

④ 9/15/2011

to make things easier we also look at
 the case $q=r=p$ (it will be the only admissible
 couple (q,q)). We finally assume that
 \tilde{u}_0 lives on $|k| \leq N$, so we will be looking
 at

$$S_N = \{(k, |k|^{\epsilon}) / k \in \mathbb{Z}^n \text{ and } |k| \leq N\}$$

What Bourgain set out to study was

$$\left\| \sum_{\vec{f} \in S_N} a_{\vec{f}} \vec{e}^{2\pi i \vec{f} \cdot (x,t)} \right\|_{L^p(\mathbb{T}^{n+1})}$$

$$\leq K_p(S_N) \|u_0\|_2$$

$K_p(S_N)$ = Smallest constant depending on
 N that will make (*) true.

Conjecture (Bourgain)

$$K_p(S_N) \leq C_p \quad p < \frac{2(n+\epsilon)}{n}$$

$$K_p(S_N) \ll N^\epsilon \quad p = \frac{2(n+\epsilon)}{n} \quad (*)$$

$$K_p(S_N) \geq C_p N^{\frac{n-\epsilon}{2} - \frac{n+\epsilon}{p}} \quad p > \frac{2(n+\epsilon)}{n}$$

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Then we will prove (*) for $n=1, 2$. Some other pieces of the conjecture is solved, but not all.

General Set up

~~Hard case of $n=2$ etc~~

Since $p \geq 2$ we write $p = 2r$, $r \geq 1$.

then if $\text{supp } \hat{u}_0 \subseteq B_N(0)$ \hookrightarrow integer

$$\|S(t)u_0\|_{L^p_{T^{n+1}}}^p = \left\| \sum_{|k| \leq N} a_k e^{i(|k|^2 t + \langle x, k \rangle)} \right\|_{L^p_{T^{n+1}}}^p$$

Note: Fourier sum $\widehat{\Delta f}(k) = |k|^2 \widehat{f}(k)$

$$= \left\| \left[\sum_{|k| \leq N} a_k e^{i(|k|^2 t + \langle k, x \rangle)} \right]^r \right\|_{L^2_{T^{n+1}}}^r$$

This allows us to use the F.T.

$$\cong \left\| \mathcal{F}_{t,x} \left([]^r \right) \right\|_{L^2_{\mathbb{Z}^n}}^r$$

We have

$$\left([]^r \right) = \sum_{k,m} b_{k,m} e^{i(\langle x, k \rangle + m t)}$$

$$b_{k,m} = \sum_{k=k_1+k_2+\dots+k_r}$$

$$m = |k_1|^2 + \dots + |k_r|^2$$

We will consider the case $n=1, p=6, r=3$

$$b_{k,m} = \sum_{\substack{k=k_1+k_2+k_3 \\ m=|k_1|^2+|k_2|^2+|k_3|^2}} |b_{k,m}|^c$$

and if we continue above we have

$$\|S(t)u_0\|_{L^p}^p = \left[\left(\sum_{k,m} |b_{k,m}|^c \right)^{\frac{1}{c}} \right]^{\frac{p}{2}}$$

$$\begin{cases} k = k_1 + k_2 + k_3 & k_3 = k - k_1 - k_2 \\ m = |k_1|^2 + |k_2|^2 + |k_3|^2 & m = \left(\sqrt{2}k_1 - \frac{k}{\sqrt{2}} \right)^2 + \left(\sqrt{2}k_2 - \frac{k}{\sqrt{2}} \right)^2 \\ & + 2k^2 \end{cases}$$

and from here

$$2m - 4k^2 = (2k_1 - k)^2 + (2k_2 - k)^2$$

~~.....~~

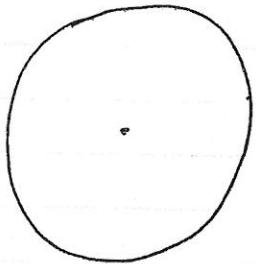
$$\begin{aligned} \sum_{m,k} \left(\sum_{\substack{k=k_1+k_2+k_3 \\ m=|k_1|^2+|k_2|^2+|k_3|^2}} a_{k_1} a_{k_2} a_{k_3} \right)^2 &\leq \\ \leq \sup_{k_1, m} \# \left\{ (k_1, k_2) \in \mathbb{Z}^2 / [2m - 4k^2] = (2k_1 - k)^2 + (2k_2 - k)^2 \right\} & \\ \prod_{i=1}^3 \left(\sum_{k_i} a_{k_i}^2 \right) & \end{aligned}$$

$$\text{Set } 2k_i - k = j_i \quad 2m - 4k^2 = a$$

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$a \leq 0$ empty set

$a > 0 \left\{ \delta_1^2 + \delta_2^2 = a \right\}$ what is the cardinality



number of lattice points on
circle of radiuses $\sqrt{a} = R$

$$\approx \exp \frac{\log R}{\log \log R} \ll R^\varepsilon$$

How large can R be in our case?

$$R^2 = |2m - 4k^2| \lesssim \max |k_i|^2 \leq N^2$$

So we proceed says $\vec{u}_0 \in B_N$

$$\left\| S(f) \vec{u}_0 \right\|_{L^2_{\prod^{n+1}}}^6 \leq N^\varepsilon \left\| \vec{u}_0 \right\|_{L^2}^6$$

If \vec{u}_0 is not says in B_N we write

$$\vec{u}_0 = \sum_N P_N \vec{u}_0 \quad \text{dyadic decomposition}$$

$$\left\| S(f) \vec{u}_0 \right\|_{L^2_{\prod^{n+1}}}^6 \leq \sum_N N^\varepsilon \left\| P_N \vec{u}_0 \right\|_{L^2}^{N \sim 2^n}$$

$$\lesssim \sum_N N^\varepsilon \| P_N D^{\varepsilon} \vec{u}_0 \|_{L^2}^{\varepsilon}$$

$$\leq \left(\sum_N \| P_N D^{\varepsilon} \vec{u}_0 \|_{L^2}^2 \right)^{\frac{1}{2}} \simeq \| \vec{u}_0 \|_{H^{\varepsilon}}$$

loss of
derivative

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~~Case h=2 p=4~~

Exercise

Show that for $n=1$ $p=4$

$$\|S(t)u_0\|_{L^4_{[-1,1]}} \leq C \|u_0\|_{L^2}$$

Case $h=2$ $p=4$

If we run the same argument in this case we obtain

$$b_{k,m} = \sum_{\substack{k = k_1 + k_2 \\ m = |k_1|^2 + |k_2|^2}} a_{k_1} a_{k_2}$$

$$|k_i| \in \mathbb{N}$$

$k_i \in \mathbb{Z}^2$. We need to look at

$$\begin{cases} k_2 = k - k_1 \\ m = |k_1|^2 + |k - k_1|^2 \end{cases} \Leftrightarrow \begin{cases} |k_1|, |k| \in \mathbb{N} \\ 2m - |k|^2 = |k - 2k_1|^2 \end{cases}$$

$$\sup_{m,k} \# \left\{ k_1 \in \mathbb{Z}^2 \mid 2m - |k|^2 = |k - 2k_1|^2 \right\}$$

$$\leq \sup_{m,n} \# \left\{ z \in \mathbb{Z}^2 \mid 2m - |k|^2 = |z|^2 \right\}$$

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Note that for fixed m and k

$$R^2 = 2m - \|k\|^2 \leq N^2$$

$$\text{So again } \leq \exp\left(\frac{\log N}{\log \log N}\right) \leq N^\varepsilon$$

for any $\varepsilon > 0$

Remark: In some cases it may be useful to an improvement of this 2D L^2 estimate.

In fact we have the following:

Proposition: [Improved Strichartz Estimate]

Assume ϕ_i supp on $B_{N_i} \subseteq \mathbb{Z}^2$. Then

if $N_1 \ll N_2$ it follows that

$$\|S(t)\phi_1 \cdot S(t)\phi_2\|_{L^2} \leq N_1^\varepsilon \|\phi_1\|_1 \|\phi_2\|_1$$

Remark: Using dyadic decomposition this means that if we have a product of two linear solutions and one of them is only defined on smaller(band) sup norm then there is no loss of derivative!

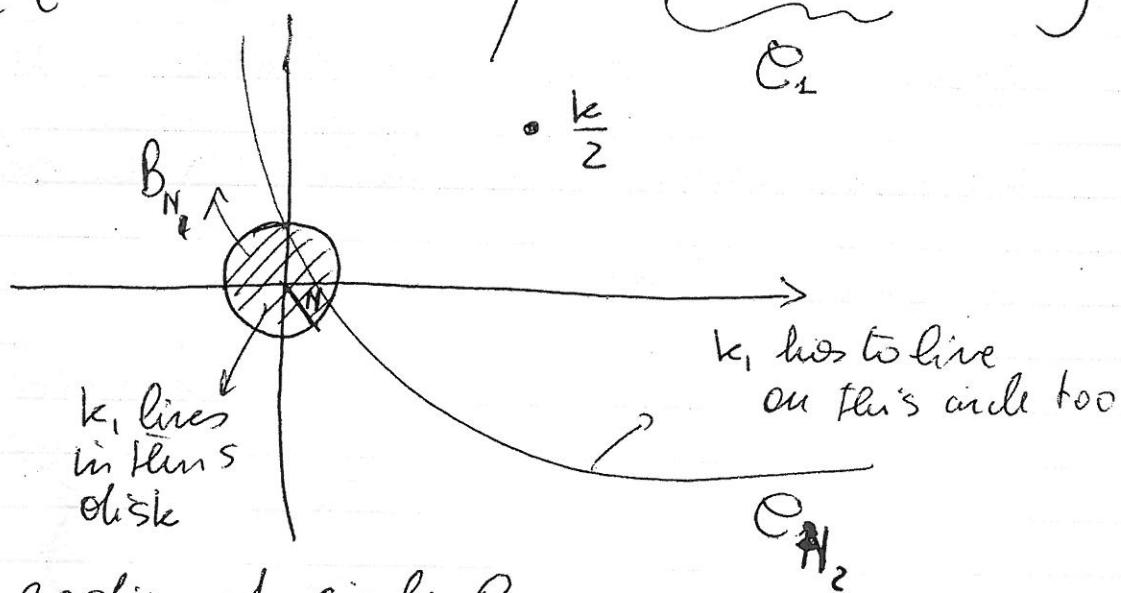
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Proof: We repeat the argument and we see that

$$K = k_1 + k_2 \Rightarrow |K| \approx |k_1| \approx N_1 \text{ (large)}$$

So we have to count

$$\#\left\{k_1 \in \mathbb{Z}^2, |k_1| \approx N_1, \quad / \quad 2m - |k|^2 = |k - 2k_1|^2\right\}$$



$R = \text{radius of circle } C_{N_1}$

$$R = 2m - |k|^2 \leq N_2^2$$

$$\#\left\{k_1 \in \mathbb{Z}^2 \cap B_{N_1} \cap C_{N_2}\right\} = ?$$

Lemma: let C be a circle of radius R . If

γ is an arc on C of length $|\gamma| < R^{\frac{1}{3}}$

then γ contains at most 2 lattice points.

Assume the lemma for a moment. Then we have

that if $|y| = P_{N_2} \cap B_N \Rightarrow |y| \approx N$,

Cor 1 $|y| \cdot N_4 < R^{\frac{1}{3}} \leq N_2^{\frac{1}{3}}$. Then in this case

\exists 2 lattice points.

Cor 2 $N_1 \geq R^{\frac{1}{3}} \approx N_2^{\frac{1}{3}}$ Then in this case

$\#$ lattice points on the whole circle

$$\# \leq N_2^{\varepsilon} \leq N_1^{3\varepsilon} \quad \text{O.k.}$$

Proof of lemma

We start with another well-known lemma

lemma [Pick's lemma]

let A_r be the area of a simply connected
lattice polygon. Let ~~number of lattice points~~

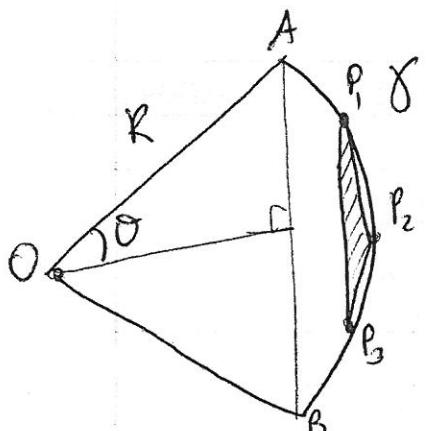
$E = \#$ lattice points on perimeter

$I = \# \approx \#$ in interior.

$$\text{Then } A_r = I + \frac{1}{2}E - 1$$

Using Pick's lemma we can now proceed
by contour addition.

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Suppose the arc \hat{AB} has 3 collinear points.

$$\frac{\pi}{2} + \frac{3}{2} - 1 = A_R$$

$$\frac{\pi}{2} + \frac{1}{2} = A_R > \frac{1}{2}$$

On the other hand

$$\text{Area } \triangle OAB = R^2 \theta$$

$$\text{Area } \triangle OAB = R^2 \sin \theta \cos \theta$$

$$\text{here } A_R \leq R^2 \theta - R^2 \sin \theta \cos \theta = \\ = R^2 (\theta - \frac{1}{2} \sin(2\theta))$$

$$\text{Since } \theta - \frac{1}{2} \sin(2\theta) \leq \frac{2}{3} \theta^3$$

$$\text{and } |AB| = 2R\theta$$

~~$$\text{Area } A_R \leq \frac{2}{3} R^2 \theta^3 = \frac{|AB|^3}{3 \cdot 4 R} \leq \frac{R^3}{3 \cdot 4 R} < \frac{1}{2}$$~~

Remark What happens if Π is such that $\#$

The symbol of Δ on it is

$$a_1 k_1^2 + a_2 k_2^2 + \dots + a_d k_d^2 \quad a_i > 0 ?$$

If $a_i \in \mathbb{N}$ we can still conclude with the same kind of results.

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Assume now $a_i \in \mathbb{R}^+ \setminus \mathbb{N}$, what can we say?

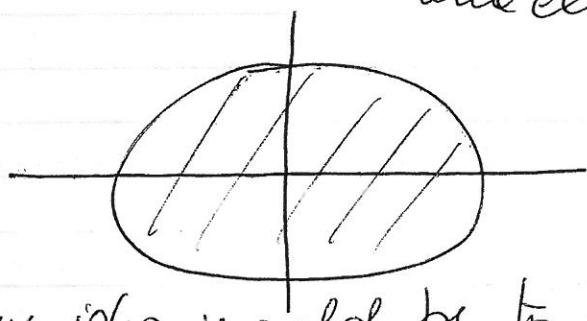
- There is a result of Bruggeman for generic a_i : proved for $d=3$. There is extra loss of derivative compared to the Conjecture stated before.

In $d=2$ we can ~~use~~ make a very rough estimate for # lattice points on ~~circle~~ ellipses:

$$\text{Count} \quad \mathfrak{B}(R) = \# \left\{ (x, y) \in \mathbb{Z}^2 / a_1 x^2 + a_2 y^2 \leq R^2 \right\}$$

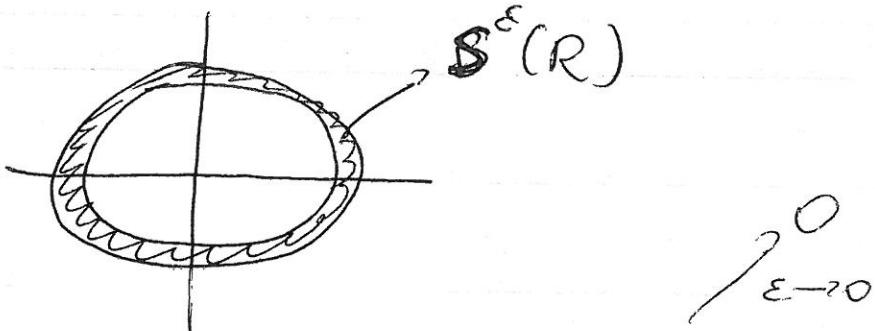
Then we have that

$$\mathfrak{B}(R) = \underbrace{\frac{2\pi}{\sqrt{4a_1^2 a_2^2}}}_{\text{one ellipse}} R^2 + P(R)$$



So one idea would be to compute # lattice points in annulus

$$\mathfrak{B}^\epsilon(R) = \left\{ (x, y) / a_1 x^2 + a_2 y^2 \leq (R+\epsilon)^2 \right. \\ \left. a_1 x^2 + a_2 y^2 \geq R^2 \right\}$$



$$\text{So } \# S_\varepsilon = E(R+\varepsilon) - E(R) = (\varepsilon R + \varepsilon^2) + P(R+\varepsilon) - P(R)$$

Unfortunately we do not much about $P(R)$,
only that

$$P(R) \ll R^{\frac{131}{208}} \log(R)^{\frac{315}{146}}$$

(This is a result of Novak and Huxley)

So with this bound we have

$$\# S_\varepsilon \ll R^{\frac{131}{208}} \approx R^{1-\frac{77}{208}}$$

By inserting this into our estimate for
the Stieltjes we get for $\bar{u}_0 \in \text{supp } B_N$

$$\| \mathcal{S}(f) u_0 \|_{L^4} \lesssim N^{\frac{1}{4}-\frac{77}{832}} \| u_0 \|_2$$

Remark: This is better than the result of
Burq-Gerard-Tzvetkov and Z.Han.