

equations. We will be interested on Strichartz estimates on tori, but recent results have been obtained on other kinds of manifolds. These estimates were first obtained for linear equations and then extended.

• General idea for a L.U.P. proof 9/13/11

$$(IVP) \begin{cases} i\partial_t u + \Delta u + F(u, \bar{u}) = 0 \\ u|_{t=0} = u_0 \in \mathcal{B}^s \end{cases}$$

$\Updownarrow$  Duhamel Principle

$$u = S(t)u_0 + \int_0^t S(t-t') F(u)(t') dt'$$

where  $S(t)u_0$  is the linear solution, that

$$\text{is solves } \begin{cases} i\partial_t v + \Delta v = 0 \\ v|_{t=0} = u_0 \end{cases}$$

then find ~~some~~ estimates for  $S(t)u_0$

in certain appropriate space-time spaces norms  
(then on the Strichartz estimates)

and use these norms to define a space  
 $X_T^s \subseteq C([-T, T], B^s)$ . Then define

the operator

$$Lw = \chi_T(t) S(t) u_0 + \chi_T(t) \int_0^t S(t-t') F(u(t')) dt'$$

$$\text{where } \chi_T(t) = \begin{cases} 1 & \text{on } [-\frac{T}{2}, \frac{T}{2}] \\ 0 & \text{on } [-2T, 2T]^c \end{cases}$$

smooth

and prove that

$$\|Lw\|_{X_T^s} \leq C \|u_0\|_{B^s} + T^\alpha \mathcal{G}(\|w\|_{X_T^s})$$

where  $\mathcal{G}(z) \nearrow$  and  $\|Lw - Lv\|_{X_T^s} \leq T^\beta \mathcal{G}(\|w\|_{X_T^s}, \|v\|_{X_T^s})$

Then if we set  $B(R) = \{w \in X_T^s \mid \|w\|_{X_T^s} \leq R\}$

on  $R = 2C \|u_0\|_{B^s}$

then we want

$$T^\alpha \mathcal{G}(R) \leq C \|u_0\|_{B^s}$$

Similarly for the contraction.

So if  $\epsilon$  is small enough in terms of  $\|u_0\|_{B^s}$  we have a contraction and hence a unique solution and continuity w.r.t. the initial data.

Simple Case:  $M = \mathbb{R}^n$

Let's start by actually writing the solution for

$$\begin{cases} i v_t + \Delta v = 0 \\ v|_{t=0} = u_0 \end{cases} \xrightarrow{\text{FT}} \begin{cases} i \hat{v}_t + (-ik)^2 \hat{v} = 0 \\ \hat{v}|_{t=0} = \hat{u}_0(k) \end{cases}$$

$$\hat{v}(t, k) = \hat{u}_0(k) e^{it|k|^2}$$

$$v(t, x) = \mathcal{F}^{-1}(\hat{v}) = \frac{1}{c_n t^{\frac{n}{2}}} \int e^{-\frac{i|x-y|^2}{4t}} u_0(y) dy$$

Dispersive Estimate:

$$\|v(t)\|_{L_x^\infty} \leq C_n \frac{1}{|t|^{\frac{n}{2}}} \|u_0\|_{L^1} \xrightarrow{|t| \rightarrow \infty} 0$$

$$\|v(t)\|_{L_x^2} = \|u_0\|_{L_x^2} \quad \uparrow \text{interpolate}$$

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Theorem: Assume that  $(q, r)$  is admissible in  $\mathbb{R}^n$

$$\frac{2}{q} = n \left( \frac{1}{2} - \frac{1}{r} \right)$$

( $2 \leq r \leq \frac{2n}{n-2}$  for  $n \geq 3$ ;  $2 \leq r < \infty$  if  $n=2$ ;  $2 \leq r < \infty$  if  $n=1$ )

then

$$\| S(t) u_0 \|_{L_t^q L_x^r} \leq C \| u_0 \|_{L^2}$$

Moreover if  $(\tilde{q}, \tilde{r})$  is another admissible pair then

$$\left\| \int_0^t S(t-t') F(t', x) dt' \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \leq C \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

(Note that  $\int_0^t S(t-t') F(t', x) dt'$  solves the non-homogeneous equation

$$\begin{cases} i w_t + \Delta w = F \\ w|_{t=0} = 0 \end{cases}$$

Proof: See Cozenave's book.

Question:

Now that we have our Strichartz estimates in  $\mathbb{R}^n$

can we say that we can solve any IVP of type

$$\begin{cases} i\partial_t u + \Delta u = F(u, \bar{u}) \\ u|_{t=0} = u_0 \end{cases} \quad \text{with } u_0 \text{ of any regularity?}$$

Answer: The answer is no. There are still cases that we cannot solve yet. But in some cases

~~Answer~~ we can at least tell if the well-posedness problem is hard or easy. For this we use scaling when possible.

Scaling: Assume  $F(u) = |u|^{p-1} u$  in  $\mathbb{R}^n$

then we set

$$u_\mu(x, t) = \mu^{-\frac{2}{p-1}} u\left(\frac{t}{\mu^2}, \frac{x}{\mu}\right)$$

and we can check that  $u_\mu$  solves the same equation and  $u_\mu(x, 0) = \mu^{-\frac{2}{p-1}} u_0\left(\frac{x}{\mu}\right)$

Now assume  $\mathcal{B}^s = H^s$  (Sobolev space)

$$\|u_0\|_{H^s} = \mu^{-\left(s + \frac{2}{p-1} - \frac{n}{2}\right)}$$

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Define  $S_c = \frac{N}{2} - \frac{2}{p-1}$  (critical exponent)

Then  $\sqrt{\text{subcritical}}$

i) if  $S > S_c$  the norm of the initial data can be made small while the time interval is made longer.

ii) if  $S = S_c$   $\sqrt{\text{critical}}$  the norm is invariant while the time interval is made longer

iii) if  $S < S_c$  (supercritical) the norm grows as the time ~~gap~~ interval is made longer.

An even better intuition comes from using scaling to relate the dispersive part  $\Delta u$  to the nonlinear part  $|u|^{p-1}u$ . Consider

a special initial data  $u_0$

$u_0$  is  $\left\{ \begin{array}{l} \text{localized in frequency} \\ \text{at large scales } N \gg 1 \end{array} \right\}$   
 $\left\{ \begin{array}{l} \text{localized in space in a} \\ \text{small ball of radius } \frac{1}{N} \end{array} \right\}$   
 $|u_0| \sim A$  amplitude

We have

$$\|u_0\|_{L^2} \sim AN^{-\frac{n}{2}} \quad \|u_0\|_{H^s} \sim AN^{-\frac{n}{2}+s}$$

If we want  $\|u_0\|_{H^s}$  small we need  
 $A \ll N^{\frac{n}{2}-s}$

If we only had a linear <sup>solution</sup> equation then  
 $\|\Delta u\| \sim AN^2$  and  $\|u\|^p \sim A^p$

So if  $AN^2 \gg A^p$  then we believe that the  
linear behavior will win. ~~Otherwise~~ otherwise

the nonlinear:

$$\text{if } A^{p-1} \ll N^2 \text{ and } A \ll N^{\frac{n}{2}-s} \Rightarrow (S > S_c) \text{ (and)}$$

(subcritical, more linear)

$$\text{if } A^{p-1} \gg N^2 \text{ and } A \ll N^{\frac{n}{2}-s} \Rightarrow (S < S_c)$$

(supercritical, more nonlinear)

Examples of subcritical / critical results

When we consider the NLS (IVP)

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$$(NLS) \begin{cases} i\partial_t u + \Delta u = \lambda |u|^{p-1} u & p \geq 1 \\ u|_{t=0} = u_0 \end{cases}$$

We would like to start with simple and natural assumptions on the data. For example  
Energy Bounded (Hamiltonian)

$$\text{Hamiltonian} = H(u) = \frac{1}{2} \int |\Delta u|^2 dx + \frac{\lambda}{p+1} \int |u|^{p+1} dx$$

Mass Bounded

$$\text{Mass} = M(u) = \frac{1}{2} \int |u|^2 dx$$

Note: if  $\lambda = 1$  (defocusing)  $\Rightarrow H(u) > 0$

and

$$\|u\|_{H^1}^2 \leq \text{Mass} + \text{Hamiltonian}$$

Fact: If (NLS) is energy subcritical

$$S_c < 1 \Leftrightarrow \frac{n}{2} - \frac{2}{p-1} < 1$$

then one sees Cazenave (handling small)

$$\text{If } S_c = 1$$

then see:

Bourgain, Guletskii's :  $n=3$  and radial  
 Colliander - Keel - Stafflen - Takaoka - Tao :  $n=3$  general  
 Vison :  $n \geq 4$  2000 - 2005

~~if  $l=0$  (focusing), then see~~

if  $l=-1$  (focusing), then see

[Kenig - Merle, Killip - Vison] ~2006

if  $S_c = 0$  (Cubic,  $n=2$ )

Killip - Vison - Tao - Zhang, Doolson 2010

Supercritical open!

Stuchartz Estimates in other Manifolds

Short Summary:

- On the sphere  $S^n$  one has a loss of  $\frac{1}{n}$  derivative:

$$\|S(f)\|_0 \left\| \right\|_{L^q_{[-T,T]} L^r_x} \leq C \|u\|_{H^{\frac{1}{n}}}$$

$(q,r)$  admissible.

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For compact manifold in general see:

[Bump - Girard - Tzvetkov] and more recently

[Z. Heni]

• On hyperbolic spaces  $\mathbb{H}^n$ :

there is a larger class of exponent that are admissible!

[Bourice, Bourice - Coles - S; Tomson - S,  
Bourice - Coles - Dujckovits, Ancher - Piepiche,  
Tomson - Pasadum - Stofflau]

Sturckatz - Estimates on tori

Bourice started his analysis in mid 90

We recall that <sup>almost</sup> all results we know at the moment are for

$(\mathbb{T}^n, \Delta)$  s.t. the symbol for  $\Delta$  is

$$\Delta f(k) = \sum_{i=1}^n n_i k_i^2 \quad n_i \in \mathbb{N}.$$

Some notation We write the solution to the linear problem

$$\begin{cases} i\partial_t v + \Delta v = 0 \\ v|_{t=0} = u_0 \end{cases}$$

as  $S(t)u_0(x) = v(t, x)$  or

$$S(t)u_0(x) = \sum_{k \in \mathbb{Z}^n} a_k e^{2\pi i (kx + |k|^2 t)}$$

where  $\hat{u}_0(k) = a_k$  the  $k$ -th Fourier Coefficient.

If we define  $\mathcal{S} = \{ (k, |k|^2) / k \in \mathbb{Z}^n \}$ , then

$$S(t)u_0(x) = \sum_{\vec{\gamma} \in \mathcal{S}} a_{\vec{\gamma}} e^{2\pi i \vec{\gamma} \cdot (x, t)}$$

$$\vec{\gamma} = (\gamma_1, \gamma_2)$$

~~we can estimate~~

**Remark**

Since on the torus we do not expect strong dispersion we do not expect a global in time Strichartz estimate.

Here we will be looking at

(we can remove this in time!)

$$\|S(t)u_0\|_{L^q_{[-1,1]} L^r_{\mathbb{T}^n}} \leq C \|u_0\|_{H^s}$$

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to make things easier we also look at  
the case  $q=r=p$  (it will be the only admissible  
case  $(q,q)$ !). We finally assume that  
 $\hat{u}_0$  lives on  $|k| \leq N$ , so we will be looking  
at

$$S_N = \{ (k, |k|^2) \mid k \in \mathbb{Z}^n \text{ and } |k| \leq N \}$$

What Bourgain set out to study was

$$\left\| \sum_{\vec{j} \in S_N} a_{\vec{j}} e^{2\pi i \vec{j} \cdot (x,t)} \right\|_{L^p(\mathbb{T}^{n+1})} \\ \leq K_p(S_N) \|u_0\|_2$$

$K_p(S_N)$  = smallest constant depending on  
 $N$  that will make (\*) true.

Conjecture (Bourgain)

$$K_p(S_N) \leq C_p$$

$$K_p(S_N) \ll N^\varepsilon$$

$$K_p(S_N) \not\leq C_p \quad N^{\frac{n-2}{2} - \frac{n+2}{p}} \quad \begin{array}{l} p < \frac{2(n+2)}{n} \\ p = \frac{2(n+2)}{n} \quad (*) \\ p > \frac{2(n+2)}{n} \end{array}$$