

## Topics in Partial Differential Equations

The main subject of study in this course will be nonlinear Schrödinger Equations in Hamiltonian form. The reason for this is that one can think, in certain cases, about these equations as  $\infty$  dimension Hamiltonian systems.

Let's start by looking at the finite case:

## Hamiltonian Equations

$$(HE) \quad \dot{q}_i = \frac{\partial H(p, q)}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H(p, q)}{\partial q_i}$$

$$y := (q_1, \dots, q_k, p_1, \dots, p_k)^T \in \mathbb{R}^{2k} \quad (2k = d)$$

then one can write

$$(HE) \quad \frac{dy}{dt} = J \nabla H(y) \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

One first calculation that one can do is

$$\frac{dH}{dt} = \sum_i \frac{\partial H}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial H}{\partial p_i} \dot{p}_i = 0$$

So  $H(y) = H(p, q)$  is a first integral.

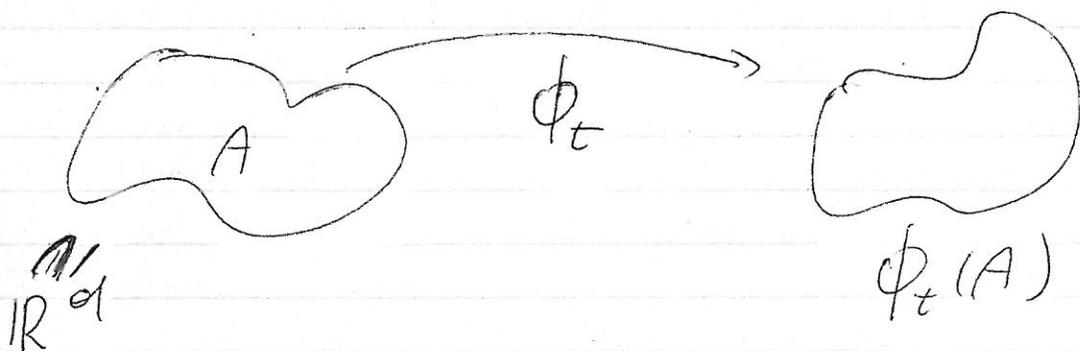
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For finite Hamiltonian system there are many simple or complicated but interesting results.

### Gibbs Measure

Define 
$$\mu := e^{-\beta H(p,q)} \prod_{i=1}^d dp_i dq_i \quad \beta > 0$$

Theorem: The measure  $\mu$  is invariant under the Hamiltonian flow: That is if



Here  $\phi_t(x_{y_0}) =$  solution to IVP 
$$\begin{cases} \frac{d}{dt} y = JH(y) \\ y|_{t=0} = y_0 \end{cases}$$

then 
$$\mu(A) = \mu(\phi_t(A))$$

Proof: The proof is based on Liouville's theorem and the fact that  $H(y(t))$  is conserved.

**Liouville's theorem**: let a vector field  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be divergence free, then <sup>if the</sup> flow map  $\phi_t$  satisfies  $\frac{d}{dt} \phi_t(y) = f(\phi_t(y))$

then it is a volume preserving map for all  $t$ .

Now by direct inspection it follows that in the case of a Hamiltonian system

$$\text{div } f = 0$$

hence for

$$d\mu = e^{-\beta H(p,q)} \prod_{i=1}^d dp_i dq_i$$

preserved since  $H(p,q)$  is first integral
preserved by Liouville's theorem.

Question: what can one say in  $2n$  dimensions?

**Symplectic non-squeezing theorem**

Consider now  $(\mathbb{R}^{2n}, \omega)$  as the symplectic phase space of the Hamiltonian PDE

$$\frac{d}{dt} y = J \nabla H(y)$$

④

where the symplectic form  $\omega$  is defined as  
$$\omega(u, v) = \langle Ju, v \rangle$$

(Here  $J$  is an almost complex structure on  $\mathbb{R}^{2n}$  since it is bounded, anti-self adjoint operator and  $J^2 = -\text{Id}$ ).

Because of this symplectic structure the Hamiltonian flow is a symplectic flow (actually a symplectomorphism)

"  
diffeomorphism between 2 symplectic manifolds."

Theorem: Non squeezing theorem (Bourgin 1985)

Let  $B(R)$  be a ball of radius  $R$

and  $Z(r)$  be a cylinder of radius  $r$

and  $\phi: B(R) \rightarrow Z(r)$  a symplectomorphism

Then  $R \leq r$ .

So  $f: (M, \omega) \rightarrow (N, \omega')$  is a symplectomorphism

iff  $f$  is a diffeomorphism and

$$f^* \omega' = \omega$$

$f^*$  = pull back of  $f$

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As a consequence, if  $\phi_t$  is an Hamiltonian flow then ~~the~~ it cannot squeeze a ball into a cylinder!

Question: Is this true in the  $\infty$  dimensional case?

Example of  $\infty$  dimensional Hamiltonian system

Consider the nonlinear Schrödinger equation (NLS)

~~$u_t = i u_{xx} + i |u|^{p-2} u$~~   $u_t = i u_{xx} + i |u|^{p-2} u$   $p \geq 1$   $u$  complex.  $k = \pm 1$

Assume  $x \in \mathbb{T}$  ( $u$  is periodic).

then if we define

$$H(u, \bar{u}) = \frac{1}{2} \int_{\mathbb{T}} |u_x|^2 dx + \frac{1}{p+1} \int_{\mathbb{T}} |u|^{p+1} dx$$

then one can write formally

$u_t = i \nabla H(u)$

and if we think of identifying

$u \rightsquigarrow \hat{u}(n) = a_n + i b_n \quad n \in \mathbb{Z}$

↳ Fourier coefficient

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then one understands that the (NLS) introduced is indeed an example of an  $n$  or  $n$ -dim Hamiltonian system.

Question 1: If  $\lambda = 1$ , can we define

$$d\mu = e^{-\beta H(u)} \prod_{n \in \mathbb{Z}} da_n db_n ?$$

Is  $\mu$  invariant?

Question 2: Does the non-squeezing theorem hold here? In which symplectic manifold?

Before we are able to set up the stage to answer these kinds of questions we need to make sure that we first show that the flow map  $\phi_t$  is well defined here. In other words we need to prove existence, uniqueness, stability of solutions to NLS!

## Well-posedness for NLS

Consider the Initial Value Problem

$$(IVP) \quad \begin{cases} iu_t + \Delta u + F(u) = 0 & x \in M \text{ (manifold)} \\ u|_{t=0} = u_0 \in B^s & \Delta = \text{Laplace-Beltrami} \end{cases}$$

Here  $B^s =$  Besov space of regularity  $s$ . Usually we will take  $H^s(M)$ .

Definition: (Well-posedness)

① We say that ~~the~~ (IVP) is locally well-posed (l.w.p.) in  $B^s$  if for any  $u_0 \in B^s \exists T = T(u_0)$  and a space  $X_T^s \subseteq C([-T, T], B^s)$  and a unique "solution"  $u$  to (IVP) s.t.  $u \in X^s$  and in the appropriate topologies there is continuity w.r.t. initial data

② We say that (IVP) is globally well posed if  $T$  can be taken arbitrarily long.

This definition will be our starting point!

⑧

Many more questions can then be asked :

Question	Meaning
well-posedness	$\exists!$ Solutions, stability
blow-up	If solutions stop to exist is there some non explosion? Rate of blow up? Shape of blow up?
Growth of Sobolev norms, turbulence	If solutions exist globally in time do their properties change in time?
Study of Kenigsas $\infty$ dim Hamiltonian systems	Invariant measures Almost surely well-posedness symplectic structure and related theorems
special solutions	Existence of solitons, soliton interactions ...

**Strichartz Estimates** then on going to be the most powerful tool in the proof of (local) well-posedness and other results. They are typical of dispersive and wave