

Lecture #18

Construction of Weighted
Wiener Measure

(1)

The we

the $\tilde{E}(v)$ to construct our measure.

We first write

$$\tilde{E}(v) = \frac{1}{2} \int |Ux|^2 dx + dP(v)$$

where $dP(v)$ is coming from the nonlinear part of the equation. Then our condition is

$$d\mu = Z^{-1} e^{-\beta dP(v)} \cdot e^{\left(\int |Ux|^2 dx + \int |Uv|^2 dv \right)} dv$$

d μ Gaussian measure

$\beta > 0$. Let's also recall that

The support is on $\mathcal{FL}^{s,r}$ for s, r s.t.

$$\beta = s + \frac{1}{r} - \frac{1}{2} < \frac{1}{2} \quad H^0 \underset{\text{Scaling}}{\sim} \mathcal{FL}^{s,r}.$$

Since $\mathcal{FL}^{s,r}$ is not a Hilbert space we have

to say few words on how to generalize d μ

Abstract Wiener Space

Assume H is a separable Hilbert space with norm $\|\cdot\|$.

Let $\mathcal{F} = \{P_I / I \text{ finite dim orthogonal}\}$
 $\{P_{II} / II \text{ finite dim orthonormal basis projection}\}$

~~All these sets model the Hahn-Banach theorem~~

with the probability of these. Thus

$E \subset H$ is a cylinder iff

$$E = \{x \in H / P_x \in F\}, \quad F \in \mathcal{F}$$

$F \subseteq \mathbb{R}^{\text{number of basis fields}}$. Let R be the collection of all basis fields but not a σ -field. Cylinders. Then the Gaussian measure on H is

$$g(E) = (2\pi)^{-\frac{m}{2}} \int_F e^{-\frac{\|x\|^2}{2}} dx$$

$\forall E \in R, g(E) = \lim P_E H$.

Remark: The Gaussian measure g on H defined this way is finitely additive but not countably additive.

(Remember we had to use the Grothendieck's theorem for this!)

Definition: (Gross). A seminorm $M \cdot M$ in H is called measurable if for every $\epsilon > 0$ $\exists P_\epsilon \in F = \{\text{set of finite dimensional orthogonal projections}\}$ s.t. \exists

$$g(\|Px\| > \varepsilon) < \varepsilon$$

for all $P \in \mathcal{F}$ orthogonal to P_ε

let $\mathcal{B} = \text{completion of } H \text{ w.r.t. } \| \cdot \| \text{ and}$

let $i : H \hookrightarrow \mathcal{B}$ the inclusion map.

The triple (i, H, \mathcal{B}) is called an abstract Wiener space.

We now embed \mathcal{B}^* in H by regarding $y \in \mathcal{B}^*$ as an element in $H^* = H$.

The extension of g onto \mathcal{B} is defined as follows:

For a Borel set $F \subseteq \mathbb{R}^n$ set

$$g(x \in \mathcal{B} : ((x, y_1), \dots, (x, y_n)) \in F) :=$$

$$g(\{x \in H : (\langle x, y_1 \rangle_H, \dots, \langle x, y_n \rangle) \in F\})$$

where $y_i \in \mathcal{B}^*$. Let $\mathcal{Q}_\mathcal{B}$ = collections of cylinder sets $\{x \in \mathcal{B} : ((x, y_1), \dots, (x, y_n)) \in F\}$.

(4)

Proposition (Gross) \mathfrak{g} is countably additive
in the σ -field generated by $\otimes \mathcal{B}_B$.

Example: $H = H'(\Pi)$, $B = \mathcal{L}^{s,r}(\Pi)$

$$2 \leq r < \infty \quad (s-1)r < -1$$

We have to prove that under the
conditions (i, H, B) ~~$(i, H, \mathcal{L}^{s,r})$~~ $= (i, H', \mathcal{L}^{s,r})$
is an abstract Wiener space if

$$(s-1)r < -1.$$

it's enough
to do so ~~we need~~ to show that

~~$\int_{\mathbb{R}^d} e^{-\|v\|_{\mathcal{L}^{s,r}}^2} dv < \infty$~~

Proposition: let $2 \leq r < \infty$ and assume
 $(s-1)r < -1$. Then the seminorm $\| \cdot \|_{\mathcal{L}^{s,r}}$
is measurable. Moreover we have the
exponential tail estimate:

$\exists c > 0$ and $c > 0$ s.t. $\forall k > 0$

$$\mathfrak{g}(\|v\|_{\mathcal{L}^{s,r}} > k) \leq Ce^{-ck^2}$$

Definition: Given two measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ and σ -field \mathcal{G} .
 Let map $T: \Omega_1 \rightarrow \Omega_2$ s.t. $w_1 = T(w_2)$ is measurable ~~also~~ if $\forall A \in \mathcal{B}_2$ measurable $T^{-1}(A) \in \mathcal{B}_1$. If $(\Omega_1, \mathcal{B}_1)$ has a probability measure P then then Q is an induced probability measure in $(\Omega_2, \mathcal{B}_2)$ if $Q(A) = P(T^{-1}(A)) \quad \forall A \in \mathcal{B}_2$

Remark: The measure \mathbb{P}_N s.t.

$$d\mathbb{P}_N = Z_{0,N} e^{-\frac{1}{2} \sum_{|n| \leq N} (1 + n_1^2) |\tilde{v}_n|^2} \prod_{|n| \leq N} da_n db_n$$

$$\tilde{v}_n = a_n + i b_n$$

can be regarded as induced probability

Measure on \mathcal{F}^{2N+1} under the map

$$w \mapsto \left\{ \frac{g_n(w)}{\sqrt{1+n_1^2}} \right\}_{|n| \leq N}$$

where $g_n(w)$ are independent standard complex Gaussian random variables

on a probability space $\{\Omega, \mathcal{F}, P\}$, and

$$\tilde{v}_n = \frac{g_n(w)}{\sqrt{1+n_1^2}}$$

$$\begin{aligned} \mathbb{E} \tilde{v}_n \tilde{v}_r &= \sum_{|n|, |r|} \left(\frac{|g_n(w)|}{\sqrt{1+n_1^2}} \right)^s \left(\frac{|g_r(w)|}{\sqrt{1+r_1^2}} \right)^r \\ &\approx \sum_{|n|=s-1, |r|=r-1} \end{aligned}$$

Proof of proposition We need a lemma

Lemma: let $\{g_n\}$ as above. Then, for M

dyadic and $\delta < \frac{1}{2}$, we have

$$\lim_{M \rightarrow \infty} M^{\frac{2\delta}{2\delta-1}} \frac{\max_{|n|=M} |g_n|^2}{\sum_{|n|=M} |g_n|^2} = 0 \quad \text{a.s.}$$

(6)

Assume the lemma and our proposition.

Let $2 \leq r < \infty$ ($s-1 < r < 1$). We need to show that for given $\epsilon_0 \exists M_0$ large

s.t.

$$g(\|P_{M_0}^+ v\|_{\ell^{s,r}}) > \epsilon$$

where $P_{M_0}^+$ = projection onto frequencies $|n| > M_0$.

In fact if $P \in \mathcal{F}$ = {finite dim projections} and $P \perp P_{M_0}^+$ then

$$\|Pv\|_{\ell^{s,r}} \leq \|P_{M_0}^+ v\|_{\ell^{s,r}}$$

Assume $v = \sum_n \frac{g_n}{\sqrt{1+n^2}} e^{inx}$

By Ergoff's theorem $\exists E \subset \mathbb{R}$ s.t.

$g(E^c) < \frac{1}{2}\epsilon$ and the convergence on E is uniform : $\exists M_0 \gg 1$ s.t.

$$\frac{\|\{g_n(\omega)\}_{|n| \geq M}\|_{\ell^\infty_n}^\infty}{\|\{g_n(\omega)\}_{|n| \geq M} \|_{L^2_n}} < M^{-\delta} \quad \begin{array}{l} \text{if } M \geq M_0 \\ \text{if } \omega \in E \end{array}$$

(7)

From now on all events are $\cap E$ even though we will not say it.

Let $\{b_j\}_{j \geq 1}$ s.t. $\sum b_j = 1$, $M_j = M_0 2^j$

We can write $b_j = C 2^{-lj} = CM_0^{-l} M_j^{-l}$

for some $l > 0$ to be determined

$$P(\|P_M^+ v(\omega)\|_{\ell^{r,s}} > \varepsilon) \leq \sum_{j=1}^{\infty} P\left(\|\{c_n g_n(\omega)\}_{n \sim M_j}\|_{L_n^r} > b_j \varepsilon\right)$$

Interpolating $r < r < \infty$ where

$$\|\{c_n g_n\}_{n \sim M_j}\|_{L_n^r} \sim M_j^{s-1} \|\{g_n\}_{n \sim M_j}\|_{L_n^r} \leq M_j^{s-1} \|\{g_n\}_{n \sim M_j}\|_{L_n^2}^{\frac{2}{r}} \|\{g_n\}_{n \sim M_j}\|_{L_n^\infty}^{\frac{r-2}{r}}$$

interpolation

$$\leq M_j^{s-1} \|\{g_n\}_{n \sim M_j}\|_{L_n^2} \left(\frac{C^\infty}{C^2} \right)^{\frac{r-2}{r}} \leq$$

$$M_j^{s-1} M_j^{-\frac{2(r-2)}{r}} \|\{g_n\}_{n \sim M_j}\|_{L_n^2}$$

So if we have

$$\|\{c_n g_n\}_{n \sim M_j}\|_{L_n^r} > b_j \varepsilon \text{ then}$$

$$\|\{g_n\}_{n \sim M_j}\|_{L_n^2} \geq R_j$$

(8)

$$\text{When } R_j := \delta_j \varepsilon M_j^{-\alpha}, \quad \alpha := -s + 1 + \delta \frac{r-\varepsilon}{n}$$

By picking δ close to $\frac{1}{2}$ and δ small

$$R_j \geq C \varepsilon M_0^{\lambda} M_j^{\frac{1}{2} + \varepsilon}$$

Now

$$\mathbb{P}(\| \{g_n\}_{n \sim M_j} \|_{L^2} \geq R_j) \sim$$

$$\sim \int_{B^c(0, R_j)} e^{-\frac{1}{2} \|g_n\|^2} \prod_{n \sim M_j} dg_n$$

$$\leq \int_{R_j}^{\infty} e^{-\frac{1}{2} s^2} s^{2 \# \{n \sim M_j\} - 1} ds$$

We dropped the surface measure of the unit sphere of dim $2 \# \{n \sim M_j\} - 1$

Since $\sigma(S^m) = 2\pi^{\frac{m}{2}} / \Gamma(\frac{m}{2}) \leq 1$ uniformly

After a change of variables and calculations

$$\begin{aligned} &\leq C \int_{R_j}^{\infty} e^{-\frac{1}{4} s^2} s ds \leq e^{-CR_j^2} \\ &= e^{-C\varepsilon^2 M_0^{2\lambda} M_j^{1+\varepsilon^2}} \end{aligned}$$

(9)

and for adding everything to get the $\sum_{j=1}^{\infty} e^{-c^2 M_0^{1+2j} (2^j)^{1+\frac{1}{2j}} \varepsilon^2}$

$$g(\|P_{M_0}^+ v\|_{L^{\infty}} > \varepsilon) \leq \sum_{j=1}^{\infty} e^{-c^2 M_0^{1+2j} (2^j)^{1+\frac{1}{2j}} \varepsilon^2} \leq \frac{1}{2} \varepsilon \quad \text{by choosing } M_0 \text{ large enough.}$$

Remark: In doing all this one needs

$$\boxed{(s-1)r < 1}$$

At this point we need to construct the Wiener measure. We have the Gaussian measure all set so we need to take care of

$$R(\omega) := \chi_{\{\|\omega\|_{L^2} \leq B\}} e^{-\frac{1}{2} d\mu(\omega)}, \quad \boxed{d\mu = R(\omega) d\omega}$$

for B small; then we will get to this

~~and~~ via a mesh limit of

$$R_N(v) := R(v^N) = \chi_{\{\|v^N\|_{L^2} \leq 3\}} e^{-\frac{1}{2} d\mu_N(v)}$$

where

$$d\mu_N(v) = \text{ADD} \mathcal{N}(v^N)$$

$$= F_N(v) + G_N(v) + K_N(v) \quad \text{and}$$

$$F_N(v) = -\frac{1}{2} \operatorname{Im} \int (v^N)^2 \overline{v^N v_x^N} dx$$

$$G_N(v) = \frac{1}{4\pi} \left(\int_{\mathbb{T}} |v^N|^2 dx \right) \left(\int_{\mathbb{T}} |v^N|^4 dx \right)$$

$$K_N(v) = \frac{1}{\pi} \left(\int_{\mathbb{T}} |v^N|^2 dx \right) \left(\Im \int_{\mathbb{T}} v^N (\bar{v}_x^N) dx \right) + \frac{1}{4\pi^2} \left(\int_{\mathbb{T}} |v^N|^2 dx \right)^3$$

, we will show

$$d\mu_N = Z_N^{-1} R_N(v) d\mu_N \xrightarrow{\substack{\text{weak} \\ \downarrow \text{renormalization}}} d\mu = Z_0^{-1} R(v) d\mu$$

Lemma 1:

a) The sequence F_N converges in $L^2(d\mu)$ to

$$F(v) = -\frac{1}{2} \Im \int v^2 \bar{v} v_x dx$$

b) Monotone for $\alpha < \frac{3}{4}$ $\exists C, \delta > 0$ s.t.

$\forall M > N \geq 1$ and $\lambda > 0$

$$\rho(|F_M(v) - F_N(v)|) \leq C e^{-\delta M^\alpha \lambda^{\frac{1}{2}}}$$

b) Let $p \in (2, \infty)$ $\exists \alpha, c$ s.t. $\forall M \geq N \geq 1, d > 0$

$$\rho(\|P_N v\|_{L^p(\mathbb{T})} > \lambda) \leq C e^{-cd^2}$$

$$\rho(\|P_M v - P_N v\|_{L^p(\mathbb{T})} > \lambda) \leq C e^{-cN^{\alpha} \lambda^2}$$