrow$  this fails the loge frequencies

But  $E^S(u(t)) = \|u(t)\|_{H^S}^2$  and if scattering is happening then if  $t \rightarrow +\infty$  ( $-\infty$ )

$$\|u(t)\|_{H^S}^2 \leq c \|S(t)u_+ - u(t)\|_{H^S} + \|S(t)u_+\|_{H^S}$$

$\downarrow 0$        $\uparrow$   
 $t \rightarrow +\infty$        $\|u_+\|_{H^S} \leq c$

As an example of this kind of questions we will prove the following 2 theorems:

~~Wellposedness~~ Consider the cubic defocusing NLS in  $\mathbb{T}^2$

$$(NLS)_3 \quad \left\{ \begin{array}{l} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \in H^S \quad S \gg 1 \end{array} \right.$$

Theorem 1: Let  $u$  be the global solution to  $(NLS)_3$ .

Then  $\exists C_S$ , continuous w.r.t.  $u_0 \in H^1$ , s.t.

$$\|u(t)\|_{H^S(\mathbb{T}^2)} \leq C(\|u_0\|_{H^1}) (1 + |t|)^{\frac{S}{S+1}} \|u_0\|_{H^S},$$

for  $\varepsilon > 0$

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 \* Theorem 2:  $\Theta K \gg 1$ ,  ~~$\delta \ll 1$~~ ,  ~~$S > 1$~~ ,  $\exists u_0 \in H^s(\mathbb{T}^2)$

such that  $T = T(K, \delta)$  s.t.

$$\|u_0\|_{H^s} \lesssim \delta \quad \|u(T)\|_{H^s} \geq k.$$

Remark: this is not a good lower bound, but it is the best "growth" result available so far.

Set up for proof of Theorem 1

Remark:

1) In  $\mathbb{R}^2$  one obtains an uniform bound

$$\|u(t)\|_{H^s} \leq C(s, \|u_0\|_{H^s})$$

thanks to the  $L^2$  scattering result of Dodson in 2010.

2) The original proof is due to

[Bougain, Zhang, Schinger]

3) From local p. one obtain the local bound

$$(2) \quad \|u(t+\varepsilon)\|_{H^s} \leq C \|u(t)\|_{H^s} \quad C > 1$$

where we can take  $\varepsilon = \varepsilon(\|u_0\|_{H^s})$  so uniform

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from iterating this we obtain

$$(D) \quad \|u(t)\|_{H^s} \leq C_1 e^{C_2 |t|} \quad t \in \mathbb{R}$$

**Question:** Can one do better than (D)?

Bouygein did better and was able to replace

(D) with

$$(D') \quad \|u(t+c)\|_{H^s} \leq \|u(t)\|_{H^s} + C \|u(t)\|_{H^s}^{1-r}$$

for some  $r \in (0,1)$ . From (D') one gets

$$(D) \quad \|u(t)\|_{H^s} \leq C_1 |t|^{\frac{1}{r}}$$

**Question:** What is the guess? One believes that then one solutions that could grow at least as  $\log(|t|)$  ~~but~~ in the periodic setting. But this is not proved. On the other hand there is the following theorem

Theorem (Bouygein - Delort)

Consider the linear (anti potential) equation

$$\begin{cases} i u_t + \Delta u = V u & \text{in } \mathbb{T}^d \\ u|_{t=0} = u_0 \end{cases}$$

+ )  
some  $V = V(x, t)$  ~~is~~ is smooth and bounded with all its derivatives, then

$$\|U(t)\|_{H^s} \leq (1+|t|)^{\epsilon}$$

Idea of the proof [the idea is link to a method

that I introduced with Collander - he - Tukehe  
and has to extend local well-posedness  
to global when one does not have a  
conservation law that can be used to  
iterate R.e.p. to g.e.p. this method

which is now known as the I-method.

In our case in stead of looking at how

$\|U(t)\|_{H^s}^2$  changes in time we will  
be looking at

$$\|\mathcal{D}_n U(t)\|_{L^2}^2 \approx \|U(t)\|_{H^s}^2$$

and later at

$$E^2(U(t)) \approx \|\mathcal{D}_n U(t)\|_{H^s}^2$$

but such that  $E^2$  will grow even slower than

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$$\|D_N u(t)\|_{L^2}^2.$$

To be precise the  $D_N$  operator is defined as a multiplier operator:

$$\widehat{D_N f}(n) := \theta(n) \widehat{f}(n) \quad \theta(n) = \begin{cases} \left(\frac{|n|}{N}\right)^s & \text{if } |n| > N \\ 1 & \text{if } |n| \leq N \end{cases}$$

Clearly

$$\|D_N f\|_{L^2} \lesssim \|f\|_{H^s} \lesssim N^s \|D_N f\|_{L^2}$$

We will be using the FTA:

$$\|D_N u(t+c)\|_{L^2}^2 = \|D_N u(c)\|_{L^2}^2 + \underbrace{\int_t^{t+c} \frac{d}{ds} \|D_N u(s)\|_{L^2}^2}_{J(t,c)}$$

and we will estimate  $J(t, c)$

using Fourier transform. To understand the kind of cancellation that goes on here let's prove the well known fact

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 = 0$$

(conservation of mass!)

using Fourier Transform:

$$\begin{aligned}
 \frac{d}{dt} \|u(t)\|_{L^2}^2 &= \frac{d}{dt} \int_{\mathbb{T}^2} u(x) \bar{u}(x) dx = \\
 \cancel{\frac{d}{dt} \sum_n \hat{u}_n(n) \bar{\hat{u}}_n(n)} &= \frac{d}{dt} \sum_{h_1+h_2=0} \hat{u}(h_1) \bar{\hat{u}}(h_2) \\
 &= 2 \sum_{h_1+h_2=0} \hat{u}_t(h_1) \bar{\hat{u}}(h_2) = \\
 \hat{u}_t^{(n)} &= -i |h_1|^2 \hat{u}(n) - i \sum_{h_1, h_2, h_3} \hat{u}(h-h_1) \hat{u}(h_1, -h_2) \bar{\hat{u}}(-h_2+h_3) \\
 &= 2 \operatorname{Re} \left[ \sum_{h_1+h_2=0} -i |h_1|^2 \hat{u}(h_1) \bar{\hat{u}}(h_2) \right] \\
 &\quad + 2 \operatorname{Re} i \sum_{h_1+h_2+h_3+h_4=0} \hat{u}(h_1) \hat{u}(h_2) \bar{\hat{u}}(h_3) \bar{\hat{u}}(h_4) / \\
 &\quad \quad \quad \text{by using symmetry}
 \end{aligned}$$

In a similar manner

$$\frac{d}{dt} \|Du(t)\|_{L^2}^2 = i c \sum_{h_1+h_2+h_3+h_4} \left[ (\partial h_1)^2 - (\partial h_2)^2 + (\partial h_3)^2 - (\partial h_4)^2 \right] \\
 \hat{u}(h_1) \partial(h_2) \hat{u}(h_3) \bar{\hat{u}}(h_4)$$

We will show that

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Lemma

$$\int_{t_0}^{t+\varepsilon} \left\| \frac{d}{ds} D u(s) \right\|_{L^2}^2 ds \leq \frac{C(\varepsilon)}{N^\beta} \| D u(t) \|_{L^2}^2$$

for some  $\beta > 0$ .

As a consequence

$$\| D u(t) \|_{L^2}^2 \leq \left( 1 + \frac{C}{N^\beta} \right) \| D u(t_0) \|_{L^2}^2$$

This says that in  $[0, T]$  when  $T \sim N^\beta$ we basically "double" the quantity  $\| D u_0 \|_{L^2}^2$ .

As a consequence

$$\| u(T) \|_{H^s} \leq N^s \| u_0 \|_{H^s} \leq N^s \| D u_0 \|_{L^2} \approx N^s \| u_0 \|_{H^s}$$

and so

$$\| u(T) \|_{H^s} \leq T^{\frac{s}{\beta}} \| u_0 \|_{H^s}$$

If we prove that  $\beta = 1+$  then we proved the theorem. But unfortunately we will obtain  $\beta < 1$  and to get  $\beta = 1$  we need to work harder.

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To prove the main lemma we need to recall<sup>1</sup>/prove few facts

**Fact 1:** Suppose  $B(N)$  is a ball in  $\mathbb{Z}^2$  of radius  $N$  and  $u$  is such that  $\text{Supp } \hat{u} \subseteq B(N)$ . Then if  $b_+, s_+ \in \mathbb{R}$  and  $\frac{1}{4} < b_+ < \frac{1}{2}^+$ ,  $s_+ > 1 - 2b_+$ , ~~then~~ one has

$$\|u\|_{L_I^{s_+} L_T^{s_+}} \leq N^{s_+} \|u\|_{X_I^{0, b_+}}$$

**Fact 2:**  $\exists C = C(\text{mon, energy})$ ,  $\exists C = C(s, \text{mon, energy}) > 0$  s.t.  $\forall t_0 \in \mathbb{R}$   $\exists$  a solution to cubic, defocusing, periodic NLS s.t.

$$\|u\|_{X_{[t_0, t_0+C]}^{1, \frac{1}{2}^+}} \leq C$$

$$\|Du\|_{X_{[t_0, t_0+C]}^{0, \frac{1}{2}^+}} \leq C \|Du(t_0)\|_{L^2}$$

Proof: We proved Fact 1 already. The proof of Fact 2 is by fixed point theorem.  
Note that one can take  $C$  uniform since

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$S \geq 1$  and one can express the boundaries of the  
 $H'$  norm in terms of the mass and energy.

(mass =  $(L^2 \text{ norm})^{1/2}$ , energy = Hamiltonian here!)