BOUNDS ON THE GROWTH OF HIGH SOBOLEV NORMS OF SOLUTIONS TO THE 2D DEFOCUSING PERIODIC DEFOCUSING CUBIC NLS

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1. Introduction.

1.1. **Statement of the problem and the main result.** We study the defocusing cubic Nonlinear Schrödinger equation on the two-dimensional torus:

$$\begin{cases} iu_t + \Delta u = |u|^2 u, \ x \in \mathbb{T}^2 \\ u|_{t=0} = \Phi \in H^s(\mathbb{T}^2), \ s > 1. \end{cases} \tag{1}$$

The equation (1) has the following conserved quantities:

$$M(u(t)) := \int |u(x,t)|^2 dx, \ \textit{(Mass)}$$

$$E(u(t)) := \frac{1}{2}\int |\nabla u(x,t)|^2 dx + \frac{1}{4}\int (|u|^2)(x,t)|u(x,t)|^2 dx, \ \textit{(Energy)}.$$

By using the periodic variants of Strichartz estimates [3], we can show that the (1) is locally well-posed in H^1. Furthermore, by using the two conservation laws, we can deduce global existence of (1) in H^1 and a priori bounds on the H^1 norm of a solution. By persistence of regularity, we obtain global existence in H^s, for $s > 1$. Hence, it makes sense to analyze the behavior of $\|u(t)\|_{H^s}$.

Given a real number x, we denote by $x+$ and $x-$ expressions of the form $x+\epsilon$ and $x-\epsilon$ respectively, where $0 < \epsilon \ll 1$. With this notation, the result that we prove for (**??**) on $\mathbb{T}^2$ is:

Theorem 1.1. *(Bound for the defocusing cubic NLS on $\mathbb{T}^2$; [45]) Let u be the global solution of (1) on $\mathbb{T}^2$. Then, there exists a function C_s, continuous on $H^1(\mathbb{T}^2)$ such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq C_s(\Phi)(1+|t|)^{s+}\|\Phi\|_{H^s(\mathbb{T}^2)}. \tag{2}$$

Remark 1.2. *Let us observe that, when s is an integer, or when Φ is smooth, essentially the same bound as in Theorem 1.1 was proved by using different techniques in the work of Zhong* [50]*. The reason why one uses the fact that s is an integer is because one wants to use exact formulae for the (Fractional) Leibniz Rule for D^s. By using an exact Leibniz Rule, one sees that certain terms which are difficult to estimate are in fact equal to zero.*

Remark 1.3. *Let us note that, if we consider the spatial domain to be $\mathbb{R}^2$, one can obtain uniform bounds on $\|u(t)\|_{H^s}$ for solutions $u(t)$ of the defocusing cubic NLS by the recent scattering result of Dodson* [27]*.*

1.2. **Motivation for the problem and previously known results:** The growth of high Sobolev norms has a physical interpretation in the context of the *Low-to-High frequency cascade*. In other words, we see that $\|u(t)\|_{H^s}$ weighs the higher frequencies more as s becomes larger, and hence its growth gives us a quantitative estimate for how much of the support of $|\widehat{u}|^2$ has transferred from the low to the high frequencies. This sort of problem also goes under the name *weak turbulence* [1, 2, 49].

By local well-posedness theory [6, 14, 33, 48], it can be observed that there exist $C, \tau_0 > 0$, depending only on the initial data Φ such that for all t:

$$\|u(t+\tau_0)\|_{H^s} \leq C\|u(t)\|_{H^s}. \tag{3}$$

Iterating (3) yields the exponential bound:

$$\|u(t)\|_{H^s} \leq C_1 e^{C_2|t|}. \tag{4}$$

Here, $C_1, C_2 > 0$ again depend only on Φ.

For a wide class of nonlinear dispersive equations, the analogue of (4) can be improved to a polynomial bound, as long as we take $s \in \mathbb{N}$, or if we consider sufficiently smooth initial data. This observation was first made in the work of Bourgain [4], and was continued in the work of Staffilani [46, 47].

The crucial step in the mentioned works was to improve the iteration bound (3) to:

$$\|u(t+\tau_0)\|_{H^s} \leq \|u(t)\|_{H^s} + C\|u(t)\|_{H^s}^{1-r}. \tag{5}$$

As before, $C, \tau_0 > 0$ depend only on Φ. In this bound, $r \in (0,1)$ satisfies $r \sim \frac{1}{s}$. One can show that (5) implies that for all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq C(\Phi)(1+|t|)^{\frac{1}{r}}. \tag{6}$$

In [4], (5) was obtained by using the *Fourier multiplier method*. In [46, 47], the iteration bound was obtained by using multilinear estimates in $X^{s,b}$-spaces. Similar estimates were used in the work of Kenig-Ponce-Vega [40] in the study of well-posedness theory. The key was to use a multilinear estimate in an $X^{s,b}$-space with negative first index. Such a bound was then used as a smoothing estimate. A slightly different approach, based on the analysis in the work of Burq-Gérard-Tzvetkov [11], is used to obtain (5) in the context of compact Riemannian manifolds in the work of Catoire-Wang [13], and Zhong [50].

An alternative iteration bound, based on the use of the *upside-down I-method*, which was used in our previous work [43, 44], gave better polynomial bounds for solutions of nonlinear Schrödinger equations on S^1 and $\mathbb{R}$. The main idea was to consider the operator $\mathcal{D}$, related to D^s such that $\|\mathcal{D}u\|_{L^2}$ was *slowly varying*. This is the technique which we will apply in these notes.

In the case of the linear Schrödinger equation with potential on $\mathbb{T}^d$, better results are known. In [7], Bourgain studies the equation:

$$iu_t + \Delta u = Vu. \tag{7}$$

The potential V is taken to be jointly smooth in x and t with uniformly bounded partial derivatives with respect to both of the variables. It is shown that solutions to (7) satisfy for all $\epsilon > 0$ and all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \lesssim_{s,\Phi,\epsilon} (1+|t|)^{\epsilon}. \tag{8}$$

The proof of (8) is based on separation properties of the eigenvalues of the Laplace operator on $\mathbb{T}^d$.

Recently, a new proof of (8) was given in the work of Delort [25]. The argument given in this paper is based on an iterative change of variable. In addition to recovering the result (8) on any

d-dimensional torus, the same bound is proved for the linear Schrödinger equation on any Zoll manifold, i.e. on any compact manifold whose geodesic flow is periodic. So far, it is an open problem to adapt any of these techniques to obtain bounds like (8) for nonlinear equations.

We finally mention that the problem of Sobolev norm growth was also recently studied in [24], but in the sense of bounding the growth from below. In this paper, the authors exhibit the existence of smooth solutions of the cubic defocusing nonlinear Schrödinger equation on $\mathbb{T}^2$, whose H^s norm is arbitrarily small at time zero, and is arbitrarily large at some large finite time. One should note that behavior at infinity is still an open problem. However, it is good to note that the equation (1) on $\mathbb{T}^2$ has non-trivial solutions which have all Sobolev norms uniformly bounded in time. Similarly as on S^1 [43], given $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}^2$, the function:

$$u(x,t) := \alpha e^{-i|\alpha|^2 t} e^{i(\langle n,x\rangle - |n|^2 t)}$$

is a solution to (1) on $\mathbb{T}^2$ with initial data $\Phi = \alpha e^{i\langle n,x\rangle}$. A similar construction was used in [10] to prove instability properties in Sobolev spaces of negative index.

1.3. **Techniques of the proof.** As was mentioned in the previous section, the main idea is to define $\mathcal{D}$ to be an *upside-down I-operator*. This operator is defined as a Fourier multiplier operator. By construction, we will be able to relate $\|u(t)\|_{H^s}$ to $\|\mathcal{D}u(t)\|_{L^2}$, so we consider the growth of the latter quantity. Following the ideas of the construction of the standard *I-operator*, as defined by Colliander, Keel, Staffilani, Takaoka, and Tao [17, 18, 19], our goal is to show that the quantity $\|\mathcal{D}u(t)\|_{L^2}^2$ is *slowly varying*. This is done by applying a Littlewood-Paley decomposition and summing an appropriate geometric series. Let us remark that a similar technique was applied in the low-regularity context in [18].

We will use *higher modified energies*, i.e. quantities obtained from $\|\mathcal{D}u(t)\|_{L^2}^2$ by adding an appropriate multilinear correction. In this way, we will obtain $E^2(u(t)) \sim \|\mathcal{D}u(t)\|_{L^2}^2$, which is even more slowly varying. Due to more a more complicated resonance phenomenon in two dimensions, the construction of E^2 is going to be more involved than it was in one dimension. In the periodic setting, E^2 is constructed in Subsection 3.3. In the non-periodic setting, E^2 is constructed in Subsection **??**.

We prove Theorem 1.1 for initial data Φ, which we assume lies only in $H^s(\mathbb{T}^2)$ and $H^s(\mathbb{R}^2)$, respectively. We don't assume any further regularity on the initial data. However, in the course of the proof, we work with Φ which is smooth, in order to make our formal calculations rigorous. The fact that we can do this follows from an appropriate Approximation Lemma.

2. Function spaces and techniques from Harmonic analysis.

An important tool in our work will also be $X^{s,b}$ spaces. We recall that these spaces come from the norm defined for $s, b \in \mathbb{R}$:

$$\|u\|_{X^{s,b}(\mathbb{T}^2\times\mathbb{R})} := \Big(\sum_{n\in\mathbb{Z}^2} \int_{\mathbb{R}} |\widetilde{u}(n,\tau)|^2 \langle n\rangle^{2s} \langle \tau + |n|^2\rangle^{2b} d\tau \Big)^{\frac{1}{2}}.$$

When there is no confusion, we write these spaces just as $X^{s,b}$.

In our proofs, we will frequently have to use Littlewood-Paley decompositions. Given a function $u \in L^2(\mathbb{T}^2)$ and a dyadic integer N, we define by u_N the function obtained from u by restricting its Fourier transform to the dyadic annulus $|n| \sim N$. Hence, we have:

$$u = \sum_N u_N.$$

We analogously define v_N for $v \in L^2(\mathbb{R}^2)$.

2.1. **Estimates on $\mathbb{T}^2$.** Having defined the spaces in which we will be working, let us recall some estimates which we will use in our analysis. The idea is that we want to estimate appropriate $X^{s,b}$ norms by $L^q_t L^p_x$ norms.

A key fact, which is proved in [37], is the following fact

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}}. \tag{9}$$

(A similar local-in-time estimate was earlier noted in [3].)

An additional estimate we will use is:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{4}+}}. \tag{10}$$

The estimate (10) is a consequence of the following:

Lemma 2.1. *(from [6]) Suppose that Q is a ball in $\mathbb{Z}^2$ of radius N, and center n_0. Suppose that u satisfies $supp\,\widehat{u} \subseteq Q$. Then, one has:*

$$\|u\|_{L^4_{t,x}} \lesssim N^{\frac{1}{2}}\|u\|_{X^{0,\frac{1}{4}+}}. \tag{11}$$

Lemma 2.1 is proved in [6] by using the Hausdorff-Young inequality and Hölder's inequality. We omit the details. We can now interpolate between (9) and (10) to deduce:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{s_1,b_1}}, \tag{12}$$

whenever $\frac{1}{4} < b_1 < \frac{1}{2}+, s_1 > 1 - 2b_1$.

By using an appropriate change of summation, as in [6], we see that (12) implies:

Lemma 2.2. *Suppose that u is as in the assumptions of Lemma 2.1, and suppose that $b_1, s_1 \in \mathbb{R}$ satisfy $\frac{1}{4} < b_1 < \frac{1}{2}+, s_1 > 1 - 2b_1$. Then, one has:*

$$\|u\|_{L^4_{t,x}} \lesssim N^{s_1}\|u\|_{X^{0,b_1}}. \tag{13}$$

Let us give some useful notation for multilinear expressions, which can also be found in [17, 21]. Let us first consider the periodic setting. For $k \geq 2$, an even integer, we define the hyperplane:

$$\Gamma_k := \{(n_1, \ldots, n_k) \in (\mathbb{Z}^2)^k : n_1 + \cdots + n_k = 0\},$$

endowed with the measure $\delta(n_1 + \cdots + n_k)$.

Given a function $M_k = M_k(n_1, \ldots, n_k)$ on Γ_k, i.e. a *k-multiplier*, one defines the *k-linear functional* $\lambda_k(M_k; f_1, \ldots, f_k)$ by:

$$\lambda_k(M_k; f_1, \ldots, f_k) := \int_{\Gamma_k} M_k(n_1, \ldots, n_k) \prod_{j=1}^{k} \widehat{f}_j(n_j).$$

As in [17], we adopt the notation:

$$\lambda_k(M_k; f) := \lambda_k(M_k; f, \bar{f}, \ldots, f, \bar{f}). \tag{14}$$

We will also sometimes write n_{ij} for $n_i + n_j$.

3. Proof of Theorem 1.1.

3.1. **Definition of the $\mathcal{D}$-operator.** Our proof is based on the definition of an *upside-down I operator*. This is idea is similar to that of the standard *I operator* used by Colliander, Keel, Staffilani, Takaoka, and Tao [17].

Suppose $N > 1$ is given. Let $\theta : \mathbb{Z}^2 \to \mathbb{R}$ be given by:

$$\theta(n) := \begin{cases} \left(\frac{|n|}{N}\right)^s, \text{if } |n| \geq N \\ 1, \text{ if } |n| \leq N \end{cases} \tag{15}$$

Then, if $f : \mathbb{T}^2 \to \mathbb{C}$, we define $\mathcal{D}f$ by:

$$\widehat{\mathcal{D}f}(n) := \theta(n)\hat{f}(n). \tag{16}$$

We observe that:

$$\|\mathcal{D}f\|_{L^2} \lesssim_s \|f\|_{H^s} \lesssim_s N^s \|\mathcal{D}f\|_{L^2}. \tag{17}$$

Our goal is to then estimate $\|\mathcal{D}u(t)\|_{L^2}$, from which we can estimate $\|u(t)\|_{H^s}$ by (17). In order to do this, we first need to have good local-in-time bounds.

3.2. **Local-in-time bounds.** Let u denote the global solution to (1) on $\mathbb{T}^2$. One then has:

Proposition 3.1. *(Local-in-time bounds for the Hartree equation on $\mathbb{T}^2$) There exist $\delta = \delta(s, E(\Phi), M(\Phi)), C = C(s, E(\Phi), M(\Phi)) > 0$, which are continuous in energy and mass, such that for all $t_0 \in \mathbb{R}$, there exists a globally defined function $v : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{C}$ such that:*

$$v|_{[t_0, t_0+\delta]} = u|_{[t_0, t_0+\delta]}. \tag{18}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)). \tag{19}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \|\mathcal{D}u(t_0)\|_{L^2}. \tag{20}$$

The proof of Proposition 3.1 is based on a fixed-point argument. We need to use the estimates in $X^{s,b}$ spaces mentioned above in order to show that we obtain a contraction.

Although our statements concern functions which are only assumed to belong to $H^s(\mathbb{T}^2)$, we can work with smooth functions and deduce the general result from the following:

Proposition 3.2. *(Approximation Lemma for the cubic NLS on $\mathbb{T}^2$)*
If Φ satisfies:

$$\begin{cases} iu_t + \Delta u = |u|^2 u, \\ u(x, 0) = \Phi(x). \end{cases} \tag{21}$$

and if the sequence $(u^{(n)})$ satisfies:

$$\begin{cases} iu_t^{(n)} + \Delta u^{(n)} = |u^{(n)}|^2 u^{(n)}, \\ u^{(n)}(x, 0) = \Phi_n(x). \end{cases} \tag{22}$$

where $\Phi_n \in C^\infty(\mathbb{T}^2)$ and $\Phi_n \xrightarrow{H^s} \Phi$, then, one has for all t:

$$u^{(n)}(t) \xrightarrow{H^s} u(t).$$

We remark that it is crucial that none of our estimates depend on higher Sobolev norms than H^s, which allows us to argue by density.

3.3. **A higher modified energy and an iteration bound.** We define the following *modified energy*:

$$E^1(u(t)) := \|\mathcal{D}u(t)\|_{L^2}^2.$$

By a calculation, we obtain that for some $c \in \mathbb{R}$, one has:

$$\frac{d}{dt}E^1(u(t)) = ic \sum_{n_1+n_2+n_3+n_4=0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - ((\theta(n_4))^2\big)$$

$$\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4) \tag{23}$$

One then considers the *higher modified energy*:

$$E^2(u) := E^1(u) + \lambda_4(M_4; u) \tag{24}$$

The quantity M_4 will be determined soon.

The modified energy E^2 is obtained by adding a "multilinear correction" to the modified energy E^1 we considered earlier. In order to find $\frac{d}{dt}E^2(u)$, we need to find $\frac{d}{dt}\lambda_4(M_4; u)$. If we fix a multiplier M_4, we obtain:

$$\frac{d}{dt}\lambda_4(M_4; u) =$$

$$-i\lambda_4(M_4(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2); u)$$

$$-i \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \big[M_4(n_{123}, n_4, n_5, n_6)$$

$$-M_4(n_1, n_{234}, n_5, n_6) + M_4(n_1, n_2, n_{345}, n_6)$$

$$-M_4(n_1, n_2, n_3, n_{456})\big]\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)\widehat{u}(n_5)\widehat{\bar{u}}(n_6). \tag{25}$$

We can compute that for $(n_1, n_2, n_3, n_4) \in \Gamma_4$, one has:

$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 2n_{12} \cdot n_{14}. \tag{26}$$

We notice that the numerator vanishes not only when $n_{12} = n_{14} = 0$, but also when n_{12} and n_{14} are orthogonal. Hence, on Γ_4, it is possible for $|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2$ to vanish, but for $(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2$ to be non-zero. Consequently, unlike in the 1D setting [43, 44], we can't cancel the whole quadrilinear term in (23). We remedy this by canceling the *non-resonant part* of the quadrilinear term. A similar technique was used in [22]. There, it was given the name *resonant decomposition*. More precisely, given $\beta_0 \ll 1$, which we determine later, we decompose:

$$\Gamma_4 = \Omega_{nr} \sqcup \Omega_r.$$

Here, the set Ω_{nr} of *non-resonant* frequencies is defined by:

$$\Omega_{nr} := \{(n_1, n_2, n_3, n_4) \in \Gamma_4; n_{12}, n_{14} \neq 0, |cos\angle(n_{12}, n_{14})| > \beta_0\} \tag{27}$$

and the set Ω_r of *resonant* frequencies Ω_r is defined to be its complement in Γ_4.

We now define the multiplier M_4 by:

$$M_4(n_1,n_2,n_3,n_4):=\begin{cases} c\frac{((\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2)}{|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2}\text{, if } (n_1,n_2,n_3,n_4)\in\Omega_{nr} \\ 0\text{, if } (n_1,n_2,n_3,n_4)\in\Omega_r. \end{cases} \tag{28}$$

Let us now define the multiplier M_6 on Γ_6 by:

$$\begin{aligned} M_6(n_1,n_2,n_3,n_4,n_5,n_6) &:= M_4(n_{123},n_4,n_5,n_6)-M_4(n_1,n_{234},n_5,n_6)+ \\ &+M_4(n_1,n_2,n_{345},n_6)-M_4(n_1,n_2,n_3,n_{456}). \end{aligned} \tag{29}$$

We now use (23) and (25), and the construction of M_4 and M_6 to deduce that:

$$\begin{aligned} &\frac{d}{dt}E^2(u)= \\ &\sum_{n_1+n_2+n_3+n_4=0,|cos\angle(n_{12},n_{14})|\leq\beta_0}\big((\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2\big)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)+ \\ &+\sum_{n_1+n_2+n_3+n_4+n_5+n_6=0}M_6(n_1,n_2,n_3,n_4,n_5,n_6)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)\widehat{u}(n_5)\widehat{\bar{u}}(n_6) \\ &=: I+II. \end{aligned} \tag{30}$$

Before we proceed, we need to prove pointwise bounds on the multiplier M_4. In order to do this, let $(n_1,n_2,n_3,n_4)\in\Gamma_4$ be given. We dyadically localize the frequencies, i.e, we find dyadic integers N_j s.t. $|n_j|\sim N_j$. We then order the N_j's to obtain: $N_1^*\geq N_2^*\geq N_3^*\geq N_4^*$. We slightly abuse notation by writing $\theta(N_j^*)$ for $\theta(N_j^*,0)$.

Lemma 3.3. *With notation as above, the following bound holds:*

$$M_4=O\big(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big). \tag{31}$$

Proof. By construction of the set Ω_{nr}, we note that:

$$|M_4|\lesssim\frac{|(\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2|}{|n_{12}||n_{14}|\beta_0}. \tag{32}$$

Let us assume, without loss of generality, that:

$$|n_1|\geq|n_2|,|n_3|,|n_4|,\text{ and }|n_{12}|\geq|n_{14}|. \tag{33}$$

We now have to consider three cases:

Case 1: $|n_1|\sim|n_{12}|\sim|n_{14}|$

In this Case, one has:

$$M_4=O\big(\frac{1}{\beta_0}\frac{(\theta(n_1))^2}{|n_1|^2}\big)=O\big(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big).$$

Case 2: $|n_1|\sim|n_{12}|\gg|n_{14}|$

We use the *Mean Value Theorem*, and monotonicity properties of the function $\frac{(\theta(n))^2}{|n|}$ to deduce:

$$(\theta(n_1))^2-(\theta(n_4))^2=(\theta(n_1))^2-(\theta(n_1-n_{14}))^2=O\big(|n_{14}|\frac{(\theta(n_1))^2}{|n_1|}\big). \tag{34}$$

$$(\theta(n_2))^2-(\theta(n_3))^2=(\theta(n_3+n_{14}))^2-(\theta(n_3))^2=$$

$$O\big(|n_{14}|\sup_{N\leq|z|\lesssim|n_1|}\frac{(\theta(z))^2}{|z|}\big)=O\big(|n_{14}|\frac{(\theta(n_1))^2}{|n_1|}\big). \tag{35}$$

Using (32), (34), (35), and the fact that $|n_{12}|\sim|n_1|$, it follows that:

$$M_4=O\big(\frac{(\theta(n_1))^2}{|n_1|^2\beta_0}\big)=O\big(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big).$$

Case 3: $|n_1|\gg|n_{12}|,|n_{14}|$

We write:

$$(\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2=(\theta(n_1))^2-(\theta(n_1-n_{12}))^2+(\theta(n_1-n_{12}-n_{14}))^2-(\theta(n_1-n_{14}))^2.$$

By using the *Double Mean-Value Theorem*, it follows that this expression is $O\big(\frac{(\theta(n_1))^2}{|n_1|^2}|n_{12}||n_{14}|\big)$.

Consequently:

$$M_4=O\big(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big).$$

The Lemma now follows.

□

Let us choose:

$$\beta_0\sim\frac{1}{N}. \tag{36}$$

The reason why we choose such a β_0 will become clear later. For details, see Remark 3.6.

Hence Lemma 3.3 implies:

$$M_4=O\big(\frac{N}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big). \tag{37}$$

The bound from (37) allows us to deduce the equivalence of E^1 and E^2. We have the following bound:

Proposition 3.4. *For each fixed time t, one has:*

$$E^1(u(t))\sim E^2(u(t)). \tag{38}$$

Here, the constant is independent of t and N, as long as N is sufficiently large.

This claim is proved by dyadically decomposing the factors of u in frequency space and summing the appropriate components. We omit the details.

3.3.1. *The iteration bound:* Let $\delta>0, v$ be as in Proposition 3.1. For $t_0\in\mathbb{R}$, we are interested in estimating:

$$E^2(u(t_0+\delta))-E^2(u(t_0))=\int_{t_0}^{t_0+\delta}\frac{d}{dt}E^2(u(t))dt=\int_{t_0}^{t_0+\delta}\frac{d}{dt}E^2(v(t))dt.$$

The iteration bound that we will show is:

Lemma 3.5. *For all $t_0\in\mathbb{R}$, one has:*

$$\big|E^2(u(t_0+\delta))-E^2(u(t_0))\big|\lesssim\frac{1}{N^{1-}}E^2(u(t_0)).$$

Arguing similarly as in [43, 44], Theorem 1.1 will follow from Lemma 3.5. We recall the proof for completeness.

Proof. (of Theorem 1.1 assuming Lemma 3.5)
The point is that we can iterate the following bound (obtained from Lemma 3.5):

$$E^2(u(t_0+\delta)) \leq (1+\frac{C}{N^{1-}})E^2(u(t_0))$$

$\sim N^{1-}$ times with a uniform time step, and the size of $E^2(t)$ will grow by at most a constant factor (and not as an exponential function in t). We hence obtain that for $T \sim N^{1-}$, one has:

$$\|\mathcal{D}u(T)\|_{L^2} \lesssim \|\mathcal{D}\Phi\|_{L^2}.$$

By recalling (17), it follows that:

$$\|u(T)\|_{H^s} \lesssim N^s \|\Phi\|_{H^s}$$

and hence:

$$\|u(T)\|_{H^s} \lesssim T^{s+}\|\Phi\|_{H^s} \lesssim (1+T)^{s+}\|\Phi\|_{H^s}.$$

This proves Theorem 1.1 for times $t \geq 1$. The claim for times $t \in [0,1]$ follows by local well-posedness theory. The claim for negative times holds by time-reversibility.

□

We now have to prove Lemma 3.5.

Proof. (of Lemma 3.5)
Let us without loss of generality consider $t_0 = 0$. The general claim will follow by time translation, and the fact that all of the implied constants are uniform in time. Let v be the function constructed in Proposition 3.1, corresponding to $t_0 = 0$.

By (30), and with notation as in this equation, we need to estimate:

$$\int_0^\delta \Big(\sum_{n_1+n_2+n_3+n_4=0, |cos\angle(n_{12},n_{14})|\leq \beta_0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big)\widehat{v}(n_1)\widehat{\bar{v}}(n_2)\widehat{v}(n_3)\widehat{\bar{v}}(n_4) +$$

$$+ \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_6(n_1,n_2,n_3,n_4,n_5,n_6)\widehat{v}(n_1)\widehat{\bar{v}}(n_2)\widehat{v}(n_3)\widehat{\bar{v}}(n_4)\widehat{v}(n_5)\widehat{\bar{v}}(n_6)\Big)dt =$$

$$= \int_0^\delta I dt + \int_0^\delta II dt =: A + B.$$

We now have to estimate A and B separately. Throughout our calculations, let us denote by $\chi = \chi(t) = \chi_{[0,\delta]}(t)$.

3.3.2. *Estimate of A (Quadrilinear Terms).* By symmetry, we can consider without loss of generality the contribution when:

$$|n_1| \geq |n_2|, |n_3|, |n_4|, \text{and } |n_2| \geq |n_4|.$$

We note that when all $|n_j| \leq N$, one has: $(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 = 0$. Hence, we need to consider the contribution in which one has:

$$|n_1| > N, |cos\angle(n_{12},n_{14})| \leq \beta_0.$$

We dyadically localize the frequencies: $|n_j| \sim N_j; j = 1,\ldots,4$. We order the N_j to obtain $N_j^* \geq N_2^* \geq N_3^* \geq N_4^*$. Since $n_1+n_2+n_3+n_4 = 0$, we know that:

$$N_1^* \sim N_2^* \gtrsim N. \tag{39}$$

Let us note that $N_1 \sim N_2$. Namely, if it were the case that: $N_1 \gg N_2$, then, one would also have: $N_1 \gg N_4$, and the vectors n_{12} and n_{14} would form a very small angle. Hence, $cos\angle(n_{12}, n_{14})$ would be close to 1, which would be a contradiction to the assumption that $|cos\angle(n_{12}, n_{14})| \leq \beta_0$. Consequently:

$$N_1 \sim N_2 \sim N_1^* \gtrsim N. \tag{40}$$

We denote the corresponding contribution to A by A_{N_1,N_2,N_3,N_4}. In other words:

$$A_{N_1,N_2,N_3,N_4} :=$$
$$\int_0^\delta \sum_{n_1+n_2+n_3+n_4=0, |cos\angle(n_{12},n_{14})|\leq\beta_0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big)$$
$$\widehat{v}_{N_1}(n_1)\widehat{\overline{v}}_{N_2}(n_2)\widehat{v}_{N_3}(n_3)\widehat{\overline{v}}_{N_4}(n_4)dt.$$

Arguing analogously as in the proof of Lemma 3.3, it follows that for the n_j that occur in the above sum, one has:

$$\big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big) = O\big(|n_{12}||n_{14}|\frac{\theta(N_1^*)\theta(N_2^*)}{(N_1^*)^2}\big). \tag{41}$$

By (40), it follows that $|n_3|, |n_4| \lesssim N_3^*$. Consequently:

$$|n_{12}| = |n_{34}| \leq |n_3| + |n_4| \lesssim N_3^*.$$

One also knows that:

$$|n_{14}| \leq |n_1| + |n_4| \lesssim N_1^*.$$

Substituting the last two inequalities into the multiplier bound (41), and using Parseval's identity in time, it follows that:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \sum_{n_1+n_2+n_3+n_4=0, |cos\angle(n_{12},n_{14})|\leq\beta_0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} N_3^* N_1^* \frac{\theta(N_1^*)\theta(N_2^*)}{(N_1^*)^2}$$
$$|\widetilde{v}_{N_1}(n_1,\tau_1)||\widetilde{\overline{v}}_{N_2}(n_2,\tau_2)||\widetilde{v}_{N_3}(n_3,\tau_3)||(\chi\overline{v})\widetilde{\ }_{N_4}(n_4,\tau_4)|d\tau_j.$$

$$\lesssim \frac{1}{N_1^*} \sum_{n_1+n_2+n_3+n_4=0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} |(\mathcal{D}v)\widetilde{\ }_{N_1}(n_1,\tau_1)||(\mathcal{D}\overline{v})\widetilde{\ }_{N_2}(n_2,\tau_2)||(\nabla v)\widetilde{\ }_{N_3}(n_3,\tau_3)||(\chi\overline{v})\widetilde{\ }_{N_4}(n_4,\tau_4)|d\tau_j.$$

Let us define $F_j; j = 1, \ldots, 4$ by:

$$\widetilde{F_1} := |(\mathcal{D}v)\widetilde{\ }_{N_1}|, \widetilde{F_2} := |(\mathcal{D}v)\widetilde{\ }_{N_2}|, \widetilde{F_3} := |(\nabla v)\widetilde{\ }_{N_3}|, \widetilde{F_4} := |(\chi v)\widetilde{\ }_{N_4}|.$$

Consequently, by Parseval's identity:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1^*} \int_{\mathbb{R}} \int_{\mathbb{T}^2} F_1 \overline{F_2} F_3 \overline{F_4} dxdt$$

By using an $L^4_{t,x}, L^4_{t,x}, L^4_{t,x}, L^4_{t,x}$ Hölder inequality, the corresponding term is: [1]

$$\lesssim \frac{1}{N_1^*} \|F_1\|_{L^4_{t,x}} \|F_2\|_{L^4_{t,x}} \|F_3\|_{L^4_{t,x}} \|F_4\|_{L^4_{t,x}}$$

By using (9), and the fact that taking absolute values in the spacetime Fourier transforms doesn't change the $X^{s,b}$ norm, it follows that this term is:

[1]Strictly speaking, we are using an $L^4_{t,x}, L^4_{t,x}, L^{4+}_{t,x}, L^{4-}_{t,x}$ Hölder inequality, as well as estimates similar to (9) to estimate the $L^{4+}_{t,x}$, and $L^{4-}_{t,x}$ norm and appropriate time-localization properties of the $X^{s,b}$ spaces. We omit the details.

$$\lesssim \frac{1}{N_1^*}\|\mathcal{D}v_{N_1}\|_{X^{0+,\frac{1}{2}+}}\|\mathcal{D}v_{N_2}\|_{X^{0+,\frac{1}{2}+}}\|v_{N_3}\|_{X^{1,\frac{1}{2}+}}\|v_{N_4}\|_{X^{0+,\frac{1}{2}+}}$$

By using frequency localization, this expression is:

$$\lesssim \frac{1}{(N_1^*)^{1-}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}} \lesssim \frac{1}{(N_1^*)^{1-}}E^1(\Phi).$$

In the last inequality, we used Proposition 3.1. By using the previous inequality, and by recalling (38), it follows that:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{(N_1^*)^{1-}}E^2(\Phi). \tag{42}$$

Using (42), summing in the N_j, and using (39) to deduce that:

$$|A| \lesssim \frac{1}{N^{1-}}E^2(\Phi). \tag{43}$$

3.3.3. *Estimate of B (Sextilinear Terms).* Let us consider just the first term in B coming from the summand $M_4(n_{123}, n_4, n_5, n_6)$ in the definition of M_6. The other terms are bounded analogously. In other words, we want to estimate:

$$B^{(1)} := \int_0^\delta \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_4(n_{123}, n_4, n_5, n_6)\widehat{(v\bar{v}v)}(n_1+n_2+n_3)\widehat{\bar{v}}(n_4)\widehat{v}(n_5)\widehat{\bar{v}}(n_6)dt$$

We now dyadically localize n_{123}, n_4, n_5, n_6, i.e., we find $N_j; j = 1, \ldots, 4$ such that:

$$|n_{123}| \sim N_1, |n_4| \sim N_2, |n_5| \sim N_3, |n_6| \sim N_4.$$

Let us define:

$$B^{(1)}_{N_1,N_2,N_3,N_4} := \int_0^\delta \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_4(n_{123}, n_4, n_5, n_6)\widehat{(v\bar{v}v)}_{N_1}(n_1+n_2+n_3)\widehat{\bar{v}}_{N_2}(n_4)\widehat{v}_{N_3}(n_5)\widehat{\bar{v}}_{N_4}(n_6)dt$$

We now order the N_j to obtain: $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. As before, we know the following localization bound:

$$N_1^* \sim N_2^* \gtrsim N. \tag{44}$$

In order to obtain a bound on the wanted term, we have to consider two cases, depending on whether N_1 is among the two larger frequencies or not. An argument similar to the estimate of the quadrilinear terms gives:

$$|B_{N_1,N_2,N_3,N_4}| \lesssim \frac{N}{(N_1^*)^{2-}}E^2(\Phi). \tag{45}$$

We now use (45), sum in the N_j, and recall (44) to deduce that:

$$|B| \lesssim \frac{1}{N^{1-}}E^2(\Phi). \tag{46}$$

The Lemma now follows from (43) and (46).

□

3.4. Further remarks on the equation.

Remark 3.6. *The quantity β_0 was chosen as in* (36) *in order to get the same decay factor in the quantities A and B. We note that the quantity β_0 only occurred in the bound for B, whereas in the bound for A, we only used the fact that the terms corresponding to the largest two frequencies in the multiplier $(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2$ appear with an opposite sign.*

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