

better #6

Theorem: Define the flow map

$$W(t) : L^2(\mathbb{C}) \rightarrow L^2(\mathbb{C})$$

$u_0 \mapsto$  solution of IVP  
at time  $t$ .

Then if  $W(t)(B_{(P_0, Q_0)}^{(r)}) \subseteq T_k^k(\mathbb{R})$  it has

to be that  $P_0, Q_0 \in \mathbb{R}$

{ From const. of  $L^2$  norm  
This is trivial if  $(P_0, Q_0) = (0, 0)$   
 $(Q, g) = (0, 0)$

Remark: 1) This kind of abstract theorem says something about the solution of the PDE.

For example if at time  $t$  we have that

the coefficient at place  $m$  are such that

$$|\alpha_m^0 + \alpha_m(t)|^2 + |\beta_m^0 + \beta_m(t)|^2 < g^2$$

then it has to be that at time  $t=0$

the wave has all frequencies smaller than  $g$ :

$$\sum_m |\alpha_m^0 + \alpha_m(t)|^2 + |\beta_m^0 + \beta_m(t)|^2 < g^2!$$

2) If  $(\alpha^0, \beta^0) = (0, 0)$  then the theorem is a consequence of  $L^2$  conservation law.

Proof: (See Kuksin for compact perturbations

of linear ~~dispersive~~ dispersive eqs).



All proofs of this type start by projecting the  $\infty$ -dimensional Hamiltonian system to finite subspaces:

$$(IVP) \left\{ \begin{array}{l} iu_t + u_{xx} + \mu |u|^2 = 0 \\ u|_{t=0} = u_0 \end{array} \right. \xrightarrow{\text{Proj}_N} \left\{ \begin{array}{l} iv_t + v_{xx} + P_N(v|v|^2) = 0 \\ v|_{t=0} = P_N u_0 \end{array} \right.$$

↑  
This is also an Hamiltonian system but with finite dimension

$$\hat{v}(n,t) = \begin{cases} 0 & \text{if } n > N \\ \hat{v}(n,t) & \text{otherwise.} \end{cases}$$

Proposition 1 Fix  $N'$  and  $t$ . Then  $\forall \epsilon > 0$

$$\| P_{N'}(w(t)) - v(t) \|_2 < \epsilon \quad \text{provided } N > N(N', |t|, \epsilon, \|u_0\|_2) \quad w = V(t) P_N u_0$$

From here we have the following consequence:

let  $V_N(t)$  be the flow of  $(IVP)_N$

Corollary 1: Fix  $k$  and  $t$ . Then  $\forall \epsilon > 0$

$$\exists N_0 = N_0(k, |t|, \epsilon, \|\cdot\|_2) \text{ s.t. } \forall N \geq N_0$$

$$\| P_k V_N(t) P_N - P_k V_{N_0}(t) P_{N_0} \|_2 < \epsilon$$

$\hookrightarrow$  projection on  $k$  coordinate

Here  $\| \cdot \|_{B_r}$  is the operator norm on the ball  $B_r$ .

[Proof] ; let  $u_0 \in B_r$ , then  $P_N u_0$  is a date that works for both (IVP) and  $(IVP)_N$ . Now

$$w(t) = U(t) P_N u_0 \text{ solve (IVP)}$$

$$v(t) = P_N U(t) P_N u_0 = (IVP)_N$$

Use the ex Proposition 1 and we get the result.

Proof of theorem

~~continuity~~

~~continuity~~

~~continuity~~

~~continuity~~

Fix an  $\epsilon > 0$  and fix  $t$  of theorem

i) By continuity w.r.t. initial state  $\exists N = N(t, \epsilon)$

$$\text{s.t. } \| U(t) u_0 - W(t) P_N u_0 \|_{L^2} < \epsilon$$

and in particular  $\| P_n U(t) u_0 - P_n W(t) P_N u_0 \|_{L^2} < \epsilon$

Now we look at

$$\| P_n U(t) P_N u_0 - (g_0, \varphi_0) \|_{L^2} \leq$$

(4)

$$\begin{aligned}
 &\leq \| P_{N_1} W_N(t) P_N u_0 - P_{N_1} W(t) P_N u_0 \|_{L^2} \\
 &+ \| P_{N_1} W(t) P_N u_0 - P_{N_1} W(t) w_0 \|_{L^2} \\
 &+ \| P_{N_1} W(t) w_0 - (f_0, g_0) \|_{L^2} \\
 &\leq \varepsilon + \varepsilon + R \\
 &\hookrightarrow \text{Hence Corollary and possibly taking } N \text{ larger}
 \end{aligned}$$

L, since by assumption  $W(t) B_\delta \subseteq T^k(R)$

By the finite dimension result of Gronwall  
it has to be  $r \leq \varepsilon + R$  and since  $\varepsilon$  is  
arbitrary  $r \leq R$ .

### Proof of Proposition 1

Lemma 1 Consider (WP) and assume there are two  
initial state  $u_0$  and  $w_0$  s.t.

$\|u_0\|_{L^2} = \|w_0\|_{L^2}$ . Then for  $|t| < T(\|u_0\|_{L^2})$   
(small state !!)  
one has

$$\begin{aligned}
 &\| P_{N_1} (W(t) u_0 - W(t) w_0) \|_{L^2} \\
 &\leq \| P_{N_1} (u_0 - w_0) \|_{L^2} + \varepsilon \\
 &\text{provided } N_1 - N_0 > C_1 \varepsilon^{-C_1}
 \end{aligned}$$

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[Local Version of Proposition 4]

Lemma 3: Consider  $W(t)P_N u_0$  and  $U_N(t)P_N u_0$ . Fix  $N_0$ .

Then for  $|t| < T(\|u_0\|_2)$  one has

$$\|P_{N_0}(W(t)P_N u_0 - U_N(t)P_N u_0)\|_2 \leq \varepsilon$$

provided  $N - N_0 > c_1 \varepsilon^{-c_2}$

Assume the lemma to prove Proposition 1.

Fix  $[0, t]$

Assume  $\{t_i\}$  s.t.  $t_{i+1} - t_i = T(\|u_0\|)$

Now we compare the problems

$$(IVP) \quad \begin{cases} iW_t + W_{xx} + |W|^2 W = 0 \\ W(x, t_i) = W(t_i)(x) \end{cases} \quad \begin{matrix} W(t, x) = W(t) P_N u_0 \\ \text{on } [t_i, t_{i+1}] \end{matrix}$$

$$(\tilde{IVP}) \quad \begin{cases} i\tilde{W}_t + \tilde{W}_{xx} + |\tilde{W}|^2 \tilde{W} = 0 \\ \tilde{W}(x, t_i) = W(t_i)(x) \end{cases} \quad \begin{matrix} \text{here } \tilde{W}(t) \text{ is} \\ \text{solution to } (IVP)_N \\ \tilde{W}(t) = U_N(t) P_N u_0 \end{matrix}$$

$$(IVP)_N \quad \begin{cases} iV_t + V_{xx} + P_N(|V|^2 V) = 0 \\ V(x, t_i) = W(t_i)(x) \end{cases}$$

By construction ~~one~~ of mass

$$\|W(t_i)\|_2 = \|V(t_i)\|_2 = \|P_N u_0\|_2$$

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Denote by  $N_i \downarrow$  decreasing sequence of integers

$N_i < N$ . By lemma 1

$$\| P_{N_{i+1}}(w(t_{i+1}) - \tilde{w}(t_{i+1})) \|_2 \leq \| P_{N_i}[w(t_i) \\ - v(t_i)] \|_2 + (N_i - N_{i+1})^{-c_2}$$

By lemma 2

~~$$\| P_{N_{i+1}}(\tilde{w}(t_i) - v(t_i)) \|_2 \leq (N_i - N_{i+1})^{-c_2}$$~~

$$\| P_{N_{i+1}}(w(t_{i+1}) - v(t_{i+1})) \|_2 \leq \\ \| P_{N_i}(w(t_i) - v(t_i)) \|_2 + (N_i - N_{i+1})^{-c_2}$$

So

$$\| P_N(w(t) - v(t)) \|_2 \leq \sum_{\text{finite steps}} (N_i - N_{i+1})^{-c_2}$$

Since the number of steps is controlled by  $C(\|w\|_2)$  the conclusion is proved.

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Weak turbulence and growth of Sobolev norms

If we consider the (IVP)