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Lecture 11

Summary of post lecture

The theorem we want to prove is

Theorem: Given $\delta < 1$ and $k \gg 1$, given δ ,
 there exists α $u_0 \in H^s$ and T solution u
 to 2D, periodic, defocusing, Cubic NLS st.

$$\|u_0\|_{H^s} \approx \delta \text{ and } \|u(T)\|_{H^s} \geq k$$

for a certain time $T = T(k, \delta) \gg 1$.

1) Gauge equation and look for solutions

$$v(x, t) = \sum a_n(t) e^{i(t n^2 + x n)}$$

$(a_n(t))$ solves an o dimension system of ODE
 that we called KNLS

2) We restrict the system only to resonant
 frequencies that is

$$\mathcal{P}_{ch} = \left\{ (n_1, n_2, n_3) \in (\mathbb{Z}^*)^3 \text{ st. } n_1 + n_2 + n_3 \in h \right\}$$

$$\text{such that } |n_1|^2 + |n_2|^2 + |n_3|^2 = |h|^2$$

we call this system RKNLS

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Claim:

$$h_1 - h_2 + h_3 - h = 0 \quad a)$$

$$(h_1, h_2, h_3, h) \text{ s.t. } |h_1|^2 - |h_2|^2 + |h_3|^2 - |h|^2 = 0 \quad b)$$

on the vertices of a rectangle.

Proof: First notice that

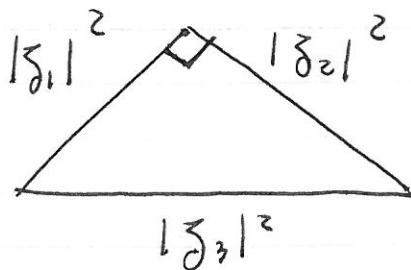
(h_1, h_2, h_3, h) is a resonant quadruplets

$\Leftrightarrow \exists n_0 \in \mathbb{Z}^2 \quad (h_1 - n_0, h_2 - n_0, h_3 - n_0, h - n_0)$ is

also resonant. Clearly a) is true. On the other hand

$$\begin{aligned} & |h_1 - n_0|^2 + |h_2 - n_0|^2 + |h_3 - n_0|^2 - |h - n_0|^2 = \\ & (|h_1|^2 + |n_0|^2 - h_1 \cdot n_0 - |h_2|^2 - |n_0|^2 + 2h_2 \cdot n_0) \\ & + (|h_3|^2 + |n_0|^2 - 2h_3 \cdot n_0 - |h|^2 - |n_0|^2 + 2h \cdot n_0) = \\ & |h_1|^2 + |h_3|^2 - |h_2|^2 - |h|^2 - 2n_0 [h_1 - h_2 + h_3 - h] \end{aligned}$$

So we can take $n - n_0 = 0$ then



$$h_1 - n_0 = \zeta_1$$

$$h_2 - n_0 = \zeta_2$$

$$h_3 - n_0 = \zeta_3$$

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3) We introduce a finite set Δ with
many properties among which

$$\Delta = \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_N$$

inner radius R , total number of frequencies

$2^N N$ and $(n_1, n_2, n_3, n_4) \in \Delta \Rightarrow (n_1, n_3) \in \Delta_j$

and $(n_1, n_2) \in \Delta_{j+1}$

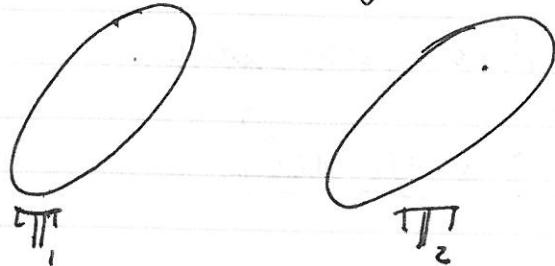
We further restrict our system of ODE so that
everything lives on Δ . We denote this
system: $(\partial \mathcal{F}N(S))_\Delta$.

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So this says that if we start on \mathbb{H}_1' then we will be stuck in them! So why is Proposition 2 true?

Let's consider the case where $N=2$. There is

an orbit connecting \mathbb{H}_1 with \mathbb{H}_2



In fact we can consider the explicit solution

$$b_1(t) = \frac{e^{-it} \omega}{\sqrt{1+e^{2\sqrt{3}t}}} \quad b_2(t) = \frac{e^{-it} \omega^2}{\sqrt{1+e^{-2\sqrt{3}t}}}$$

$\omega = e^{2\pi i/3}$ a cube root of unity

$$(b_1(t), b_2(t)) \xrightarrow[t \rightarrow \infty]{} (b_1^+, b_2^+) \text{ s.t.}$$

$$|b_1^+| = 0 \quad |b_2^+| = 1$$

$$\text{So } (b_1^+, b_2^+) \in \mathbb{H}_2'$$

Similarly

$$(b_1(t), b_2(t)) \xrightarrow[t \rightarrow -\infty]{} (b_1^-, b_2^-) \subset \mathbb{H}_1'$$

When N is large one would like to link these "sliders" ... But as is this argument cannot work since to move from $1 \pi_j$ to the next π_{j+1} it takes an ∞ . amount of time ... and we cannot wait that long. But nevertheless something along these lines works.

More remarks on the toy Model

Conservation of mass

$$\sum_j |b_j(+)|^2 = \text{const.}$$

Conservation of momentum

$$\sum_j |b_j(+)|^2 \left(\sum_{n \in A_j} n \right)$$

Conservation of Energy (Hamiltonian)

$$H(b) = \sum_j \frac{1}{4} |b_j|^4 - \text{Re} (\bar{b}_j^2 b_{j+1}^2)$$

$$\partial_t b = -2i \frac{\partial H}{\partial \bar{b}} \quad \partial_t \bar{b} = 2i \frac{\partial H}{\partial b}$$

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Lemme

(Third ingredient: approximation
lemme). let $0 < \sigma < 1$ an absolute constant.

let $B > 1$ and $T < B^2 \log B$. let

$$g(t) = (g_n(t))_{n \in \mathbb{Z}^2}$$

be a solution to

$$(*) -i\dot{g}(t) = \sigma \mathcal{P}(t)(g(t), g(t), g(t)) + \mathcal{E}(t)$$

for $0 \leq t \leq T$, where $\mathcal{P}(t)$ is the nonlinearity
of FICS and $g(0)$ is compactly supported.

Assume also

$$\|g(t)\|_{L^1(\mathbb{R}^2)} \lesssim B^{-1}$$

$$\left\| \int_0^t \mathcal{E}(s) ds \right\|_{L^1(\mathbb{R}^2)} \lesssim B^{-1.5}$$

$\forall t \in (0, T)$. Then if $a(t)$ solves (*) with $\mathcal{E}(t) = 0$
with \mathcal{F} date $g(0)$, then

$$\|a(t) - g(t)\|_{L^1(\mathbb{R}^2)} \lesssim B^{-1-\sigma/2}$$

the Glen that puts the ingredients together

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the Toy Model has solution

$$b^{(u)}(t) := \lambda^{-1} b\left(\frac{t}{\lambda^2}\right)$$

and we let us prove the main theorem that
we can rewrite in the following way:

Main theorem For any $\varepsilon <$ and $k \gg 1$

$$(a_n) \text{ s.t. } \left(\sum_{n \in \mathbb{Z}^2} |\alpha_n|^2 m_n^{2s} \right)^{\frac{1}{2}} \leq \varepsilon$$

and a solution $a_n(t)$ of (ENIS) and $T > 0$

$$\text{s.t. } \left(\sum_{n \in \mathbb{Z}^2} |\alpha_n(T)|^2 m_n^{2s} \right)^{\frac{1}{2}} > k$$

Proof: We start with the initial state

$$\alpha_n(0) = \begin{cases} b_j^1(0) & n \in \Lambda_j \\ 0 & n \in \Lambda^c \end{cases}$$

Here $b_j(0)$ is the separatrix one starts in the

Instability theorem, in particular we recall that

$$b_j(0) \approx \begin{cases} 1-\varepsilon & j=3 \\ 0 & \text{otherwise} \end{cases}$$

From this Instability theorem one knows

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that $b_j \rightsquigarrow b_j(t)$ and to go through all Λ , it will take time T_0 . Once rescaled then it will take $\lambda^2 T_0$. So the Inverse Trichotomy Theorem gives us after rescaling:

$$a_n(0) \rightsquigarrow g_n(t) = \begin{cases} b_j^\lambda(t) & n \in \Lambda_j \\ 0 & n \notin \Lambda \end{cases}$$

$$b_j^\lambda(0) \rightsquigarrow a_n(t)$$

We want to relate $g_n(t)$ and $a_n(t)$ through the perturbation lemma

$$\mathcal{E}(t) = \text{nonresonant part} := - \sum_{\substack{i=1 \\ i \neq n, n+1, n-1}} \overline{g_{n_i} g_{n_2} g_{n_3}} e^{i \omega_i t} \\ P(n) \cap P_{n+1} \cap P_{n-1}$$

Claim: If $B = \mathcal{E}(N) \lambda$ then ~~the necessary conditions for~~ ~~the~~ $\mathcal{E}(N)$ to be determined for λ large enough one obtains all the conditions for the lemma:

$$B^2 \log B > \lambda^2 T_0 \quad \checkmark$$

$$\|g(t)\|_l \leq B^{-1}$$

$$\left\| \int_0^t \mathcal{E}(s) ds \right\|_l \leq B^{-5}$$

We first observe that since $\|b(t)\|_{\ell^\infty} \approx 1 \Rightarrow$

$\|b(t)\|_{\ell^1} \approx \|A\| \approx C(N)$ and therefore

$$\|b^*(t)\|_{\ell^1} = \|g(t)\|_{\ell^1} \lesssim t^{-1} C(N)$$

Now the last condition:

Claim $\left\| \int_0^t E(s) ds \right\|_{\ell^1} \lesssim C(N) (t^{-3} + t^{-6} T)$

$$\text{where } T = t^{2/3}$$

To prove this we use integration by parts

$$\begin{aligned} \int_0^t E(s) ds &= - \int_0^t \sum_{\substack{\Gamma(n) \cap \Gamma_{n_3}(m) \cap \Lambda^3 \\ i w_4 s}} \overline{g_{n_1}(s)} g_{n_2}(s) g_{n_3}(s) \\ &\quad \cdot \underbrace{\frac{e^{i w_4 s}}{i w_4} ds}_{= \frac{d}{ds} \left[\frac{e^{i w_4 s}}{i w_4} \right]} \\ &= - \sum_{\Gamma(n) \dots} g_{n_1}(T) \overline{g_{n_2}(T)} g_{n_3}(T) \frac{e^{i w_4 T}}{i w_4} \xrightarrow{i w_4 T \rightarrow 0} t^{-3} \\ &\quad + \sum_{\Gamma(n) \dots} g_{n_1}(0) \overline{g_{n_2}(0)} g_{n_3}(0) \bullet \frac{1}{i w_4} \xrightarrow{i w_4 \rightarrow 0} t^{-3} \\ &\quad + \int_0^t \sum_{\Gamma(n) \dots} \dot{g}_{n_1}(s) \overline{g_{n_2}(s)} g_{n_3}(s) \frac{e^{i w s}}{i w_4} ds \end{aligned}$$

plug the equation $\dot{g}_n(s) \approx \frac{1}{s}$ $\xrightarrow{s \rightarrow 0} \text{quadratic}$

Estimate of the initial state

$$\left(\sum_{n \in \Lambda} |\alpha_n(0)|^2 |n|^{2s} \right)^{\frac{1}{2}} = \frac{1}{\lambda} \left(\sum_{j=1}^N |\beta_j(0)|^2 \left(\sum_{n \in \Lambda_j} |n|^{2s} \right) \right)^{\frac{1}{2}}$$

$$\sim \frac{1}{\lambda} Q_3^{\frac{1}{2}} \text{ where}$$

$$Q_j = \sum_{n \in \Lambda_j} |n|^{2s}$$

$$\text{Now } Q_3^{\frac{1}{2}} = \left(\sum_{n \in \Lambda_3} |n|^{2s} \right)^{\frac{1}{2}} \geq R^s c(N)$$

where $R = \text{inner radius}$. So since λ is picked we take R s.t.

$$\frac{1}{\lambda} R^s c(N) \sim 5$$

Estimate of $\alpha_n(\tau)$ for $\tau = \lambda^2 T_0$

$$\|\alpha_n(\tau)\|_{H^s} \geq \|\beta_j(\tau)\|_{H^s} - \|\alpha_n(\tau) - \beta_j(\tau)\|_{H^s}$$

$$= I_1 - I_2$$

$$I_2 \leq \|\alpha_n(\tau) - \beta_j(\tau)\|_{\ell^1} \left(\sum_{n \in \Lambda} |n|^{2s} \right)^{\frac{1}{2}}$$

$\leq C(N) \lambda^{-1-6} N^s \underset{2}{\zeta} \text{ by possibly taking } \lambda \text{ even bigger.}$

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On the other hand if we want to show that

$$\|f\|^2 = \|b^A(T)\|_{H^S}^2 \geq \frac{k^2}{6^2} \|b^A(0)\|_{H^S}^2 \sim k^2$$

So if we define

$$W = \frac{\sum_{n \in \Lambda} |b_n^A(T)|^2 |n|^{2s}}{\sum_{n \in \Lambda} |b_n^A(0)|^2 |n|^{2s}}$$

then we only need to show $W \geq \frac{k^2}{6^2}$.

We know that $b_j(T) = 1 - \varepsilon$ for $j = N-2$

and $b_j(T) \approx \varepsilon$ for $j \neq N-2$

$$W = \frac{\sum_{i=1}^N \sum_{n \in \Lambda_i} |b_i^A(T)|^2 |n|^{2s}}{\sum_{i=1}^N \sum_{n \in \Lambda_i} |b_i^A(0)|^2 |n|^{2s}}$$

$$\sum_{n \in \Lambda_j} |n|^{2s} = Q_j$$

$$\gtrsim \frac{Q_{N-2}(1-\varepsilon)}{(1-\varepsilon)Q_3 + \varepsilon Q_1 + \dots + \varepsilon Q_N} \sim$$

$$\frac{Q_{N-2}(1-\varepsilon)}{Q_{N-2} \left[(1-\varepsilon) \frac{Q_3}{Q_{N-2}} + \dots + \varepsilon \right]}$$

$$\gtrsim \frac{(1-\varepsilon)}{(1-\varepsilon) \frac{Q_3}{Q_{N-2}}} = \frac{Q_{N-2}}{Q_3}$$

but the last property of A was:

$$Q_{N-2} = \sum_{n \in \Lambda_{N-2}} |n|^{2s} \gtrsim \frac{k^2}{6^2} \sum_{n \in \Lambda_3} |n|^{2s} = \frac{k^2}{6^2} Q_3 !$$