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PRIMITIVE LATTICE POINTS INSIDE AN ELLIPSE

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To Professor Ekkehard Krätzel on his 70th birthday

Abstract. Let $Q(u,v)$ be a positive definite binary quadratic form with arbitrary real coefficients. For large real x, one may ask for the number $B(x)$ of primitive lattice points (integer points (m,n) with $\gcd(M,n)=1$) in the ellipse disc $Q(u,v)\leqslant x$, in particular, for the remainder term $R(x)$ in the asymptotics for $B(x)$. While upper bounds for $R(x)$ depend on zero-free regions of the zeta-function, and thus, in most published results, on the Riemann Hypothesis, the present paper deals with a lower estimate. It is proved that the absolute value or $R(x)$ is, in integral mean, at least a positive constant c time $x^{1/4}$. Furthermore, it is shown how to find an explicit value for c, for each specific given form Q.

Keywords: primitive lattice points, lattice point discrepancy, planar domains

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1. Introduction

Let $Q=Q(m,n)=am^2+bmn+cn^2$ be a positive definite binary quadratic form, where a, b, c are *arbitrary* real numbers with $a>0$, $D:=4ac-b^2>0$. For a large parameter x, we consider the lattice point quantities

$$\begin{aligned} A(x)&=\#\{(m,n)\in\mathbb{Z}^2\colon\ Q(m,n)\leqslant x\},\\ B(x)&=\#\{(m,n)\in\mathbb{Z}^2\colon\ Q(m,n)\leqslant x,\ \ \gcd(m,n)=1\},\end{aligned}\tag{1.1}$$

which count the number of *all*, resp., of all *primitive* lattice points in the ellipse disc $Q\leqslant x$. It is well known that

$$A(x)=\frac{2\pi}{\sqrt{D}}\,x+P(x),\qquad B(x)=\frac{12}{\pi\sqrt{D}}\,x+R(x),\tag{1.2}$$

where $P(x)$, $R(x)$ are error terms on which a lot of research has been done. (For an enlightening presentation of this theory, see the monograph of Krätzel [11].) As far as $P(x)$ is concerned, the sharpest published[1] results read

$$P(x) \ll x^{23/73} (\log x)^{315/146}, \tag{1.3}$$

$$\liminf_{x\to\infty} \frac{P(x)}{x^{1/4}(\log x)^{1/4}} < 0, \tag{1.4}$$

and

$$\int_0^T (P(t^2))^2 \,\mathrm{d}t \sim C_Q T^2. \tag{1.5}$$

They are due to M. Huxley [6], [7], the author [15], P. Bleher [1] and the author [16].[2] All these estimates have been proved for general convex planar domains with smooth boundary of nonvanishing curvature.

The question for analogous results about $R(x)$ remains much more enigmatic. To see why, we recall that the generating Dirichlet series corresponding to $P(x)$, resp., $A(x)$, is the Epstein zeta-function

$$\zeta_Q(s) = \sum_{(m,n)\in\mathbb{Z}_*^2} Q(m,n)^{-s} \qquad (\Re(s) > 1), \tag{1.6}$$

where $\mathbb{Z}_*^2 := \mathbb{Z}^2 \setminus \{(0,0)\}$. It possesses an analytic continuation to the whole complex plane, with the exception of a simple pole at $s = 1$, and satisfies a functional equation

$$\zeta_Q(s) = \left(\frac{2\pi}{\sqrt{D}}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)}\, \zeta_Q(1-s). \tag{1.7}$$

(See Potter [18], or, for a multivariate version, Krätzel's monograph [12], p. 202.) By Vinogradov's Lemma, the generating function of $B(x)$ reads, for $\Re(s) > 1$,

$$\sum_{\substack{(m,n)\in\mathbb{Z}_*^2 \\ \gcd(m,n)=1}} Q(m,n)^{-s} = \sum_{k=1}^{\infty} \mu(k) \sum_{(m,n)\in\mathbb{Z}_*^2} Q(km,kn)^{-s} = \frac{\zeta_Q(s)}{\zeta(2s)}. \tag{1.8}$$

[1] Actually, M. Huxley has meanwhile improved further his upper bound, essentially replacing the exponent $\frac{23}{73} = 0.315068\ldots$ by $\frac{131}{416} = 0.314903\ldots$ [8]. The author is indebted to Professor Huxley for sending him a copy of his unpublished manuscript.

[2] In this latter reference, actually a short interval version of this asymptotics is established. We omit the discussion of a possible error term in (1.5) which, for the case of a general ellipse, is by no means simple.

By Perron's formula, for every value of $x > 0$ which is not attained by $Q(m,n)$, $(m,n) \in \mathbb{Z}_*^2$,

$$B(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta_Q(s)}{\zeta(2s)} \frac{x^s}{s} \, ds.$$

Shifting the line of integration to the left, we are confronted with the lack of information about the zeros of the Riemann zeta-function[3]: These might come close to $\Re(s) = 1$, hence an estimate $R(x) \ll x^\theta$ cannot be proved for any $\theta < \frac{1}{2}$, at the present state of art. The best known upper bound is

$$R(x) = O\big(x^{1/2} \exp\big(-C(\log x)^{3/5} (\log\log x)^{-1/5}\big)\big).$$

Several authors have investigated this problem under the assumption of the Riemann Hypothesis. After previous work by Huxley & Nowak [9] and by W. Müller [14], the sharpest *conditional* results of this kind are due to W. Zhai [25] and read $R(x) \ll x^{221/608+\varepsilon}$ for a rational form Q, and $R(x) \ll x^{33349/84040+\varepsilon}$ in general. (Note that $\frac{221}{608} = 0.3634\ldots$, $\frac{33349}{84040} = 0.3968\ldots$.) See also Zhai & Cao [24] and Wu [23].

There is little hope to establish estimates for $R(x)$ which are directly analogous to (1.4) and (1.5).

Nevertheless, in the present paper we shall prove a result which says that at least the lower bound part of (1.5) holds true for $R(x)$ also.[4] Trivially, this implies a pointwise Ω-result for $R(x)$, which is comparable to, though slightly weaker than, formula (1.4).

Theorem. *The error term $R(x)$ defined in* (1.1), (1.2) *satisfies*

(1.9)
$$\frac{1}{Y} \int_1^Y |R(x)| \, dx \gg Y^{1/4},$$

as $Y \to \infty$, the $\gg$-constant depending on the form Q.

[3] For an enlightening presentation of its theory the reader is referred to the monograph of A. Ivić [10].

[4] Ironically, our analysis actually will yield this result not *although* there is the cumbersome denominator $\zeta(2s)$ in (1.8), but *because* it is there.

2. A ZERO-DENSITY BOUND FOR EPSTEIN ZETA-FUNCTIONS.[5]

Lemma. *For any positive definite binary quadratic form Q, $\sigma \in \mathbb{R}$ and $T \in \mathbb{R}^+$, denote by $N_Q^*(\sigma, T)$ the number of zeros (counted with multiplicity) of $\zeta_Q(s)$ with $\Re(s) = \sigma$, $|\Im(s)| \leqslant T$, and put $N_Q(\sigma, T) = \sum_{\sigma' \geqslant \sigma} N_Q^*(\sigma', T)$. Then, as $T \to \infty$,*

$$N_Q^*\Big(\frac{1}{4}, T\Big) = N_Q^*\Big(\frac{3}{4}, T\Big) \leqslant N_Q\Big(\frac{3}{4}, T\Big) = o(T \log T).$$

P r o o f. First of all, $N_Q^*(\frac{1}{4}, T) = N_Q^*(\frac{3}{4}, T)$ is clear by the functional equation (1.7). To establish the o-assertion, one can follow the classical example of Titchmarsh's monograph [20], section 9.15. We rewrite (1.6), for $\Re(s) > 1$, as

$$\zeta_Q(s) = \sum_{k=1}^{\infty} r_k \lambda_k^{-s} = \lambda_1^{-s}(r_1 + U(s)),$$

where $r_k \in \mathbb{N}^*$ and (λ_k) is a strictly increasing sequence of positive reals. Since $U(\sigma + \mathrm{i}t) \to 0$ as $\sigma \to \infty$, uniformly in t, there exists some $\sigma^* > 1$ (depending on Q) such that $|U(\sigma + \mathrm{i}t)| \leqslant \frac{1}{2} r_1$ for $\sigma \geqslant \sigma^*$ and all t. As a consequence,

$$|\zeta_Q(\sigma^* + \mathrm{i}t)| \geqslant \frac{1}{2} r_1 \lambda_1^{-\sigma^*} \tag{2.1}$$

for all t, and $\zeta_Q(s) \neq 0$ for $\Re(s) \geqslant \sigma^*$. Let further $\mathscr{T}_Q := \{t \in \mathbb{R}\colon\ \cos(t \log \lambda_1) \geqslant \frac{3}{4}\}$, then

$$|\Re(\zeta_Q(\sigma^* + \mathrm{i}t))| \geqslant \frac{1}{4} r_1 \lambda_1^{-\sigma^*} \tag{2.2}$$

for all $t \in \mathscr{T}_Q$.

We use a variant of formula (9.9.1) in [20] ("Littlewood's Lemma"): If $\alpha > 0$ and $T > 0$, $T \in \mathscr{T}_Q$[6] are such that there are no zeros of $\zeta_Q(s)$ on $\Re(s) = \alpha$ and on $|\Im(s)| = T$, then

$$\int_{\mathscr{R}} \log \zeta_Q(s)\, \mathrm{d}s = -2\pi \mathrm{i} \int_{\alpha}^{\sigma^*} N_Q(\sigma, T)\, \mathrm{d}\sigma + O(1), \tag{2.3}$$

[5] The result stated suffices for our purpose and will be believed at first glance by the expert. However, it is difficult to find it explicitly in the literature. Further, it cannot be improved substantially: As Davenport & Heilbronn [3], [4], and M. Voronin [21] showed, if Q is an integral form of class number exceeding 1, then $N_Q(1, T) \gg T$ and also $N_Q(\alpha, T) - N_Q(1, T) \gg T$ for $\frac{1}{2} < \alpha < 1$.

[6] Obviously, for any given $T_0 \in \mathbb{R}^+$, there exists some $T \in \mathscr{T}_Q$ with $T_0 \leqslant T \ll T_0$.

where $\mathscr{R}$ is the rectangle $(\alpha \pm \mathrm{i}T)$, $(\sigma^* \pm \mathrm{i}T)$, and the logarithm is defined (almost everywhere) by

$$\log \zeta_Q(\sigma + \mathrm{i}t) = \log \zeta_Q(\sigma^*) + \int_{\mathscr{C}} \frac{\zeta_Q'(s)}{\zeta_Q(s)}\,\mathrm{d}s$$

where $\log \zeta_Q(\sigma^*) \in \mathbb{R}$ and $\mathscr{C}$ consists of the two straight line segments from σ^* to $\sigma^* + \mathrm{i}t$ and further to $\sigma + \mathrm{i}t$. Moreover, let $\arg \zeta_Q(s) := \Im(\log \zeta_Q(s))$. Taking the imaginary part of (2.3), we get

$$\begin{aligned} 2\pi \int_\alpha^{\sigma^*} N_Q(\sigma, T)\,\mathrm{d}\sigma = {} & \int_{-T}^{T} \log|\zeta_Q(\alpha + \mathrm{i}t)|\,\mathrm{d}t - \int_{-T}^{T} \log|\zeta_Q(\sigma^* + \mathrm{i}t)|\,\mathrm{d}t \\ & + \int_\alpha^{\sigma^*} \arg \zeta_Q(\sigma + \mathrm{i}T)\,\mathrm{d}\sigma - \int_\alpha^{\sigma^*} \arg \zeta_Q(\sigma - \mathrm{i}T)\,\mathrm{d}\sigma + O(1). \end{aligned}$$

By (2.1), the second integral on the right-hand side is $O(T)$. We mimick the argument in section 9.4 of [20] to show that (at least)

$$\arg \zeta_Q(\sigma \pm \mathrm{i}T) = O(T) \tag{2.4}$$

uniformly in $\alpha \leqslant \sigma \leqslant \sigma^*$. This will readily yield

$$2\pi \int_\alpha^{\sigma^*} N_Q(\sigma, T)\,\mathrm{d}\sigma = \int_{-T}^{T} \log|\zeta_Q(\alpha + \mathrm{i}t)|\,\mathrm{d}t + O(T), \tag{2.5}$$

for any fixed $\alpha > 0$ and $T \to \infty$. To prove (2.4), we note first that $\zeta_Q'(s)/\zeta_Q(s)$ is bounded on $\Re(s) = \sigma^*$, hence $\arg \zeta_Q(\sigma^* \pm \mathrm{i}T) = O(T)$. The variation of $\arg \zeta_Q(\sigma \pm \mathrm{i}T)$ on $\alpha \leqslant \sigma \leqslant \sigma^*$ is $\ll 1 + q$, q being the number of zeros of $\Re(\zeta_Q(\sigma \pm \mathrm{i}T))$ on this line segment. Further, $q \leqslant n(\sigma^* - \alpha)$, if $n(r)$ denotes the number of zeros (counted with multiplicity) of the function $G(s) := \frac{1}{2}(\zeta_Q(s \pm \mathrm{i}T) + \zeta_Q(s \mp \mathrm{i}T))$ in the disc $|s - \sigma^*| \leqslant r$. Now

$$\int_0^{\sigma^* - \frac{1}{2}\alpha} \frac{n(r)}{r}\,\mathrm{d}r \geqslant \int_{\sigma^* - \alpha}^{\sigma^* - \frac{1}{2}\alpha} \frac{n(r)}{r}\,\mathrm{d}r \gg n(\sigma^* - \alpha),$$

and, by Jensen's theorem,

$$\int_0^{\sigma^* - \frac{1}{2}\alpha} \frac{n(r)}{r}\,\mathrm{d}r = \frac{1}{2\pi} \int_0^{2\pi} \log\Big|G\Big(\sigma^* + \Big(\sigma^* - \frac{1}{2}\alpha\Big)\mathrm{e}^{\mathrm{i}\theta}\Big)\Big|\,\mathrm{d}\theta - \log|G(\sigma^*)| \ll \log T,$$

since $|G(\sigma^*)| \gg 1$ because of $T \in \mathscr{T}_q$ and (2.2). This establishes (2.4) and thus (2.5).

According to W. Müller [14][7], Proposition 2, at least for every $\alpha \geqslant \frac{2}{3}$,

$$\int_0^T |\zeta_Q(\alpha + \mathrm{i}t)|^2 \, \mathrm{d}t \ll T^{1+\varepsilon}$$

for any $\varepsilon > 0$. Hence, by Jensen's inequality (e.g., [5], p. 1132) and the reflection principle, for suitable $\alpha \in \left]\frac{2}{3}, \frac{3}{4}\right[$,

$$\int_{-T}^T \log|\zeta_Q(\alpha + \mathrm{i}t)| \, \mathrm{d}t \leqslant T \log\left(\frac{1}{T}\int_0^T |\zeta_Q(\alpha + \mathrm{i}t)|^2 \, \mathrm{d}t\right) \ll \varepsilon T \log T.$$

Thus, by (2.5), for $\sigma_0 = \frac{1}{2}(\alpha + \frac{3}{4})$,

$$N_Q(\sigma_0, T) \leqslant \frac{1}{\sigma_0 - \alpha}\int_\alpha^{\sigma_0} N_Q(\sigma, T) \, \mathrm{d}\sigma \ll \varepsilon T \log T.$$

Since $\varepsilon > 0$ is arbitrary, this establishes the lemma. □

3. Proof of the Theorem

Following an idea due to Pintz [17], we consider the Mellin transform, for $\Re(s) > 1$,

$$\begin{aligned}
H(s) &:= \int_1^\infty R(x) x^{-s-1} \, \mathrm{d}x \\
&= \int_1^\infty \Bigg(\sum_{\substack{Q(m,n) \leqslant x \\ \gcd(m,n)=1}} 1 - \frac{12}{\pi\sqrt{D}}\, x \Bigg) x^{-s-1} \, \mathrm{d}x \\
&= \sum_{\substack{(m,n)\in\mathbb{Z}^2_* \\ \gcd(m,n)=1}} \int_{Q(m,n)}^\infty x^{-1-s} \, \mathrm{d}x - \frac{12}{\pi\sqrt{D}} \int_1^\infty x^{-s} \, \mathrm{d}x \\
&= \frac{\zeta_Q(s)}{s\zeta(2s)} - \frac{12}{\pi\sqrt{D}} \frac{1}{s-1} =: \frac{E(s)}{s(s-1)\zeta(2s)(2s-1)}.
\end{aligned} \tag{3.1}$$

Obviously $H(s)$ possesses a meromorphic continuation to all of $\mathbb{C}$, with $E(s)$ an entire function. Now choose $z_0 = \frac{1}{4} + \mathrm{i}\beta_0$ such that $2z_0$ is a zero of the Riemann zeta-function and $\zeta_Q(z_0) \neq 0$. (The existence follows from the above lemma and a

[7] In fact, Müller proves this bound more generally for the Hlawka zeta-function of a convex planar domain with smooth boundary of nonvanishing curvature. Similar results can be found in Huxley & Nowak [9] and in W. Zhai [25].

celebrated result of Selberg [19], refined further by Levinson [13] and Conrey [2].) The function

$$g(s) := \frac{s(s-1)\zeta(2s)(2s-1)}{(s-z_0)(s+2)^7} \tag{3.2}$$

is regular in $\Re(s) > -2$, and so is

$$g(s)H(s) = \frac{E(s)}{(s-z_0)(s+2)^7}, \tag{3.3}$$

apart from a simple pole at $s = z_0$, since $E(z_0) = (z_0-1)(2z_0-1)\zeta_Q(z_0) \neq 0$. By the functional equation (1.7), $\zeta_Q(-1+\mathrm{i}t) \ll |t|^3$, and similarly $\zeta(-2+2\mathrm{i}t) \asymp |t|^{5/2}$, as $|t| \to \infty$, hence the integrals $\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty} |g(s)|\,\mathrm{d}s$ and $\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty} |g(s)H(s)|\,\mathrm{d}s$ converge for $\beta \in \{-1, 2\}$. For $\eta > 0$, we define a weight function

$$w(\eta) := \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} g(s)\eta^{s+1}\,\mathrm{d}s, \tag{3.4}$$

which satisfies

$$w(\eta) = \begin{cases} O(1) & \text{for } \eta \geqslant 1, \\ 0 & \text{for } 0 < \eta < 1. \end{cases} \tag{3.5}$$

(To see this, one can shift the line of integration to $\int_{-1-\mathrm{i}\infty}^{-1+\mathrm{i}\infty}$ in the first case and to $\int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty}$, with $C \to \infty$, in the second case.) Thus, for $Y > 0$,

$$\begin{aligned} V(Y) &:= \frac{1}{Y}\int_1^\infty R(x)w\Big(\frac{Y}{x}\Big)\,\mathrm{d}x \\ &= \frac{1}{Y}\int_1^\infty R(x)\bigg(\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} g(s)\Big(\frac{Y}{x}\Big)^{s+1}\,\mathrm{d}s\bigg)\,\mathrm{d}x \\ &= \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} g(s)Y^s\bigg(\int_1^\infty R(x)x^{-s-1}\,\mathrm{d}x\bigg)\,\mathrm{d}s \\ &= \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} g(s)H(s)Y^s\,\mathrm{d}s. \end{aligned} \tag{3.6}$$

Shifting the line of integration to $\Re(s) = -1$, we get, for Y large,

$$\begin{aligned} V(Y) &= 2\pi\mathrm{i}\operatorname*{Res}_{s=z_0}(g(s)H(s)Y^s) + \int_{-1-\mathrm{i}\infty}^{-1+\mathrm{i}\infty} g(s)H(s)Y^s\,\mathrm{d}s \\ &= 2\pi\mathrm{i}\alpha_0 Y^{z_0} + O(Y^{-1}), \end{aligned} \tag{3.7}$$

where

$$\alpha_0 = \frac{E(z_0)}{(z_0+2)^7} = \frac{(z_0-1)(2z_0-1)\zeta_Q(z_0)}{(z_0+2)^7}. \tag{3.8}$$

From this it is evident that, as $Y \to \infty$,

$$|V(Y)| \gg |Y^{z_0}| = Y^{1/4} \tag{3.9}$$

and, on the other hand, in view of (3.5),

$$|V(Y)| = \left|\frac{1}{Y}\int_1^Y R(x)w\Big(\frac{Y}{x}\Big)\,\mathrm{d}x\right| \ll \frac{1}{Y}\int_1^Y |R(x)|\,\mathrm{d}x, \tag{3.10}$$

which completes the proof of our theorem. □

4. How to get an estimate with an explicit constant

The above argument was clearly non-effective, as far as the $\gg$-constant in (1.9) is concerned: In particular, our lemma only guarantees the existence of a Riemann-zeta zero $2z_0$ for which $\zeta_Q(z_0) \neq 0$, but gives no possibility to estimate it.

In this final section, we shall therefore show how to obtain a lower bound[8] for

$$K_0 := \liminf_{Y\to\infty}\left(Y^{-5/4}\int_1^Y |R(x)|\,\mathrm{d}x\right),$$

for any specific given form $Q(m,n)$. Our first step is to show that

$$|w(\eta)| \leqslant 0.33, \tag{4.1}$$

for all $\eta > 0$ and any Q. In fact, by (3.4) and (3.2),

$$\begin{aligned}|w(\eta)| &= \left|\int_{-1-\mathrm{i}\infty}^{-1+\mathrm{i}\infty} g(s)\eta^{s+1}\,\mathrm{d}s\right| \leqslant \int_{-\infty}^{\infty} |g(-1+\mathrm{i}t)|\,\mathrm{d}t \\ &\leqslant \int_{-\infty}^{\infty}\left|\frac{(-1+\mathrm{i}t)(-2+\mathrm{i}t)(-3+2\mathrm{i}t)}{(1+\mathrm{i}t)^7(-\frac{5}{4}+\mathrm{i}(t-\beta_0))}\right||\zeta(-2+2\mathrm{i}t)|\,\mathrm{d}t,\end{aligned}$$

if we recall that $z_0 = \frac{1}{4} + \mathrm{i}\beta_0$. We further use the functional equation (e.g., [20], formulæ (2.1.9), (2.1.10))

$$\zeta(-2+2\mathrm{i}t) = \pi^{-5/2+2\mathrm{i}t}\frac{\Gamma(\frac{3}{2}-\mathrm{i}t)}{\Gamma(-1+\mathrm{i}t)}\zeta(3-2\mathrm{i}t),$$

[8] However, we shall not invest too much effort to make this bound as large as possible.

along with well-known identities for the Γ-function (in particular formula 8.332 in [5]) which imply

$$\left|\frac{\Gamma(\frac{3}{2}-\mathrm{i}t)}{\Gamma(-1+\mathrm{i}t)}\right| \leqslant \left|\Big(\frac{1}{2}+\mathrm{i}t\Big)(-1+\mathrm{i}t)\right|\sqrt{|t|}.$$

Thus

$$\begin{aligned}|w(\eta)| &\leqslant \frac{\zeta(3)}{\pi^{5/2}}\int_{-\infty}^{\infty}\left|\frac{(-1+\mathrm{i}t)^2(-2+\mathrm{i}t)(-3+2\mathrm{i}t)(\frac{1}{2}+\mathrm{i}t)}{(1+\mathrm{i}t)^7(-\frac{5}{4}+\mathrm{i}(t-\beta_0))}\right|\sqrt{|t|}\,\mathrm{d}t\\ &\leqslant \frac{\zeta(3)}{\pi^{5/2}}\left(2\int_0^{\infty}\frac{(4+t^2)(9+4t^2)(\frac{1}{4}+t^2)t}{(1+t^2)^5}\,\mathrm{d}t\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\frac{25}{16}+(t-\beta_0)^2}\right)^{1/2},\end{aligned}$$

by Cauchy's inequality. The integrals are evaluated to $\frac{143}{32}$ (with a little help from *Mathematica* [22], for instance) and $\frac{4}{5}\pi$, which readily gives (4.1). By (3.10) and (3.7), it follows that

$$K_0 \geqslant 6\pi|\alpha_0|, \tag{4.2}$$

thus it remains to estimate $|\alpha_0|$ (see (3.8)), in particular $|\zeta_Q(z_0)|$, for any fixed form Q and some fixed Riemann-zeta zero $2z_0$ on the critical line. To this end, we employ a classical formula due to Potter [18], formula (2.22), which approximates the Epstein zeta-function by a partial sum of its series, throughout the half-plane $\Re(s) > -\frac{1}{4}$, $s \neq 1$. In our notation,

$$\zeta_Q(s) = F_1(Z,s) + F_2(Z,s), \tag{4.3}$$

where Z is a positive real parameter,

$$\begin{aligned}F_1(Z,s) := &\sum_{\substack{(m,n)\in\mathbb{Z}^2_*\\ Q(m,n)\leqslant Z}} Q(m,n)^{-s} + sZ^{-s-1}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ Q(m,n)\leqslant Z}} Q(m,n)\\ &-(1+s)Z^{-s}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ Q(m,n)\leqslant Z}} 1 + \frac{\pi}{\sqrt{D}}\frac{s(s+1)}{(s-1)}Z^{1-s},\end{aligned} \tag{4.4}$$

$$F_2(Z,s) := s(s+1)\int_Z^{\infty} v^{-s-2}P_1(v)\,\mathrm{d}v, \tag{4.5}$$

where, for $v > 0$,

$$P_1(v) := \int_0^v P(w)\,\mathrm{d}w = \frac{\sqrt{D}}{2\pi}\,v\sum_{(m,n)\in\mathbb{Z}^2_*} Q(m,n)^{-1}J_2\left(4\pi\sqrt{\frac{v}{D}Q(m,n)}\right),$$

J_2 being the usual Bessel function (see [18], Lemma 1). To estimate $|F_2(Z,s)|$, we use that $|J_2(x)| \leqslant x^{-1/2}$ for $x > 0$, which is easily verified by formula 8.451 in [5]. This gives

$$|P_1(v)| \leqslant \frac{D^{3/4}}{4\pi^{3/2}}\, v^{3/4} \sum_{(m,n)\in\mathbb{Z}^2_*} Q(m,n)^{-5/4}. \tag{4.6}$$

To bound this series, let $\kappa_Q := \inf_{(u,v)\in\mathbb{R}^2_*} Q(u,v)/(u^2+v^2)$, then a calculus exercise yields: If $\tau_\pm := \frac{1}{b}(a-c\pm\sqrt{(a-c)^2+b^2})$ for $b \neq 0$, then

$$\kappa_Q = \begin{cases} \min\Big(\dfrac{Q(\tau_+,1)}{\tau_+^2+1}, \dfrac{Q(\tau_-,1)}{\tau_-^2+1}\Big) & \text{if } b \neq 0, \\ \min(a,c) & \text{if } b = 0. \end{cases} \tag{4.7}$$

Hence

$$\sum_{(m,n)\in\mathbb{Z}^2_*} Q(m,n)^{-5/4} \leqslant \kappa_Q^{-5/4} \sum_{k=1}^{\infty} r(k)k^{-5/4} = 4\kappa_Q^{-5/4}\zeta\Big(\frac{5}{4}\Big)L\Big(\frac{5}{4}\Big),$$

where $r(k)$ counts the number of ways to express k as a sum of two squares, and $L(s)$ is the Dirichlet L-series[9] corresponding to the non-principal Dirichlet character mod 4. Combining this with (4.5) and (4.6), we obtain altogether, provided that[10] $\Re(s) = \frac{3}{4}$,

$$|F_2(Z,s)| \leqslant |s(s+1)| \frac{D^{3/4}}{\pi^{3/2}} \kappa_Q^{-5/4} \zeta(\frac{5}{4}) L(\frac{5}{4}) \frac{1}{Z}. \tag{4.8}$$

For a given form Q, one can therefore proceed as follows: Choose, e.g., $z_0^* = \frac{1}{4} + \mathrm{i}\beta_0^*$ with $\beta_0^* = 7.06736\ldots$ so that $\zeta(2z_0^*) = 0$, then by the functional equation (1.7),

$$|\zeta_Q(z_0^*)| = \Big(\frac{2\pi}{\sqrt{D}}\Big)^{-1/2} \frac{|\Gamma(1-z_0^*)|}{|\Gamma(z_0^*)|} |\zeta_Q(1-z_0^*)|.$$

Combining this with (4.2), (3.8), and (4.3), we arrive at

$$\begin{aligned} K_0 \geqslant{}& 6\pi\big|(z_0^*-1)(2z_0^*-1)(z_0^*+2)^7\big| \\ &\times \Big(\frac{2\pi}{\sqrt{D}}\Big)^{-1/2} \frac{|\Gamma(1-z_0^*)|}{|\Gamma(z_0^*)|} (|F_1(Z,1-z_0^*)| - |F_2(Z,1-z_0^*)|) \end{aligned} \tag{4.9}$$

[9] The evaluation of $L(\frac{5}{4})$ can be done by *Mathematica* [22], via the identity $L(s) = 2^{-s}\Phi(-1,s,\frac{1}{2})$, where Φ is the Lerch Phi-function: see [5], formula 9.550.

[10] For better convergence, our strategy is to bound $|\zeta_Q(1-z_0)|$ away from 0, and then to appeal to the functional equation.

where Z remains a free parameter and $|F_1(Z,1-z_0^*)|$, $|F_2(Z,1-z_0^*)|$ can be evaluated, resp., estimated by (4.4), (4.8). The only thing that could go wrong is that $|\zeta_Q(1-z_0^*)|$ is so small (or actually 0) that we cannot get a positive lower bound for the last bracket in (4.9). In this case, we can take one of the next Riemann-zeta zeros instead of $2z_0^*$. $\square$

Example. Let us consider the special (irrational) quadratic form

$$Q_0(m,n) = m^2 + \sqrt{2}\,mn + \sqrt{3}\,n^2.$$

Choosing $Z = 1000$ and employing *Mathematica* [22] to evaluate (4.4), resp., (4.8), we obtain $|F_1(1000, 1-z_0^*)| = 0.422182\ldots$, $|F_2(1000, 1-z_0^*)| \leqslant 0.236529\ldots$, hence $|F_1(1000, 1-z_0^*)| - |F_2(1000, 1-z_0^*)| \geqslant 0.185653\ldots$. Using this in (4.9), we finally arrive at

$$K_0 = \liminf_{Y\to\infty}\left(Y^{-5/4}\int_1^Y |R(x)|\,\mathrm{d}x\right) > 4\times 10^{-4}$$

for this particular form Q_0.

Applying to the integral $\int_{-1-\mathrm{i}\infty}^{-1+\mathrm{i}\infty} g(s)H(s)Y^s\,\mathrm{d}s$ in (3.7) similar arguments as we used to estimate $w(\eta)$, one can replace the lim inf-bound by an inequality valid for all $Y > 0$. For the form Q_0 we obtain in this way

$$Y^{-5/4}\int_1^Y |R(x)|\,\mathrm{d}x > 4\times 10^{-4} - 3.62\,Y^{-5/4},$$

which is non-trivial for $Y > 1500$.

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