LECTURES ON SEMICLASSICAL ANALYSIS

VERSION 0.2

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Preface

This book originates with a course MZ taught at UC Berkeley during the spring semester of 2003, notes for which LCE took in class. In this presentation we have tried hard to work out the full details for many proofs only sketched in the original lectures. We have reworked the order of presentation, added many additional topics, and included more heuristic commentary. We have as well introduced consistent notation, recounted in Appendix A. Relevant functional analysis and other background mathematics have been consolidated into Appendices B–D.

This version 0.2 represents our latest draft, the clarity of which we (still) hope greatly to improve in later editions, to be posted on our websites. We are quite aware that many errors remain in our exposition, and so we ask our readers to please send any comments or corrections to us at evans@math.berkeley.edu or zworski@math.berkeley.edu.

We should mention that two excellent treatments of mathematical semiclassical analysis have appeared recently. The book [D-S] by M. Dimassi and J. Sjöstrand starts with the WKB-method, develops the general semiclassical calculus, and then provides high tech spectral asymptotics. The presentation of Martinez [M] is based on a systematic development of FBI (Fourier-Bros-Iagolnitzer) transform techniques, with applications to microlocal exponential estimates and propagation estimates. These notes are intended as a more elementary and broader introduction. Except for the general symbol calculus, where we followed Chapter 7 of [D-S], there is little overlap with these other two texts, or with the early and influential book by Robert [R].

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Contents

1. Introduction

1.1 BASIC THEMES

Our primary goal is understanding the relationships between dynamical systems and the behavior of solutions to various linear PDE and pseudodifferential equations containing a small positive parameter h.

PDE with small parameters. The principal realm of motivation is quantum mechanics, in which case we understand h as denoting *Planck's constant*. With this interpretation in mind, we break down our basic task into these two subquestions:

(i) How and to what extent do classical dynamics determine the behavior as $h \to 0$ of solutions to *Schrödinger's equation*

$$ih\partial_t u = -h^2 \Delta u + V u$$

for the potential $V = V(x)$, and the related *eigenvalue equation*

$$-h^2 \Delta u + V u = E u?$$

The name "semiclassical" comes from this interpretation.

(ii) Conversely, given various mathematical objects associated with classical mechanics, for instance symplectic transformations, how can we profitably "quantize" them?

In fact the techniques of semiclassical analysis apply in many other settings and for many other sorts of PDE. For example we will later study the *damped wave equation*

$$\partial_t^2 u + a\partial_t u - \Delta u = 0 \tag{1.1}$$

for large times. A rescaling in time will introduce the requisite small parameter h.

Basic techniques. We will construct in Chapters 2–4 and 9 a wide variety of mathematical tools to address these issues, among them:

- the apparatus of symplectic geometry (to record succinctly the behavior of classical dynamical systems);
- the Fourier transform (to display dependence upon both the position variables x and the momentum variables ξ);
- stationary phase (to describe asymptotics as $h \to 0$ of various expressions involving rescaled Fourier transforms);

• pseudodifferential operators (to localize or, as is said in the trade, to *microlocalize* functional behavior in phase space).

1.2 CLASSICAL AND QUANTUM MECHANICS

In this section we introduce and foreshadow a bit about quantum and classical correspondences.

Observables. We can think of a given function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $a = a(x, \xi)$, as a *classical observable* on phase space, where as above x denotes *position* and ξ *momentum.* We will also call a a *symbol.*

Let $h > 0$ be given. We will associate with the observable a, a corresponding *quantum observable* $a^{\mathrm{w}}(x, hD)$, an operator defined by the formula

$$a^{\mathrm{w}}(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left(\tfrac{x+y}{2}, \xi\right) u(y)\, d\xi dy$$

for appropriate smooth functions u. This is *Weyl's quantization formula.*

We will later learn that if we change variables in a symbol, we preserve the principal symbol up to lower order terms (that is, terms involving high powers of the small parameter h.)

Equations of evolution. We are concerned as well with the evolution in time of classical particles and quantum states.

Classical evolution. Our most important example will concern the symbol

$$p(x, \xi) := |\xi|^2 + V(x),$$

corresponding to the phase space flow

$$\begin{cases} \dot{x} = 2\xi \\ \dot{\xi} = -\partial V. \end{cases}$$

We generalize by introducing the arbitrary Hamiltonian $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $p = p(x, \xi)$, and the corresponding *Hamiltonian dynamics*

$$\begin{cases} \dot{x} = \partial_\xi p(x, \xi) \\ \dot{\xi} = -\partial_x p(x, \xi). \end{cases} \tag{1.2}$$

It is instructive to change our viewpoint somewhat, by first introducing some more notation. Let us write

$$\varphi_t = \exp(tH_p)$$

for the solution of (1.2), where

$$H_p q := \{p, q\} = \langle \partial_\xi p, \partial_x q\rangle - \langle \partial_x p, \partial_\xi q\rangle$$

is the *Poisson bracket.* Select a symbol a and set

$$a_t(x,\xi) := a(\varphi_t(x,\xi)).$$

Then

$$\frac{d}{dt}a_t = \{p, a_t\}. \tag{1.3}$$

This equation tells us how the symbol evolves in time.

Quantum evolution. We next quantize the foregoing by putting $P = p^{\mathrm{w}}(x, hD)$, $A = a^{\mathrm{w}}(x, hD)$ and defining

$$A_t := e^{\frac{itP}{n}}\; A\; e^{-\frac{itP}{n}}.$$

Then we have the evolution equation

$$\frac{d}{dt}A_t = \frac{i}{h}[P, A_t],$$

an obvious analog of (1.3). Here is a basic principle we will later work out in some detail: an assertion about Hamiltonian dynamics, and so the Poisson bracket $\{\cdot,\cdot\}$, will involve at the quantum level the commutator $[\cdot,\cdot]$.

1.3 OVERVIEW

Chapters 2–4 and 9 develop the basic machinery, and the other chapters cover applications to PDE. Here is a quick overview, with some of the highpoints:

Chapter 2: We start with a quick introduction to symplectic analysis and geometry and their implications for classical Hamiltonian dynamical systems.

Chapter 3: This chapter provides the basics of the Fourier transform and derives also important *stationary phase* asymptotic estimates, of the sort

$$I_h = (2\pi h)^{n/2} |\det\, \partial^2\varphi(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\partial^2\varphi(x_0)} e^{\frac{i\varphi(x_0)}{h}} a(x_0) + O\left(h^{\frac{n+2}{2}}\right)$$

as $h \to 0$, for the oscillatory integral

$$I_h := \int_{\mathbb{R}^n} e^{\frac{i\varphi}{h}} a\, dx.$$

We assume here that the gradient $\partial\varphi$ vanishes only at the point x_0.

Chapter 4: Next we introduce the Weyl quantization $a^{\mathrm{w}}(x, hD)$ of the symbol $a(x,\xi)$ and work out various properties, chief among them the composition formula

$$a^{\mathrm{w}}(x, hD) \circ b^{\mathrm{w}}(x, hD) = c^{\mathrm{w}}(x, hD),$$

where the symbol $c := a\#b$ is computed explictly in terms of a and b. We will prove as well the sharp Gårding inequality, stating that if a is a nonnegative symbol, then

$$\langle a^{\mathrm{w}}(x, hD)u, u\rangle \geq -Ch\|u\|_{L^2}^2$$

for all u and sufficiently small $h > 0$.

Chapter 5: This section introduces semiclassical defect measures, and uses them to derive decay estimates for the damped wave equation (1.1), where $a \geq 0$, on the flat torus $\mathbb{T}^n$.

Chapter 6: In Chapter 6 we begin our study of the eigenvalue problem

$$P(h)u(h) = E(h)u(h),$$

for the operator

$$P(h) := -h^2\Delta + V(x).$$

We prove *Weyl's Law* for the asymptotic distributions of eigenvalues as $h \to 0$, stating for all $a < b$ that

$$\begin{aligned} &\#\{E(h) \mid a \leq E(h) \leq b\} \\ &\qquad\qquad = \frac{1}{(2\pi h)^n}(|\{a \leq |\xi|^2 + V(x) \leq b\}| + o(1)). \end{aligned}$$

Chapter 7: Chapter 7 continues the study of eigenfunctions, first establishing an exponential vanishing theorem in the "classically forbidden" region. We derive as well a *Carlemann-type* inequality

$$\|u(h)\|_{L^2(E)} \geq e^{-\frac{C}{h}}\|u(h)\|_{L^2(\mathbb{R}^n)}$$

where $E \subset \mathbb{R}^n$. This provides a quantitative estimate for quantum mechanical *tunneling*.

Chapter 8: Chapter 8 concerns the quantum implications of ergodicity for our underlying dynamical systems. A key assertion is that if the underlying dynamical system satisfies an appropriate ergodic condition, then

$$(2\pi h)^n \sum_{a \leq E_j \leq b} \left| \langle Au_j, u_j\rangle - \fint_{\{a \leq p \leq b\}} \sigma(A)\, dx d\xi \right|^2 \longrightarrow 0$$

as $h \to 0$, for a wide class of pseudodifferential operators A. In this expression the classical observable $\sigma(A)$ denotes the symbol of A.

Chapter 9: We return in Chapter 9 to the symbol calculus. We introduce the useful formalism of half-densities and use them to illustrate how changing variables in a symbol affects the Weyl quantization. We introduce also the notion of the semiclassical wave front set and show

how a natural localization in phase space leads to pointwise bounds on approximate solutions. We prove as well a semiclassical version of Beals's Theorem, characterizing pseudodifferential operators. As an application we show how quantization commutes with exponentiation at the level of order functions.

Chapter 10: The concluding Chapter 10 explains how to quantize symplectic transformations, with applications including local constructions of propagators, L^p bounds on eigenfuctions, and normal forms of differential operators.

Appendices: Appendix A records our notation in one convenient location, and Appendix B is a quick review of differential forms. Appendix C collects various useful functional analysis theorems (with selected proofs). Appendix D discusses Fredholm operators within the framework of Grushin problems, and Appendix E discusses general manifolds and modifications our the symbol calculus to cover pseudodifferential operators on manifolds.

2. Symplectic analysis

Since our task in these notes is understanding some interrelationships between dynamics and PDE, we provide in this chapter a quick discussion of the symplectric geometric structure on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ and its interplay with Hamiltonian dynamics.

The reader may wish to first review our basic notation and also the theory of differential forms, set forth respectively in Appendices A and B.

2.1 FLOWS

Let $V : \mathbb{R}^N \to \mathbb{R}^N$ denote a smooth vector field. Fix a point $z \in \mathbb{R}^N$ and solve the ODE

$$\begin{cases} \dot{z}(t) = V(z(t)) & (t \in \mathbb{R}) \\ z(0) = z. \end{cases} \tag{2.1}$$

We assume that the solution of the flow (2.1) exists and is unique for all times $t \in \mathbb{R}$.

NOTATION. We define

$$\varphi_t z := z(t)$$

and sometimes also write

$$\varphi_t =: \exp(tV).$$

We call $\{\varphi_t\}_{t\in\mathbb{R}}$ the *exponential map.*

The following lemma records some standard assertions from theory of ordinary differential equations:

LEMMA 2.1 (Properties of flow map).
(i) $\varphi_0 z = z$.
(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ *for all* $s, t \in \mathbb{R}$.
(iii) *For each time* $t \in \mathbb{R}$*, the mapping* $\varphi_t : \mathbb{R}^N \to \mathbb{R}^N$ *is a diffeomorphism, with*

$$(\varphi_t)^{-1} = \varphi_{-t}.$$

2.2 SYMPLECTIC STRUCTURE ON $\mathbb{R}^{2n}$

We henceforth specialize to the even-dimensional space $\mathbb{R}^N = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

NOTATION. We refine our previous notation and henceforth denote an element of $\mathbb{R}^{2n}$ as

$$z = (x, \xi),$$

and interpret $x \in \mathbb{R}^n$ as denoting *position*, $\xi \in \mathbb{R}^n$ as *momentum*. Alternatively, we can think of ξ as belonging to $T_x^*\mathbb{R}^n$, the cotangent space of $\mathbb{R}^n$ at x. We will likewise write

$$w = (y, \eta)$$

for another typical point of $\mathbb{R}^{2n}$.

We let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^n$, and then define this pairing on $\mathbb{R}^{2n}$:

DEFINITION. Given $z = (x, \xi)$, $w = (y, \eta)$ on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, define their *symplectic product*

$$\sigma(z, w) := \langle \xi, y \rangle - \langle x, \eta \rangle. \tag{2.2}$$

Note that

$$\sigma(z, w) = \langle Jz, w \rangle \tag{2.3}$$

for the $2n \times 2n$ matrix

$$J := \begin{pmatrix} O & I \\ -I & O \end{pmatrix}. \tag{2.4}$$

Observe

$$J^2 = -I, \quad J^T = -J.$$

We will later in Section 10.2 interpret the transformation J as the classical analog of the Fourier transform.

LEMMA 2.2 (Properties of $\boldsymbol{\sigma}$). *The bilinear form σ is antisymmetric:*

$$\sigma(z, w) = -\sigma(w, z)$$

and nondegenerate:

$$\textit{if } \sigma(z, w) = 0 \textit{ for all } w, \textit{ then } z = 0.$$

These assertions are straightforward to check.

NOTATION. We now bring in the terminology of differential forms, reviewed in Appendix B. Using the notation discussed above, we introduce for $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$ the 1-forms dx_j and $d\xi_j$ for $j = 1, \dots, n$. We then can write

$$\sigma = d\xi \wedge dx = \sum_{j=1}^{n} d\xi_j \wedge dx_j. \tag{2.5}$$

Observe also

$$\sigma = d\omega \quad \text{for} \quad \omega := \xi dx = \sum_{j=1}^{n} \xi_j dx_j. \tag{2.6}$$

It follows that

$$d\sigma = 0. \tag{2.7}$$

2.3 CHANGING VARIABLES.

Suppose next that $U, V \subseteq \mathbb{R}^{2n}$ are open sets and

$$\kappa : U \to V$$

is a smooth mapping. We will write

$$\gamma(x, \xi) = (y, \eta) = (y(x, \xi), \eta(x, \xi)).$$

DEFINITION. We call γ a *symplectic* mapping, or a *symplectomorphism*, provided

$$\gamma^* \sigma = \sigma. \tag{2.8}$$

Here the *pull-back* $\gamma^* \sigma$ of the symplectic product σ is defined by

$$(\gamma^* \sigma)(z, w) := \sigma(\gamma_*(z), \gamma_*(w)),$$

γ_* denoting the *push-forward* of vectors: see Appendix B.

NOTATION. We will usually write (2.8) in the more suggestive notation

$$d\eta \wedge dy = d\xi \wedge dx. \tag{2.9}$$

EXAMPLE 1: Linear symplectic mappings. Suppose $\kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is linear:

$$\kappa(x, \xi) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = (Ax + B\xi, Cx + D\xi) = (y, \eta),$$

where A, B, C, D are $n \times n$ matrices.

THEOREM 2.3 (Symplectic matrices). *The linear mapping γ is symplectic if and only if the matrix*

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

satisfies

$$M^T J M = J. \tag{2.10}$$

DEFINITION. We call a $2n \times 2n$ matrix M *symplectic* if (2.10) holds.

In particular the linear mapping $(x, \xi) \mapsto (\xi, -x)$ determined by J is symplectic.

Proof. Let us compute

$$\begin{aligned} d\eta \wedge dy &= (Cdx + Dd\xi) \wedge (Adx + Bd\xi) \\ &= A^T C dx \wedge dx + B^T D d\xi \wedge d\xi + (A^T D - C^T B) d\xi \wedge dx \\ &= d\xi \wedge dx \end{aligned}$$

if and only if

$$A^T C \text{ and } B^T D \text{ are symmetric, } A^T D - C^T B = I. \tag{2.11}$$

Then

$$\begin{aligned} M^T J M &= \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} A^T C - C^T A & A^T D - C^T B \\ B^T C - D^T A & B^T D - D^T B \end{pmatrix} \\ &= J \end{aligned}$$

if and only if (2.11) holds. □

EXAMPLE 2: Nonlinear symplectic mappings. Assume next that $\kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is nonlinear:

$$\kappa(x, \xi) = (y, \eta)$$

for smooth functions $y = y(x, \xi), \eta = \eta(x, \xi)$. Its linearization is the $2n \times 2n$ matrix

$$\partial \kappa = \partial_{x,\xi} \kappa = \begin{pmatrix} \partial_x y & \partial_\xi y \\ \partial_x \eta & \partial_\xi \eta \end{pmatrix}.$$

THEOREM 2.4 (Symplectic transformations). *The mapping κ is symplectic if and only if the matrix $\partial \kappa$ is symplectic at each point.*

Proof. We have

$$d\eta \wedge dy = (Cdx + Dd\xi) \wedge (Adx + Bd\xi)$$

for

$$A := \partial_x y, B := \partial_\xi y, C := \partial_x \eta, D := \partial_\xi \eta.$$

Consequently, as in the previous proof, we have $d\eta \wedge dy = d\xi \wedge dx$ if and only if (2.11) is valid, which in turn is so if and only if $\partial\kappa$ is a symplectic matrix. □

EXAMPLE 3: Lifting diffeomorphisms. Let

$$\gamma : \mathbb{R}^n \to \mathbb{R}^n$$

be a diffeomorphism on $\mathbb{R}^n$, with nondegenerate Jacobian matrix $\partial_x \gamma$. We propose to extend γ to a symplectomorphism

$$\kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

having the form

$$\kappa(x, \xi) = (\gamma(x), \eta(x, \xi)) = (y, \eta), \tag{2.12}$$

by "lifting" to the momentum variables.

THEOREM 2.5 (Extending to a symplectic mapping). *The transformation* (2.12) *is symplectic for*

$$\eta(x, \xi) := \left[\partial_x \gamma(x)^{-1}\right]^T \xi. \tag{2.13}$$

Proof. It turns out to be easier to look for ξ as a function of x and η. We compute

$$dy = A\ dx,\ \ d\xi = E\ dx + F\ d\eta,$$

for

$$A := \partial_x y, \quad E := \partial_x \xi, \quad F := \partial_\eta \xi.$$

Therefore

$$d\eta \wedge dy = d\eta \wedge (A\ dx)$$

and

$$d\xi \wedge dx = (Edx \wedge Fd\eta) \wedge dx = Edx \wedge dx + d\eta \wedge F^T dx.$$

We would like to construct $\xi = \xi(x, \eta)$ so that

$$A = F^T \quad \text{and} \quad E \text{ is symmetric},$$

the latter condition implying that $Edx \wedge dx = 0$. To do so, let us define

$$\xi(x, \eta) := (\partial_x \gamma)^T \eta.$$

Then clearly $F^T = A$, and $E = E^T = ((\gamma_{x_i x_j}))$, as required. □

INTERPRETATION: This example will prove useful later, when we quantize symbols in Chapter 4 and learn that the partial differential operator

$$P(h) = -h^2\Delta$$

is associated with the symbol $p(x,\xi) = |\xi|^2$. If we change variables $y = \gamma(x)$, it is natural to ask how $P(h)$ transforms. Now $\partial_x = (\partial_x\gamma)^T \partial_y$, and so

$$P(h) = -h^2 \ (\partial_x\gamma)^T \partial_y \left((\partial_x\gamma)^T \partial_y \right).$$

We will see later that the operator on the right is associated with the symbol

$$\left\langle (\partial_x\gamma)^T \eta, (\partial_x\gamma)^T \eta \right\rangle.$$

All this is consistent with the transformation (2.13).

Here by the way is an instance of another general principle: if we change variables in a symbol, we preserve the principal symbol, up to higher order terms. □

EXAMPLE 4: Generating functions. Our last example demonstrates that we can, locally at least, build a symplectic transformation from a real-valued *generating function*.

Suppose $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi = \varphi(x,y)$, is smooth. Assume also that

$$\det(\partial^2_{xy}\varphi(x_0,y_0)) \neq 0. \tag{2.14}$$

Define

$$\xi = \partial_x\varphi, \ \ \eta = -\partial_y\varphi, \tag{2.15}$$

and observe that the Implicit Function Theorem implies (y,η) is a smooth function of (x,ξ) near $(x_0, \partial_x\varphi(x_0,y_0))$.

THEOREM 2.6 (Generating functions and symplectic maps). *The mapping* γ *defined by*

$$(x, \partial_x\varphi(x,y)) \mapsto (y, -\partial_y\varphi(x,y)) \tag{2.16}$$

is a symplectomorphism near (x_0,ξ_0).

Proof. We compute

$$\begin{aligned} d\eta\wedge dy &= d(-\partial_y\varphi)\wedge dy \\ &= [(-\partial^2_y\varphi dy)\wedge dy] + [(-\partial^2_{xy}\varphi dx)\wedge dy] \\ &= -(\partial^2_{xy}\varphi) dx\wedge dy, \end{aligned}$$

since $\partial_y^2\varphi$ is symmetric. Likewise,

$$\begin{aligned} d\xi \wedge dx &= d(\partial_x\varphi) \wedge dx \\ &= [(\partial_x^2\varphi\ dx) \wedge dx] + [(\partial_{xy}^2\varphi\ dy) \wedge dx] \\ &= -(\partial_{xy}^2\varphi) dx \wedge dy = d\eta \wedge dy. \end{aligned}$$

□

TERMINOLOGY. In Greek, the word "symplectic" means "intertwined", This is consistent with Example 4, since the generating function $\varphi = \varphi(x, y)$ is a function of a mixture of half of the original variables (x, ξ) and half of the new variables (y, η). "Symplectic" can also be interpreted as "complex", mathematical usage due to Hermann Weyl who renamed "line complex group" the "symplectic group": see Cannas da Silva [CdS].

APPLICATION: Lagrangian submanifolds. A *Lagrangian submanifold* Λ is an n-dimensional submanifold of $\mathbb{R}^{2n}$ for which

$$\sigma|_\Lambda = 0.$$

Then

$$d\omega|_\Lambda = \sigma|_\Lambda = 0;$$

and so according to Poincaré's Theorem B.4, we locally have

$$\omega = d\varphi,$$

for some smooth function φ on Λ. We will exploit this observation in Section 10.2. □

2.4 HAMILTONIAN VECTOR FIELDS

DEFINITION. Given $f \in C^\infty(\mathbb{R}^{2n})$, we define the corresponding *Hamiltonian vector field* by requiring

$$\sigma(z, H_f) = df(z) \qquad \text{for all } z = (x, \xi). \tag{2.17}$$

This is well defined, since σ is nondegenerate. We can write explicitly that

$$H_f = \langle \partial_\xi f, \partial_x \rangle - \langle \partial_x f, \partial_\xi \rangle = \sum_{j=1}^n f_{\xi_j}\partial_{x_j} - f_{x_j}\partial_{\xi_j}. \tag{2.18}$$

LEMMA 2.7 (Differentials and Hamiltonian vector fields). *We have the relation*

$$df = -(H_f \lrcorner \sigma), \tag{2.19}$$

for the contraction $\lrcorner$ *defined in Appendix B.*

Proof. We calculate for each z that

$$(H_f \lrcorner \sigma)(z) = \sigma(H_f, z) = -\sigma(z, H_f) = -df(z).$$

□

DEFINITION. If $f, g \in C^\infty(\mathbb{R}^{2n})$, we define their *Poisson bracket*

$$\{f, g\} := H_f g = \sigma(\partial f, \partial g). \tag{2.20}$$

That is,

$$\{f, g\} = \langle \partial_\xi f, \partial_x g \rangle - \langle \partial_x f, \partial_\xi g \rangle = \sum_{j=1}^n f_{\xi_j} g_{x_j} - f_{x_j} g_{\xi_j}. \tag{2.21}$$

LEMMA 2.8 (Brackets, commutators).

(i) *We have* Jacobi's identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{2.22}$$

for all functions $f, g, h \in C^\infty(\mathbb{R}^{2n})$.

(ii) *Furthermore,*

$$H_{\{f,g\}} = [H_f, H_g]. \tag{2.23}$$

Proof. 1. A direct calculation verifies assertion (i). For an alternative proof, note that Lemma B.1 provides the identity

$$\begin{aligned} 0 &= d\sigma(H_f, H_g, H_h) \\ &= H_f \sigma(H_g, H_h) + H_g \sigma(H_h, H_f) + H_h \sigma(H_f, H_g) \\ &\quad - \sigma([H_f, H_g], H_h) - \sigma([H_g, H_h], H_f) - \sigma([H_h, H_f], H_g), \end{aligned} \tag{2.24}$$

since $d\sigma = 0$. Now (2.20) implies

$$H_f \sigma(H_g, H_h) = \{f, \{g, h\}\}$$

and

$$\begin{aligned} \sigma([H_f, H_g], H_h) = [H_f, H_g] h &= H_f H_g h - H_g H_f h \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\}. \end{aligned}$$

Similar identities hold for other terms. Substituting into (2.24) gives Jacobi's identity.

2. We observe that

$$H_{\{f,g\}}h = [H_f, H_g]h$$

is a rewriting of (2.22). $\square$

THEOREM 2.9 (Jacobi's Theorem). *If γ is a symplectomorphism, then*

$$H_f = \gamma_*(H_{\gamma^* f}). \tag{2.25}$$

In other words, the pull-back of a Hamiltonian vector field generated by f,

$$\gamma^* H_f := (\gamma^{-1})_* H_f, \tag{2.26}$$

is the Hamiltonian vector field generated by the pull-back of f.

Proof. Using the notation of (2.26),

$$\begin{aligned}\gamma^*(H_f) \lrcorner \sigma &= \gamma^*(H_f) \lrcorner \gamma^* \sigma = \gamma^*(H_f \lrcorner \sigma) \\ &= -\gamma^*(df) = -d(\gamma^* f) \\ &= H_{\gamma^* f} \lrcorner \sigma.\end{aligned}$$

Since σ is nondegenerate, (2.25) follows. $\square$

EXAMPLE. Define $\gamma = J$, so that $\gamma(x,\xi) = (\xi, -x)$; and recall γ is a symplectomorphism. We have $\gamma^* f(x,\xi) = f(\xi,-x)$, and therefore

$$H_{\gamma^* f} = \langle \partial_x f(\xi,-x), \partial_x \rangle + \langle \partial_\xi f(\xi,-x), \partial_\xi \rangle.$$

Then

$$\kappa^* H_f = \langle \partial_\xi f(\xi,-x), \partial_\xi \rangle - \langle \partial_x f(\xi,-x), \partial_{-x} \rangle = H_{\gamma^* f}.$$

$\square$

THEOREM 2.10 (Hamiltonian flows as symplectomorphisms). *If f is smooth, then for each time t, the mapping*

$$(x,\xi) \mapsto \varphi_t(x,\xi) = \exp(tH_f)$$

is a symplectomorphism.

Proof. According to Cartan's formula (Theorem B.3), we have

$$\frac{d}{dt}((\varphi_t)^* \sigma) = \mathcal{L}_{H_f} \sigma = d(H_f \lrcorner \sigma) + (H_f \lrcorner d\sigma).$$

Since $d\sigma = 0$, it follows that

$$\frac{d}{dt}((\varphi_t)^* \sigma) = d(-df) = -d^2 f = 0.$$

Thus $(\varphi_t)^*\sigma = \sigma$ for all times t. $\square$

THEOREM 2.11 (Darboux's Theorem). *Let U be a neighborhood of (x_0, ξ_0) and suppose η is a nondegenerate 2-form defined on U, satisfying*

$$d\eta = 0.$$

Then near (x_0, ξ_0) there exists a diffeomorphism γ such that

$$\gamma^*\eta = \sigma. \tag{2.27}$$

This means that all symplectic structures are identical locally, in the sense that all are equivalent to that generated by σ.

Proof. 1. Let us assume $(x_0, \xi_0) = (0, 0)$. We first find a linear mapping L so that

$$L^*\eta(0,0) = \sigma(0,0).$$

This means that we find a basis $\{e_k, f_k\}_{k=1}^n$ of $\mathbb{R}^{2n}$ such that

$$\begin{cases} \eta(f_l, e_k) = \delta_{kl} \\ \eta(e_k, e_l) = 0 \\ \eta(f_k, f_l) = 0 \end{cases}$$

for all $1 \le k, l \le n$. Then if $u = \sum_{i=1}^n x_i e_i + \xi_i f_i$, $v = \sum_{j=1}^n y_j e_j + \eta_j f_j$, we have

$$\begin{aligned} \eta(u,v) &= \sum_{i,j=1}^n x_i y_j \eta(e_i, e_j) + \xi_i \eta_j \eta(f_i, f_j) + x_i \eta_j \sigma(e_i, f_j) + \xi_i y_j \sigma(f_i, e_j) \\ &= \langle \xi, y \rangle - \langle x, \eta \rangle = \sigma((x,\xi),(y,\eta)). \end{aligned}$$

2. Next, define $\eta_t := t\eta + (1-t)\sigma$ for $0 \le t \le 1$. Our intention is to find γ_t so that $\gamma_t^*\eta_t = \sigma$ near $(0,0)$; then $\gamma := \gamma_1$ solves our problem. We will construct γ_t by solving the flow

$$\begin{cases} \dot z(t) = V_t(z(t)) \qquad (0 \le t \le 1) \\ z(0) = z, \end{cases} \tag{2.28}$$

and setting $\gamma_t := \varphi_t$.

For this to work, we must design the vector fields V_t in (2.28) so that $\frac{d}{dt}(\gamma_t^*\eta_t) = 0$. Let us therefore calculate

$$\begin{aligned} \frac{d}{dt}(\gamma_t^*\eta_t) &= \kappa_t^*\left(\frac{d}{dt}\eta_t\right) + \gamma_t^*\mathcal{L}_{V_t}\eta_t \\ &= \gamma_t^*[(\eta - \sigma) + d(V_t \lrcorner \eta_t) + V_t \lrcorner d\eta_t], \end{aligned}$$

where we used Cartan's formula, Theorem B.3. Note that $d\eta_t = td\eta + (1-t)d\sigma$. Hence $\frac{d}{dt}(\gamma_t^*\eta_t) = 0$ provided

$$(\eta - \sigma) + d(V_t \lrcorner \eta_t) = 0. \tag{2.29}$$

According to Poincaré's Theorem B.4, we can write

$$\eta - \sigma = d\alpha \quad \text{near } (0,0).$$

So (2.29) will hold, provided

$$V_t \lrcorner \eta_t = -\alpha \qquad (0 \leq t \leq 1). \tag{2.30}$$

Since $\eta = \sigma$ at $(0,0)$, $\eta_t = \sigma$ at $(0,0)$. In particular, η_t is nondegenerate for $0 \leq t \leq 1$ in a neighbourhood of $(0,0)$, and hence we can solve (2.29) for the vector field V_t. □

3. Fourier transform, stationary phase

We discuss in this chapter how to define the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ on various classes of smooth functions and nonsmooth distributions. We introduce also the rescaled semiclassical transforms $\mathcal{F}_h, \mathcal{F}_h^{-1}$ depending on the small parameter h, and develop stationary phase asymptotics to help us understand various formulas involving $\mathcal{F}_h$ in the limit as $h \to 0$.

3.1 FOURIER TRANSFORM ON $\mathcal{S}$

We begin by defining and investigating the Fourier transform of smooth functions that decay rapidly as $|x| \to \infty$.

DEFINITIONS (i) The *Schwartz space* is

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty \text{ for all multiindices } \alpha, \beta\}.$$

(ii) We say

$$\varphi_j \to \varphi \quad \text{in } \mathcal{S}$$

provided

$$\sup_{\mathbb{R}^n} |x^\alpha \partial^\beta (\varphi_j - \varphi)| \to 0$$

for all multiindices α, β.

DEFINITION. If $\varphi \in \mathcal{S}$, define the *Fourier transform*

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \varphi(x)\, dx \qquad (\xi \in \mathbb{R}^n). \tag{3.1}$$

The reader is warned that many other texts use slightly different definitions, entailing normalizing factors involving π.

EXAMPLE: Exponential of a real quadratic form.

THEOREM 3.1 (Transform of a real exponential). *Let Q be a real, symmetric, positive definite $n \times n$ matrix. Then*

$$\mathcal{F}(e^{-\frac{1}{2}\langle Qx,x\rangle}) = \frac{(2\pi)^{n/2}}{(det\ Q)^{1/2}} e^{-\frac{1}{2}\langle Q^{-1}\xi,\xi\rangle}. \tag{3.2}$$

Proof. Let us calculate

$$\begin{aligned}\mathcal{F}(e^{-\frac{1}{2}\langle Qx,x\rangle}) &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Qx,x\rangle - i\langle x,\xi\rangle} dx \\ &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Q(x+iQ^{-1}\xi),\, x+iQ^{-1}\xi\rangle} e^{-\frac{1}{2}\langle Q^{-1}\xi,\xi\rangle} dx \\ &= e^{-\frac{1}{2}\langle Q^{-1}\xi,\xi\rangle} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Qy,y\rangle} dy.\end{aligned}$$

We compute the last integral by making an orthogonal change of variables that converts Q into diagonal form $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Qy,y\rangle} dy &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_{k=1}^n \lambda_k w_k^2}\, dw = \prod_{k=1}^n \int_{-\infty}^{\infty} e^{-\frac{\lambda_k}{2} w^2} dw \\ &= \prod_{k=1}^n \frac{2^{1/2}}{\lambda_k^{1/2}} \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \frac{(2\pi)^{n/2}}{(\lambda_1 \cdots \lambda_n)^{1/2}} = \frac{(2\pi)^{n/2}}{(\det\ Q)^{1/2}}.\end{aligned}$$

□

The Fourier transform $\mathcal{F}$ lets us move from position variables x to momentum variables ξ, and we need to catalog how it converts various algebraic and analytic expressions in x into related expressions in ξ:

THEOREM 3.2 (Properties of Fourier transform).
(i) *The mapping $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism.*
(ii) *We have the* Fourier inversion formula

$$\mathcal{F}^{-1} = \frac{1}{(2\pi)^n} R \circ \mathcal{F}, \tag{3.3}$$

where $Rf(x) := f(-x)$. In other words,

$$\mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \psi(\xi)\, d\xi; \tag{3.4}$$

and therefore

$$\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{\varphi}(\xi)\, d\xi. \tag{3.5}$$

(iii) *In addition,*

$$D^\alpha_\xi(\mathcal{F}\varphi) = \mathcal{F}((-x)^\alpha\varphi) \tag{3.6}$$

and

$$\mathcal{F}(D^\alpha_x\varphi) = \xi^\alpha\mathcal{F}\varphi. \tag{3.7}$$

(iv) *Furthermore,*

$$\mathcal{F}(\varphi\psi) = \frac{1}{(2\pi)^n}\mathcal{F}(\varphi) * \mathcal{F}(\psi). \tag{3.8}$$

REMARKS. (i) In these formulas we employ the notation from Appendix A that

$$D^\alpha = \frac{1}{i^{|\alpha|}}\partial^\alpha.$$

(ii) We will later interpret the Fourier inversion formula (3.4) as saying that

$$\delta_{\{y=x\}} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}\,d\xi \quad \text{in the sense of distributions,} \tag{3.9}$$

δ denoting the Dirac measure. □

Proof. 1. Let us calculate for $\varphi \in \mathcal{S}$ that

$$\begin{aligned} D^\alpha_\xi(\mathcal{F}\varphi) &= D^\alpha_\xi \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\varphi(x)\,dx = \frac{1}{i^\alpha}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}(-ix)^\alpha\varphi(x)\,dx \\ &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}(-x)^\alpha\varphi(x)\,dx = \mathcal{F}((-x)^\alpha\varphi). \end{aligned}$$

Likewise,

$$\begin{aligned} \mathcal{F}(D^\alpha_x\varphi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}D^\alpha_x\varphi\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n} D^\alpha_x(e^{-i\langle x,\xi\rangle})\varphi\,dx \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} \frac{1}{i^{|\alpha|}}(-i\xi)^\alpha e^{-i\langle x,\xi\rangle}\varphi\,dx = \xi^\alpha(\mathcal{F}\varphi). \end{aligned}$$

This proves (iii).

2. Recall from Appendix A the notation $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. Then for all multiindices α, β, we have

$$\begin{aligned}
\sup_\xi |\xi^\beta D^\alpha_\xi \hat{\varphi}| &= \sup_\xi |\xi^\beta \mathcal{F}((-x)^\alpha \varphi)| \\
&= \sup_\xi |\mathcal{F}(D^\beta_x((-x)^\alpha \varphi)| \\
&= \sup_\xi \left| \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \frac{1}{\langle x\rangle^{n+1}} \langle x\rangle^{n+1}\ D^\beta_x((-x)^\alpha \varphi)\, dx \right| \\
&\le \sup_x |\langle x\rangle^{n+1} D^\beta_x((-x)^\alpha \varphi)| \int_{\mathbb{R}^n} \frac{dx}{\langle x\rangle^{n+1}} < \infty.
\end{aligned}$$

Hence $\mathcal{F} : \mathcal{S} \to \mathcal{S}$, and a similar calculation shows that $\varphi_i \to \varphi$ in $\mathcal{S}$ implies $\mathcal{F}(\varphi_j) \to \mathcal{F}(\varphi)$.

3. To show $\mathcal{F}$ is invertible, note that

$$\begin{aligned}
R \circ \mathcal{F} \circ \mathcal{F} \circ D_{x_j} &= R \circ \mathcal{F} \circ M_{\xi_j} \circ \mathcal{F} \\
&= R \circ (-D_{x_j}) \circ \mathcal{F} \circ \mathcal{F} \\
&= D_{x_j} \circ R \circ \mathcal{F} \circ \mathcal{F},
\end{aligned}$$

where M_{ξ_j} denotes multiplication by ξ_j. Thus $R \circ \mathcal{F} \circ \mathcal{F}$ commutes with D_{x_j} and it likewise commutes with the multiplication operator M_λ. According to Lemma 3.3, stated and proved below, $R \circ \mathcal{F} \circ \mathcal{F}$ is a multiple of the identity operator:

$$R \circ \mathcal{F} \circ \mathcal{F} = cI. \tag{3.10}$$

From the example above, we know that

$$\mathcal{F}(e^{-\frac{|x|^2}{2}}) = (2\pi)^{n/2} e^{-\frac{|\xi|^2}{2}}.$$

Thus $\mathcal{F}(e^{-\frac{|\xi|^2}{2}}) = (2\pi)^{n/2} e^{-\frac{|x|^2}{2}}$. Consequently $c = (2\pi)^n$, and hence

$$\mathcal{F}^{-1} = \frac{1}{(2\pi)^n} R \circ \mathcal{F}.$$

4. Lastly, since

$$\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{\varphi}(\xi)\, d\xi, \quad \psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\eta\rangle} \hat{\psi}(\eta)\, d\eta,$$

we have

$$\begin{aligned}\varphi\psi &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi+\eta\rangle}\hat{\varphi}(\xi)\hat{\psi}(\eta)\,d\xi d\eta \\ &= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n} e^{i\langle x,\rho\rangle}\left(\int_{\mathbb{R}^n}\hat{\varphi}(\xi)\hat{\psi}(\rho-\xi)\,d\rho\right)\,d\xi \\ &= \frac{1}{(2\pi)^n}\mathcal{F}^{-1}(\hat{\varphi}*\hat{\psi}).\end{aligned}$$

But $\varphi\psi = \mathcal{F}^{-1}\mathcal{F}(\varphi\psi)$, and so assertion (iv) follows. □

LEMMA 3.3 (Commutativity). *Let $M_f : g \mapsto fg$ be the multiplication operator. Suppose that $L : \mathcal{S} \to \mathcal{S}$ is linear, and that*

$$L \circ M_{x_j} = M_{x_j} \circ L,\ \ L \circ D_{x_j} = D_{x_j} \circ L \tag{3.11}$$

$j = 1, \dots, n$. Then

$$L = cI$$

for some constant c, where I denotes the identity operator.

Proof. 1. Choose $\varphi \in \mathcal{S}$, fix $y \in \mathbb{R}^n$, and write

$$\varphi(x) - \varphi(y) = \sum_{j=1}^{n}(x_j - y_j)\psi_j(x)$$

for

$$\psi_j(x) := \int_0^1 \varphi_{x_j}(y + t(x-y))\,dt.$$

Since typically $\psi_j \notin \mathcal{S}$, we select a smooth function χ with compact support such that $\chi \equiv 1$ for x near y. Write

$$\varphi_j(x) := \chi(x)\psi_j(x) + \frac{(x_j - y_j)}{|x-y|^2}(1-\chi(x))\varphi(x).$$

Then

$$\varphi(x) - \varphi(y) = \sum_{j=1}^{n}(x_j - y_j)\varphi_j(x) \tag{3.12}$$

with $\varphi_j \in \mathcal{S}$.

2. We claim next that if $\varphi(y) = 0$, then $L\varphi(y) = 0$. This follows from (3.12), since

$$L\varphi(x) = \sum_{j=1}^{n}(x_j - y_j)L\varphi_j = 0$$

at $x = y$.

Therefore $L\varphi(x) = c(x)\varphi(x)$ for some function c. Taking $\varphi(x) = e^{-|x|^2}$, we deduce that $c \in C^\infty$. Finally, since L commutes with differentiation, we conclude that c must be a constant. $\square$

THEOREM 3.4 (Integral identities). *If $\varphi, \psi \in \mathcal{S}$, then*

$$\int_{\mathbb{R}^n} \hat{\varphi}\psi \, dx = \int_{\mathbb{R}^n} \varphi\hat{\psi} \, dy \tag{3.13}$$

and

$$\int_{\mathbb{R}^n} \varphi\bar{\psi} \, dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{\varphi}\bar{\hat{\psi}} \, d\xi. \tag{3.14}$$

In particular,

$$\|\varphi\|_{L^2}^2 = \frac{1}{(2\pi)^n}\|\hat{\varphi}\|_{L^2}^2. \tag{3.15}$$

Proof. Note first that

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{\varphi}\psi \, dx &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-i\langle x,y\rangle}\varphi(y) \, dy\right)\psi(x) \, dx \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-i\langle y,x\rangle}\psi(x) \, dx\right)\varphi(y) \, dy = \int_{\mathbb{R}^n}\hat{\psi}\varphi \, dy.\end{aligned}$$

Replace ψ by $\bar{\hat{\psi}}$ in (3.13):

$$\int_{\mathbb{R}^n} \hat{\varphi}\bar{\hat{\psi}} \, d\xi = \int_{\mathbb{R}^n} \varphi(\bar{\hat{\psi}})^\wedge \, dx.$$

But $\bar{\hat{\psi}} = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\bar{\psi}(x) \, dx = (2\pi)^n\mathcal{F}^{-1}(\bar{\psi})$ and so $(\bar{\hat{\psi}})^\wedge = (2\pi)^n\bar{\psi}$. $\square$

We record next some elementary estimates that we will need later:

LEMMA 3.5 (Useful estimates).
(i) *We have the bounds*

$$\|\hat{u}\|_{L^\infty} \le \|u\|_{L^1} \tag{3.16}$$

and

$$\|u\|_{L^\infty} \le \frac{1}{(2\pi)^n}\|\hat{u}\|_{L^1}. \tag{3.17}$$

(ii) *There exists a constant C such that*

$$\|\hat{u}\|_{L^1} \le C \sup_{|\alpha|\le n+1}\|\partial^\alpha u\|_{L^1}. \tag{3.18}$$

Proof. Estimates (3.16) and (3.17) follow easily from (3.1) and (3.5). Furthermore,

$$\begin{aligned}\|\hat{u}\|_{L^1} &= \int_{\mathbb{R}^n} |\hat{u}|\langle\xi\rangle^{n+1}\langle\xi\rangle^{-n-1}\,d\xi \;\leq C\sup(|\hat{u}|\langle\xi\rangle^{n+1})\\ &\leq C\sup_{|\alpha|\leq n+1}|\xi^\alpha\hat{u}| = C\sup_{|\alpha|\leq n+1}|(\partial^\alpha u)^\wedge| \leq C\sup_{|\alpha|\leq n+1}\|\partial^\alpha u\|_{L^1}.\end{aligned}$$

This proves (3.18). □

We close this section with an application showing that we can sometimes use the Fourier transform to solve PDE with variable coefficients.

EXAMPLE: Solving a PDE. Consider the initial-value problem

$$\begin{cases}\partial_t u = x\partial_y u + \partial_x^2 u & \text{on } \mathbb{R}^2\times(0,\infty)\\ \quad u = \delta_{(x_0,y_0)} & \text{on } \mathbb{R}^2\times\{t=0\}.\end{cases}$$

Let $\hat{u} := \mathcal{F}u$ denote the Fourier transform of u in the variables x, y (but not in t). Then

$$(\partial_t + \eta\partial_\xi)\hat{u} = -\xi^2\hat{u}.$$

This is a linear first-order PDE we can solve by the method of characteristics:

$$\begin{aligned}\hat{u}(t,\xi+t\eta,\eta) &= \hat{u}(0,\xi,\eta)e^{-\int_0^t(\xi+s\eta)^2ds}\\ &= \hat{u}(0,\xi,\eta)e^{-\xi^2t-\xi\eta t^2-\frac{\eta^2t^3}{3}}\\ &= \hat{u}(0,\xi,\eta)e^{-\frac{1}{2}\langle B_t(\xi,\eta),(\xi,\eta)\rangle},\end{aligned}$$

for

$$B_t := \begin{pmatrix}2t & t^2\\ t^2 & 2t^3/3\end{pmatrix}.$$

Furthermore, $\hat{u}(0,\xi,\eta) = \hat{\delta}_{(x_0,y_0)}$. Taking $\mathcal{F}^{-1}$, we find

$$\begin{aligned}u(t,x,y-tx) &= \delta_{(x_0,y)} * \mathcal{F}^{-1}(e^{-\frac{1}{2}\langle B_t(\xi,\eta),(\xi,\eta)\rangle})\\ &= \frac{\sqrt{3}}{2\pi t^3}\exp\Big(-\frac{(x-x_0)^2}{t}+\frac{3(x-x_0)(y-y_0)}{t^2}-\frac{3(y-y_0)^2}{t^3}\Big);\end{aligned}$$

and hence

$$\begin{aligned}&u(t,x,y)\\ &= \frac{\sqrt{3}}{2\pi t^3}\exp\Big(-\frac{(x-x_0)^2}{t}+\frac{3(x-x_0)(y+tx-y_0)}{t^2}-\frac{3(y+tx-y_0)^2}{t^3}\Big).\end{aligned}$$

□

3.2 FOURIER TRANSFORM ON $\mathcal{S}'$

Next we extend the Fourier transform to $\mathcal{S}'$, the dual space of $\mathcal{S}$. We will then be able to study the Fourier transforms of various important, but nonsmooth, expressions.

DEFINITIONS.

(i) We write $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ for the space of *tempered distributions*, which is the dual of $\mathcal{S}$. That is, $u \in \mathcal{S}'$ provided $u : \mathcal{S} \to \mathbb{C}$ is linear and $\varphi_j \to \varphi$ in $\mathcal{S}$ implies $u(\varphi_j) \to u(\varphi)$.

(ii) We say

$$u_j \to u \quad \text{in } \mathcal{S}'$$

if

$$u_j(\varphi) \to u(\varphi) \quad \text{for all } \varphi \in \mathcal{S}.$$

DEFINITION. If $u \in \mathcal{S}'$, we define

$$D^\alpha u, x^\alpha u, \mathcal{F}u \in \mathcal{S}'$$

by the rules

$$\begin{aligned} D^\alpha u(\varphi) &:= (-1)^{|\alpha|} u(D^\alpha \varphi) \\ (x^\alpha u)(\varphi) &:= u(x^\alpha \varphi) \\ (\mathcal{F}u)(\varphi) &:= u(\mathcal{F}\varphi) \end{aligned}$$

for $\varphi \in \mathcal{S}$.

EXAMPLE 1: Dirac measure. It follows from the definitions that

$$\hat{\delta}_0(\varphi) = \delta_0(\hat{\varphi}) = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi \, dx.$$

We interpret this calculation as saying that

$$\hat{\delta}_0 \equiv 1.$$

□

EXAMPLE 2: Exponential of an imaginary quadratic form. The *signature* of a real, symmetric, nonsingular matrix Q is

$$\begin{aligned} \operatorname{sgn} Q := {} & \text{number of positive eigenvalues of } Q \\ & - \text{number of negative eigenvalues of } Q. \end{aligned} \tag{3.19}$$

THEOREM 3.6 (Transform of an imaginary exponential). *Let Q be a real, symmetric, nonsingular $n \times n$ matrix. Then*

$$\mathcal{F}\left(e^{\frac{i}{2}\langle Qx,x\rangle}\right) = \frac{(2\pi)^{n/2} e^{\frac{i\pi}{4}\operatorname{sgn}(Q)}}{|\det Q|^{1/2}} e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}. \tag{3.20}$$

Compare this carefully with the earlier formula (3.2). The extra *phase shift* term $e^{\frac{i\pi}{4}\operatorname{sgn} Q}$ in (3.20) arises from the complex exponential.

Proof. 1. Let $\epsilon > 0$, $Q_\epsilon := Q + \epsilon i I$. Then

$$\begin{aligned}
\mathcal{F}\left(e^{\frac{i}{2}\langle Q_\epsilon x, x\rangle}\right) &= \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\epsilon x, x\rangle - i\langle x,\xi\rangle}\, dx \\
&= \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\epsilon (x - Q_\epsilon^{-1}\xi), x - Q_\epsilon^{-1}\xi\rangle} e^{-\frac{i}{2}\langle Q_\epsilon^{-1}\xi,\xi\rangle}\, dx \\
&= e^{-\frac{i}{2}\langle Q_\epsilon^{-1}\xi,\xi\rangle} \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\epsilon y, y\rangle}\, dy.
\end{aligned}$$

Now change variables, to write Q in the form $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$, with $\lambda_1, \dots, \lambda_r > 0$ and $\lambda_{r+1}, \dots, \lambda_n < 0$. Then

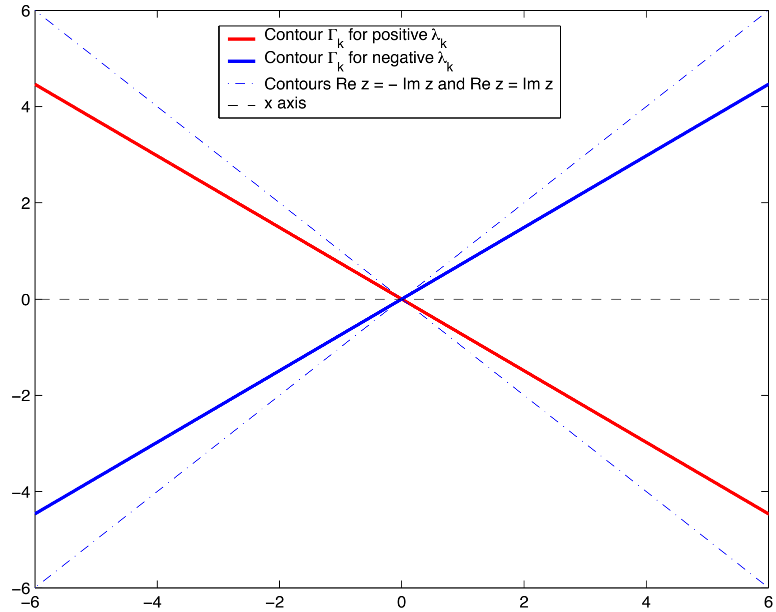
$$\int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\epsilon y, y\rangle}\, dy = \int_{\mathbb{R}^n} e^{\sum_{k=1}^n \frac{1}{2}(i\lambda_k - \epsilon) w_k^2}\, dw = \prod_{k=1}^n \int_{-\infty}^{\infty} e^{\frac{1}{2}(i\lambda_k - \epsilon) w^2}\, dw.$$

2. If $1 \le k \le r$, then $\lambda_k > 0$ and we set $z = (\epsilon - i\lambda_k)^{1/2} w$, and we take the branch of the square root so that $\operatorname{Im}(\epsilon - i\lambda_k)^{1/2} < 0$. Then

$$\int_{-\infty}^{\infty} e^{\frac{1}{2}(i\lambda_k - \epsilon) w^2}\, dw = \frac{1}{(\epsilon - i\lambda_k)^{1/2}} \int_{\Gamma_k} e^{-\frac{z^2}{2}}\, dz,$$

for the contour Γ_k as drawn.

Since $e^{-\frac{z^2}{2}} = e^{\frac{y^2 - x^2}{2} - ixy}$ and $x^2 > y^2$ on Γ_k, we can deform Γ_k into the real axis.



Hence

$$\int_{\Gamma_k} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}.$$

Thus

$$\prod_{k=1}^{r} \int_{-\infty}^{\infty} e^{\frac{1}{2}(i\lambda_k - \epsilon)w^2} dw = (2\pi)^{r/2} \prod_{k=1}^{r} \frac{1}{(\epsilon - i\lambda_k)^{1/2}}.$$

Also for $1 \leq k \leq r$:

$$\lim_{\epsilon \to 0^+} \frac{1}{(\epsilon - i\lambda_k)^{1/2}} = \frac{1}{(-i)^{1/2}\lambda_k^{1/2}} = \frac{e^{\frac{i\pi}{4}}}{\lambda_k^{1/2}},$$

since we take the branch of the square root with $(-i)^{1/2} = e^{-i\pi/4}$.

3. Similarly for $r+1 \leq k \leq n$, we set $z = (\epsilon - i\lambda_k)^{1/2}w$, but now take the branch of square root with $\operatorname{Im}(\epsilon - i\lambda_k)^{1/2} > 0$. Hence

$$\prod_{k=r+1}^{n} \int_{-\infty}^{\infty} e^{\frac{1}{2}(i\lambda_k - \epsilon)w^2} dw = (2\pi)^{\frac{n-r}{2}} \prod_{k=r+1}^{n} \frac{1}{(\epsilon - i\lambda_k)^{1/2}};$$

and for $r+1 \leq k \leq n$

$$\lim_{\epsilon \to 0^+} \frac{1}{(\epsilon - i\lambda_k)^{1/2}} = \frac{1}{(-i\lambda_k)^{1/2}} = \frac{e^{-\frac{i\pi}{4}}}{|\lambda_k|^{1/2}},$$

since we take the branch of the square root with $i^{1/2} = e^{\frac{i\pi}{4}}$.

4. Combining the foregoing calculations gives us

$$\begin{aligned}
\mathcal{F}\left(e^{\frac{i}{2}\langle Qx,x\rangle}\right) &= \lim_{\epsilon\to 0}\mathcal{F}\left(e^{\frac{i}{2}\langle Q_\epsilon x,x\rangle}\right)\\
&= e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}\frac{(2\pi)^{n/2}e^{\frac{i\pi}{4}(r-(n-r))}}{|\lambda_1\lambda_2\dots\lambda_n|^{1/2}}\\
&= e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}\frac{(2\pi)^{n/2}e^{\frac{i\pi}{4}\operatorname{sgn}Q}}{|\det\, Q|^{1/2}}.
\end{aligned}$$

□

3.3 SEMICLASSICAL FOURIER TRANSFORM

DEFINITION. The *semiclassical Fourier transform* for $h > 0$ is

$$\hat{\varphi}(\xi) = \mathcal{F}_h\varphi(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle}\varphi(x)\,dx \tag{3.21}$$

and its inverse is

$$\mathcal{F}_h^{-1}\psi(x) := \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle}\psi(\xi)\,d\xi. \tag{3.22}$$

Consequently

$$\delta_{\{y=x\}} = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}\;d\xi \quad \text{in } \mathcal{S}'. \tag{3.23}$$

This is a rescaled version of (3.9).

We record for future reference some formulas involving the parameter h:

THEOREM 3.7 (Properties of $\mathcal{F}_h$). *We have*

$$(hD_\xi)^\alpha\mathcal{F}_h\varphi = \mathcal{F}_h((-x)^\alpha\varphi); \tag{3.24}$$

$$\mathcal{F}_h((hD_x)^\alpha\varphi) = \xi^\alpha\mathcal{F}_h\varphi; \tag{3.25}$$

and

$$\|\varphi\|_{L^2} = \frac{1}{(2\pi h)^{n/2}}\|\mathcal{F}_h\varphi\|_{L^2}; \tag{3.26}$$

We present next a scaled version of the uncertainty principle, which in its various guises limits the extent to which we can simultaneously localize our calculations in both the x and ξ variables.

THEOREM 3.8 (Uncertainty principle). *We have*

$$\frac{h}{2}\|f\|_{L^2}\,\|\mathcal{F}_h f\|_{L^2} \le \|x_j f\|_{L^2}\,\|\xi_j \mathcal{F}_h f\|_{L^2} \quad (j = 1, \cdots, n). \tag{3.27}$$

Proof. To see this, note first that

$$\xi_j \mathcal{F}_h f(\xi) = \mathcal{F}_h(hD_{x_j} f).$$

Also, if A, B are self-adjoint opertors, then

$$\operatorname{Im}\langle Af, Bf\rangle = \frac{1}{2i}\langle [B, A]f, f\rangle.$$

Let $A = hD$, $B = x$. Therefore

$$[x, hD]f = \frac{h}{i}[\langle x, \partial f\rangle - \partial(xf)] = inhf.$$

Thus

$$\begin{aligned}
\|x_j f\|_{L^2}\,\|\xi_j \mathcal{F}_h f\|_{L^2} &= \|x_j f\|_{L^2}\,\|\mathcal{F}_h(hD_{x_j} f)\|_{L^2} \\
&= (2\pi h)^{n/2}\|x_j f\|_{L^2}\,\|hD_{x_j} f\|_{L^2} \\
&\ge (2\pi h)^{n/2}|\langle hD_{x_j} f, x_j f\rangle| \\
&\ge (2\pi h)^{n/2}|\operatorname{Im}\langle hD_{x_j} f, x_j f\rangle| \\
&= \frac{(2\pi h)^{n/2}}{2}|\langle [x_j, hD_{x_j}]f, f\rangle| \\
&= \frac{(2\pi h)^{n/2}}{2}h\|f\|_{L^2}^2 \\
&= \frac{h}{2}\|f\|_{L^2}\,\|\mathcal{F}_h f\|_{L^2}.
\end{aligned}$$

□

3.4 STATIONARY PHASE IN ONE DIMENSION

Understanding the right hand side of (3.21) in the limit $h \to 0$ requires our studying integral expressions with rapidly oscillating integrands. We begin with one dimensional problems.

DEFINITION. Given functions $a \in C_c^\infty(\mathbb{R})$, $\varphi \in C^\infty(\mathbb{R})$, we define for $h > 0$ the *oscillatory integral*

$$I_h = I_h(a, \varphi) := \int_{-\infty}^{\infty} e^{\frac{i\varphi}{h}}\, a\, dx.$$

LEMMA 3.9 (Rapid decay). *If $\varphi' \neq 0$ on $K := \operatorname{spt}(a)$, then*

$$I_h = O(h^\infty) \quad \text{as } h \to 0. \tag{3.28}$$

NOTATION. As explained in Appendix A, the identity (3.28) means that for each positive integer N, there exists a constant C_N such that

$$|I_h| \leq C_N h^N \quad \text{for all } 0 < h \leq 1.$$

Proof. We will integrate by parts N times. For this, observe that the operator

$$L := \frac{h}{i}\frac{1}{\varphi'}\partial_x$$

is defined on K, since $\varphi' \neq 0$ there. Notice also that

$$L\left(e^{\frac{i\varphi}{h}}\right) = e^{\frac{i\varphi}{h}}.$$

Hence $L^N(e^{i\varphi/h}) = e^{i\varphi/h}$, for $N = 1, 2, \dots$. Consequently

$$|I_h| = \left|\int_{-\infty}^{\infty} L^N\left(e^{\frac{i\varphi}{h}}\right) a \, dx\right| = \left|\int_{-\infty}^{\infty} e^{i\varphi/h}(L^*)^N a \, dx\right|,$$

L^* denoting the adjoint of L. Since a is smooth, $L^* a = -\frac{h}{i}\partial_x\left(\frac{a}{\varphi'}\right)$ is of order h. We deduce that $|I_h| \leq C_N h^N$. □

Suppose next that φ' vanishes at some point within $K := \operatorname{spt}(a)$, in which case the oscillatory integral is no longer of order h^∞. We instead want to expand I_h in an asymptotic expansion in powers of h:

THEOREM 3.10 (Stationary phase). *Let $a \in C_c^\infty(\mathbb{R})$. Suppose that $x_0 \in K = \operatorname{spt}(a)$ and*

$$\varphi'(x_0) = 0, \ \ \varphi''(x_0) \neq 0.$$

Assume further that $\varphi'(x) \neq 0$ on $K - \{x_0\}$.

(i) *There exist for each $k = 0, 1, \dots$ differential operators $A_{2k}(x, D)$, of order less than or equal to $2k$, such that for all N*

$$\left| I_h - \left(\sum_{k=0}^{N-1} A_{2k}(x,D)a(x_0)h^{k+\frac{1}{2}}\right) e^{\frac{i}{h}\varphi(x_0)} \right| \leq C_N h^{N+\frac{1}{2}} \sum_{0 \leq m \leq 2N+2} \sup_{\mathbb{R}} |a^{(m)}|. \tag{3.29}$$

(i) *In particular,*

$$A_0 = (2\pi)^{1/2}|\varphi''(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\varphi''(x_0)}; \tag{3.30}$$

and consequently

$$I_h = (2\pi h)^{1/2} |\varphi''(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\varphi''(x_0)} e^{\frac{i\varphi(x_0)}{h}} a(x_0) + O(h^{3/2}) \tag{3.31}$$

as $h \to 0$.

NOTATION. We will sometimes write (3.29) in the less precise form

$$I_h \sim e^{\frac{i}{h}\varphi(x_0)} \sum_{k=0}^{\infty} A_{2k}(x, D) a(x_0) h^{k+\frac{1}{2}}. \tag{3.32}$$

We present two proofs of this important theorem. The second proof is more complicated, but provides us with explicit expressions for the terms of the expansion (3.32): see (3.35).

First proof of Theorem 3.10. 1. We may without loss assume $x_0 = 0$, $\varphi(0) = 0$. Then $\varphi(x) = \frac{1}{2}\psi(x)x^2$, for

$$\psi(x) := 2\int_0^1 (1-t)\varphi''(tx)\,dt.$$

Notice that $\psi(0) = \varphi''(0) \neq 0$. We change variables by writing

$$y := |\psi(x)|^{1/2} x$$

for x near 0. Thus

$$\partial_y x = |\varphi''(0)|^{-1/2} \quad \text{at } x = y = 0.$$

Now select a smooth function $\chi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ near 0, and $\operatorname{sgn}\varphi''(x) = \operatorname{sgn}\varphi''(0) \neq 0$ on the support of χ. Then Lemma 3.9 implies

$$\begin{aligned} I_h &= \int_{-\infty}^{\infty} e^{i\varphi(x)/h}\chi(x)a(x)\,dx + \int_{-\infty}^{\infty} e^{i\varphi(x)/h}(1-\chi(x))a(x)\,dx \\ &= \int_{-\infty}^{\infty} e^{\frac{i\epsilon}{2h}y^2} u(y)\,dy + O(h^\infty), \end{aligned}$$

for $\epsilon := \operatorname{sgn}\varphi''(0) = \pm 1$, $u(y) := \chi(x(y))a(x(y))|\det \partial_y x|$.

2. The Fourier transform formula (3.20) tells us that

$$\mathcal{F}\left(e^{-\frac{i\epsilon y^2}{2h}}\right) = (2\pi h)^{1/2} e^{-\frac{i\pi\epsilon}{4}} e^{\frac{i\epsilon h\xi^2}{2}}.$$

Applying (3.14), we see that consequently

$$I_h = \left(\frac{h}{2\pi}\right)^{1/2} e^{\frac{i\pi\epsilon}{4}} \int_{-\infty}^{\infty} e^{-\frac{i\epsilon h\xi^2}{2}} \hat{u}(\xi)\,d\xi + O(h^\infty).$$

The advantage is that the small parameter h, and not h^{-1}, occurs in the exponential.

3. Next, write

$$J(h,u) := \int_{-\infty}^{\infty} e^{-\frac{i\epsilon h\xi^2}{2}}\hat{u}(\xi)\,d\xi, \quad J(0,u) = 2\pi u(0).$$

Then

$$\partial_h J(h,u) = \int_{-\infty}^{\infty} e^{-\frac{i\epsilon h\xi^2}{2}}\left(\frac{\epsilon\xi^2}{2i}\hat{u}(\xi)\right)d\xi = J(h,Pu)$$

for $P := (i\epsilon/2)\,\partial^2$. Continuing, we discover

$$\partial_h^k J(h,u) = J(h,P^k u).$$

Therefore

$$J(h,u) = \sum_{k=0}^{N-1}\frac{h^k}{k!}J(0,P^k u) + \frac{h^N}{N!}R_N(h,u),$$

for the remainder term

$$R_N(h,u) := N\int_0^1 (1-t)^{N-1}J(th,P^N u)\,dt.$$

Thus Lemma 3.5 implies

$$|R_N| \le C_N\|\widehat{P^N u}\|_{L^1} \le C_N \sum_{0\le k\le 2}\sup_{\mathbb{R}}|\partial^k(P^N u)|.$$

4. Since the definition of J gives

$$h^k J(0,P^k u) = h^2 P^k u(0) = (h/2i)^k u^{(2k)}(0)$$

and since $u = \chi(x(y))a(x(y))|\det\partial_y x|$, the expansion follows. □

The second proof of stationary phase asymptotics will employ this

LEMMA 3.11 (More on rapid decay). *Suppose that $a \in C_c^\infty(\mathbb{R})$ and that $\varphi \in C^\infty(\mathbb{R})$. For each positive integer m, there exists a constant C_m such that*

$$\left|\int_{-\infty}^{\infty} e^{i\varphi/h}a\,dx\right| \le C_m h^m \sum_{0\le k\le m}\sup_{\mathbb{R}}(|a^{(k)}||\varphi'|^{k-2m}). \tag{3.33}$$

This inequality will be useful at points where φ' is small, provided $a^{(m)}$ is also small.

Proof. The proof is an induction on m, the case $m = 0$ being obvious. Assume the assertion for $m-1$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\varphi/h} a\, dx &= \frac{h}{i}\int_{-\infty}^{\infty} \left(e^{i\varphi/h}\right)' \frac{a}{\varphi'} dx \\ &= -\frac{h}{i}\int_{-\infty}^{\infty} e^{i\varphi/h}\left(\frac{a}{\varphi'}\right)' dx = -\frac{h}{i}\int_{-\infty}^{\infty} e^{i\varphi/h}\tilde{a}\, dx,\end{aligned}$$

for

$$\tilde{a} := (a/\varphi')'.$$

Observe that

$$|\tilde{a}^{(k)}| = |(a/\varphi')^{(k+1)}| \le C \sum_{0\le j\le k+1} a^{(j)}|\varphi'|^{j-k-2}.$$

The induction hypothesis therefore implies

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{i\varphi(x)/h} a\, dx\right| &\le h\left|\int_{-\infty}^{\infty} e^{i\varphi(x)/h}\tilde{a}\, dx\right| \\ &\le hC_{m-1}h^{m-1}\sum_{0\le k\le m-1}\sup_{\mathbb{R}}(|\tilde{a}^{(k)}||\varphi'|^{k-2(m-1)}) \\ &\le C_m h^m \sum_{0\le j\le m}\sup_{\mathbb{R}}(|a^{(j)}||\varphi'|^{j-2m}).\end{aligned}$$

□

Second proof of Theorem 3.10. 1. As before, we may assume $x_0 = 0$, $\varphi(0) = \varphi'(0) = 0, \varphi''(0) \neq 0$. To find the expansion in h of our integral

$$I_h = \int_{-\infty}^{\infty} e^{i\varphi/h} a\, dx,$$

we write

$$\varphi_s(x) := \varphi''(0)x^2/2 + sg(x)$$

for $0 \le s \le 1$, where

$$g(x) := \varphi(x) - \varphi''(0)x^2/2.$$

Then $\varphi = \varphi_1$ and $g = O(x^3)$ as $x \to 0$. Furthermore,

$$\varphi_s'(x) = \varphi''(0)x + O(x^2),$$

and therefore

$$|x| \le |\varphi''(0)|^{-1}|\varphi_s'(x) + O(x^2)| \le 2|\varphi''(0)|^{-1}|\varphi'(x)|$$

for sufficiently small x. Consequently, using a cutoff function χ as in the first proof, we may assume that

$$\frac{x}{\varphi_s'(x)} \text{ is bounded on } K = \operatorname{spt}(a). \tag{3.34}$$

2.We also write

$$I_h(s) := \int_{-\infty}^{\infty} e^{i\varphi_s/h} a\, dx.$$

Let us calculate

$$\frac{d^{2m}}{ds^{2m}} I_h(s) = (i/h)^{2m} \int_{-\infty}^{\infty} e^{i\varphi_s/h} g^{2m} a\, dx.$$

Lemma 3.11, with $3m$ replacing m, implies

$$|I_h^{(2m)}(s)| \le \frac{C}{h^{2m}} h^{3m} \sum_{0 \le k \le 3m} \sup_{\mathbb{R}} (|(ag^{2m})^{(k)}||\varphi_s'|^{k-6m}).$$

Now the amplitude ag^{2m} vanishes to order $6m$ at $x = 0$. Consequently, for each $0 \le k \le 3m$ we recall (3.34) to estimate

$$|(ag^{2m})^{(k)}||\varphi_s'|^{k-6m} \le C|x|^{6m-k}|x|^{k-6m} \le C.$$

Therefore

$$|I_h^{(2m)}(s)| \le M h^m.$$

It follows that

$$\begin{aligned} I_h = I_h(1) &= \sum_{l=0}^{2m-1} I_h^{(l)}(0)/l! + \frac{1}{(2m-1)!} \int_0^1 (1-s)^{2m-1} I_h^{(2m)}(s)\, ds \\ &= \sum_{l=0}^{2m-1} I_h^{(l)}(0)/l! + O(h^m). \end{aligned}$$

3. It remains to compute the expansions in h of the terms

$$I_h^{(l)}(0) = (i/h)^l \int_{-\infty}^{\infty} e^{i\varphi_0/h} g^l a\, dx$$

for $l = 0, \dots, 2m-1$. But this follows as in the first proof, since the phase $\varphi_0(x) = \varphi''(0)x^2/2$ is purely quadratic. Up to constants, the terms in the expansion are

$$h^{\frac{1}{2}+k-l}(g^l a)^{(2k)}(0)$$

for $l < 2m$ and $k = 0, 1, \cdots$.

This at first first looks discouraging because of $-l$ in the power of h. Recall however that $g = O(x^3)$ near 0; so that $(g^l a)^{(2k)}(0) = 0$ unless $2k \ge 3l$. Also, if $k - l = j$, then

$$3j = 3k - 3l \ge k, \quad 2j = 2k - 2l \ge l.$$

This means that there are at most finitely many values of k and l in the expansion corresponding to the term $h^{\frac{1}{2}+j} = h^{\frac{1}{2}+k-l}$. $\square$

REMARK. This second proof is a one dimensional, variant of that in Hörmander [H1, Section 7.7]. It avoids the Morse Lemma (see Theorem 3.14 below), but at some considerable technical expense. But this proof in fact provides the explicit expansion

$$\begin{aligned}\int_{\mathbb{R}} e^{i\varphi/h} a\,dx \sim e^{\frac{i\pi}{4}\operatorname{sgn}\varphi''(x_0)} \left(\frac{2\pi h}{|\varphi''(0)|}\right)^{\frac{1}{2}} \\ \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \left(\frac{h}{2i\varphi''(0)}\right)^k \frac{1}{l!}\frac{1}{k!}\frac{d^{2k}}{dx^{2k}}((i/h)^l g^l a)(0).\end{aligned} \tag{3.35}$$

□

3.5 STATIONARY PHASE IN HIGHER DIMENSIONS

We turn next to n-dimensional integrals.

DEFINITION. We call the expression

$$I_h = I_h(a,\varphi) = \int_{\mathbb{R}^n} e^{i\varphi/h} a\,dx,$$

where $a \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$ are real-valued.

3.5.1 Quadratic phase function. We begin with the case of a quadratic phase

$$\varphi(x) = \frac{1}{2}\langle Qx, x\rangle,$$

where Q is a nonsingular, symmetric matrix.

THEOREM 3.12 (Quadratic phase asymptotics). *For each postive integer N, we have the expansion*

$$I_h = (2\pi h)^{\frac{n}{2}} \frac{e^{\frac{i\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{\frac{1}{2}}} \left(\sum_{k=0}^{N-1} \frac{h^k}{k!}\left(\frac{\langle Q^{-1}D, D\rangle}{2i}\right)^k a(0) + O(h^N)\right). \tag{3.36}$$

Proof. 1. The Fourier transform formulas (3.20) and (3.14) imply

$$I_h = \left(\frac{h}{2\pi}\right)^{n/2} \frac{e^{\frac{i\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{\frac{1}{2}}} \int_{\mathbb{R}^n} e^{-\frac{ih}{2}\langle Q^{-1}\xi,\xi\rangle} \hat{a}(\xi)\,d\xi.$$

Write

$$J(h,a) := \int_{\mathbb{R}^n} e^{-\frac{ih}{2}\langle Q^{-1}\xi,\xi\rangle} \hat{a}(\xi)\,d\xi;$$

then

$$\partial_h J(h,a) = \int_{\mathbb{R}^n} e^{-\frac{ih}{2}\langle Q^{-1}\xi,\xi\rangle} \left(-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle \hat{a}(\xi)\right) d\xi = J(h, Pa)$$

for

$$P := -\frac{i}{2}\langle Q^{-1}D, D\rangle.$$

Therefore

$$J(h,a) = \sum_{k=0}^{N-1} \frac{h^k}{k!} J(0, P^k a) + \frac{h^N}{N!} R_N(h,a),$$

for the remainder term

$$R_N(h,a) := N \int_0^1 (1-t)^{N-1} J(th, P^N a)\, dt.$$

2. Now (3.5) gives

$$J(0, P^k a) = \int_{\mathbb{R}^n} \left(-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle\right)^k \hat{a}(\xi)\, d\xi = (2\pi)^n P^k a(0).$$

Furthermore, Lemma 3.5,(ii) implies

$$|R_N| \le C_N \|\widehat{P^N a}\|_{L^1} \le C_N \sup_{|\alpha| \le 2N+n+1} |\partial^\alpha a|.$$

□

3.5.2 General phase function. Assume next that the phase φ is a smooth function.

LEMMA 3.13 (Rapid decay again). *If $\partial\varphi \neq 0$ on $K := \operatorname{spt}(a)$, then*

$$I_h = O(h^\infty).$$

In particular, for each positive integer N

$$|I_h| \le C h^N \sum_{|\alpha| \le N} \sup_{\mathbb{R}^n} |\partial^\alpha a|, \tag{3.37}$$

where C depends upon only K and n.

Proof. Define the operator

$$L := \frac{h}{i} \frac{1}{|\partial\varphi|^2} \langle \partial\varphi, \partial\rangle$$

for $x \in K$, and observe that

$$L\left(e^{i\varphi/h}\right) = e^{i\varphi/h}.$$

Hence $L^N\left(e^{i\varphi/h}\right) = e^{i\varphi/h}$, and consequently

$$|I_h| = \left|\int_{\mathbb{R}^n} L^N\left(e^{i\varphi/h}\right) a\,dx\right| = \left|\int_{\mathbb{R}^n} e^{i\varphi/h}(L^*)^N a\,dx\right| \le Ch^N.$$

□

DEFINITION. We say $\varphi : \mathbb{R}^n \to \mathbb{R}$ has a *nondegenerate critical point* at x_0 if

$$\partial\varphi(x_0) = 0, \ \det \partial^2\varphi(x_0) \neq 0.$$

We also write

$$\begin{aligned}\operatorname{sgn}\partial^2\varphi(x_0) := &\text{ number of postive eigenvalues of } \partial^2\varphi(x_0)\\ &- \text{number of negative eigenvalues of } \partial^2\varphi(x_0).\end{aligned}$$

Next we change variables locally to convert the phase function φ into a quadratic:

THEOREM 3.14 (Morse Lemma). *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be smooth, with a nondegenerate critical point at x_0. Then there exist neighborhoods U of 0 and V of x_0 and a diffeomorphism*

$$\gamma : V \to U$$

such that

$$(\varphi \circ \gamma^{-1})(x) = \varphi(x_0) + \frac{1}{2}(x_1^2 + \cdots + x_r^2 - x_{r+1}^2 \cdots - x_n^2), \tag{3.38}$$

where r is the number of positive eigenvalues of $\partial^2\varphi(x_0)$.

Proof. 1. As usual, we suppose $x_0 = 0$, $\varphi(0) = 0$. After a linear change of variables, we have

$$\varphi(x) = \frac{1}{2}(x_1^2 + \cdots + x_r^2 - x_{r+1}^2 \cdots - x_n^2) + O(|x|^3);$$

and so the problem is to design a further change of variables that removes the cubic and higher terms.

2. Now

$$\varphi(x) = \int_0^1 (1-t)\partial_t^2\varphi(tx)\,dt = \frac{1}{2}\langle x, Q(x)x\rangle,$$

where

$$Q(0) = \partial^2\varphi(0) = \begin{pmatrix} I_r & O \\ O & -I_{n-r}\end{pmatrix}.$$

In this expression the upper indentity matrix is $r \times r$ and the lower identity matrix is $(n-r)\times(n-r)$. We want to find a smooth mapping A from $\mathbb{R}^n$ to $\mathbb{M}^{n\times n}$ such that

$$\langle A(x)x, Q(0)A(x)x\rangle = \langle x, Q(x)x\rangle. \tag{3.39}$$

Then

$$\gamma(x) = A(x)x$$

is the desired change of variable.

Formula (3.39) will hold provided

$$A^T(x)Q(0)A(x) = Q(x). \tag{3.40}$$

Let $F : \mathbb{M}^{n\times n} \to \mathbb{S}^{n\times n}$ be defined by

$$F(A) = A^T Q(0)A.$$

We want to find a *right inverse* $G : \mathbb{S}^{n\times n} \to \mathbb{M}^{n\times n}$, so that

$$F \circ G = I \quad \text{near } Q(0).$$

Then

$$A(x) := G(Q(x))$$

will solve (3.40).

3. We will apply a version of the Implicit Function Theorem (Theorem C.2). To do so, it suffices to find $A \in L(\mathbb{S}^{n\times n}, \mathbb{M}^{n\times n})$ such that

$$\partial F(I)A = I.$$

Now

$$\partial F(I)(C) = C^T Q(0) + Q(0)C.$$

Define

$$A(D) := \frac{1}{2}Q(0)^{-1}D$$

for $D \in \mathbb{S}^{n\times n}$. Then

$$\begin{aligned}
\partial F(I)A(D) &= \frac{1}{2}\partial F(I)(Q^{-1}(0)D)\\
&= \frac{1}{2}[(Q(0)^{-1}D)^T Q(0) + Q(0)(Q(0)^{-1}D)]\\
&= D.
\end{aligned}$$

□

Given now a general phase function φ, we apply the Morse Lemma to convert locally to a quadratic phase for which the asymptotics provided by Theorem 3.12 apply:

THEOREM 3.15 (Stationary phase asymptotics). *Assume that* $a \in C_c^\infty(\mathbb{R}^n)$. *Suppose* $x_0 \in K := \operatorname{spt}(a)$ *and*

$$\partial\varphi(x_0) = 0,\ \det \partial^2\varphi(x_0) \neq 0.$$

Assume further that $\partial\varphi(x) \neq 0$ *on* $K - \{x_0\}$.

(i) *Then there exist for* $k = 0, 1, \dots$ *differential operators* $A_{2k}(x, D)$ *of order less than or equal to* $2k$, *such that for each* N

$$\left| I_h - \left(\sum_{k=0}^{N-1} A_{2k}(x,D)a(x_0)h^{k+\frac{n}{2}} \right) e^{\frac{i\varphi(x_0)}{h}} \right| \leq C_N h^{N+\frac{n}{2}} \sum_{|\alpha| \leq 2N+n+1} \sup_{\mathbb{R}^n} |\partial^\alpha a|. \tag{3.41}$$

(ii) *In particular,*

$$A_0 = (2\pi)^{n/2}|\det\partial^2\varphi(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\partial^2\varphi(x_0)}; \tag{3.42}$$

and therefore

$$I_h = (2\pi h)^{n/2}|\det\partial^2\varphi(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\partial^2\varphi(x_0)} e^{\frac{i\varphi(x_0)}{h}} a(x_0) + O\left(h^{\frac{n+2}{2}}\right) \tag{3.43}$$

as $h \to 0$.

Proof. Without loss $x_0 = 0$, $\varphi(x_0) = \partial\varphi(x_0) = 0$. Introducing a cutoff function χ and applying the Morse Lemma, Theorem 3.14, and then Lemma 3.13, we can write

$$I_h = \int_{\mathbb{R}^n} e^{i\varphi(x)/h} a\, dx = \int_{\mathbb{R}^n} e^{\frac{i}{2h}\langle Qx,x\rangle} u\, dx + O(h^\infty),$$

where

$$Q = \begin{pmatrix} I_r & O \\ O & -I_{n-r} \end{pmatrix}$$

and

$$u(x) := a(\kappa^{-1}(x))|\det\partial\kappa^{-1}(x)|, \quad \det\partial\kappa^{-1}(0) = \det\partial\phi(x_0)^{-1}.$$

and

$$Q = \begin{pmatrix} I_r & O \\ O & -I_{n-r} \end{pmatrix}$$

Note that $\operatorname{sgn} Q = \operatorname{sgn}\partial^2\varphi(x_0)$ and $|\det Q| = 1$. We invoke Theorem 3.12 to finish the proof. □

3.6 IMPORTANT EXAMPLES.

In Chapter 4 we will be interested in asymptotics in h of various expressions involving the Fourier transform. These involve the particular phase function

$$\varphi(x,y) = \langle x, y\rangle$$

on $\mathbb{R}^n \times \mathbb{R}^n$, corresponding to the Euclidean inner product. We will also encounter important applications with the phase

$$\varphi(z,w) = \sigma(z,w) = \langle Jz, w\rangle$$

on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, corresponding to the symplectic structure. We therefore record in this section the stationary phase expansions corresponding to these special cases.

THEOREM 3.16 (Important phase functions).
(i) *Assume that $a \in C_c^\infty(\mathbb{R}^{2n})$. Then for each postive integer N,*

$$(3.44) \quad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,y\rangle} a(x,y)\,dxdy = (2\pi h)^n\left(\sum_{k=0}^{N-1}\frac{h^k}{k!}\left(\frac{\langle D_x, D_y\rangle}{i}\right)^k a(0,0) + O(h^N)\right)$$

as $h \to 0$.
(ii) *Assume that $a \in C_c^\infty(\mathbb{R}^{4n})$. Then for each postive integer N,*

$$(3.45) \quad \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\sigma(z,w)} a(z,w)\,dzdw = (2\pi h)^{2n}\left(\sum_{k=0}^{N-1}\frac{h^k}{k!}\left(\frac{\sigma(D_x, D_\xi, D_y, D_\eta)}{i}\right)^k a(0,0) + O(h^N)\right),$$

where $z = (x,\xi)$, $w = (y,\eta)$, and

$$\sigma(D_x, D_\xi, D_y, D_\eta) := \langle D_\xi, D_y\rangle - \langle D_x, D_\eta\rangle.$$

Proof. 1. We write (x,y) to denote a typical point of $\mathbb{R}^{2n}$, and let

$$Q := \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix}.$$

Then Q is symmetric, $Q^{-1} = Q$, $|\det Q| = 1$, $\operatorname{sgn}(Q) = 0$ and $Q(x,y) = (y,x)$. Consequently $\frac{1}{2}\langle Q(x,y),(x,y)\rangle = \langle x,y\rangle$.

Furthermore, since $D = (D_x, D_y)$,

$$\frac{1}{2}\langle Q^{-1}D, D\rangle = \langle D_x, D_y\rangle.$$

Hence Theorem 3.12 gives (3.44).

2. We write (z,w) to denote a typical point of $\mathbb{R}^{4n}$, where $z=(x,\xi)$, $w=(y,\eta)$. Set

$$Q := \begin{pmatrix} O & -J \\ J & O \end{pmatrix}.$$

Then Q is symmetric, $Q^{-1}=Q$, $|\det Q|=1$, $\operatorname{sgn}(Q)=0$ and $Q(z,w)=(-Jw,Jz)$. Consequently $\frac{1}{2}\langle Q(z,w),(z,w)\rangle = \langle Jz,w\rangle = \sigma(z,w)$.

We have $D=(D_z,D_w)=(D_x,D_\xi,D_y,D_\eta)$, and therefore

$$\frac{1}{2}\langle Q^{-1}D,D\rangle = \sigma(D_x,D_\xi,D_y,D_\eta).$$

Theorem 3.12 now provides us with the expansion (3.45). $\square$

4. Quantization of symbols

The Fourier transform and its inverse allow us to move at will between the position x and momentum ξ variables, but what we really want is to deal with both sets of variables simultaneously. This chapter therefore introduces the quantization of symbols, that is, of appropriate functions of both x and ξ. The resulting operators applied to functions entail information in the full (x, ξ) phase space, and particular choices of the symbol will later prove very useful, allowing us for example to "localize" in phase space.

The plan is to introduce quantization and then to work out the resulting *symbol calculus*, meaning the systematic rules for manipulating symbols and their associated operators.

4.1 QUANTIZATION FORMULAS

NOTATION. For this section we take $h > 0$ and $a \in \mathcal{S}(\mathbb{R}^{2n})$, $a = a(x, \xi)$. We hereafter call a a *symbol*.

To *quantize* this symbol means to associate with it an h-dependent linear operator acting on functions $u = u(x)$. There are several standard ways to do so:

DEFINITIONS.

(i) We define the *Weyl quantization* to be the operator $a^{\mathrm{w}}(x, hD)$ acting on $u \in \mathcal{S}(\mathbb{R}^n)$ by the formula

$$a^{\mathrm{w}}(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left(\tfrac{x+y}{2}, \xi\right) u(y)\, dyd\xi. \tag{4.1}$$

(ii) We define also the *standard quantization*

$$a(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x, \xi) u(y)\, dyd\xi \tag{4.2}$$

for $u \in \mathcal{S}$.

(iii) More generally, for $u \in \mathcal{S}$ and $0 \leq t \leq 1$, we set

$$\text{(4.3)} \quad \operatorname{Op}_t(a)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(tx + (1-t)y, \xi) u(y) \, dy d\xi.$$

Hence

$$\operatorname{Op}_{\frac{1}{2}}(a) = a^{\mathrm{w}}(x, hD), \quad \operatorname{Op}_1(a) = a(x, hD).$$

REMARKS. (i) Observe that

$$a(x, hD)u = \mathcal{F}_h^{-1}(a\mathcal{F}_h u).$$

This simple expression makes most of the subsequent calculations much easier for the standard quantization, as opposed to the Weyl quantization. However the later has many better properties and will be our principal concern.

(ii) We will only rarely be directly interested in the operators Op_t for $t \neq \frac{1}{2}, 1$; but they will prove useful for interpolating between the Weyl and standard quantizations.

EXAMPLES.

(i) If $a(x, \xi) = \xi^\alpha$, then

$$\operatorname{Op}_t(a)u = (hD)^\alpha u \qquad (0 \leq t \leq 1).$$

(ii) If $a(x, \xi) = V(x)$, then

$$\operatorname{Op}_t(a)u = V(x)u \qquad (0 \leq t \leq 1).$$

(iii) If $a(x, \xi) = \langle x, \xi \rangle$, then

$$\operatorname{Op}_t(a)u = (1-t)\langle hD, xu\rangle + t\langle x, hDu\rangle \qquad (0 \leq t \leq 1).$$

(iv) If $a(x, \xi) = \sum_{|\alpha| \leq N} a_\alpha(x)\xi^\alpha$ and $t = 1$, then

$$a(x, hD) = \sum_{|\alpha| \leq N} a_\alpha(x)(hD)^\alpha u.$$

These formulas follow straightforwardly from the definitions.

THEOREM 4.1 (Schwartz class symbols). *Assume $a \in \mathcal{S}$.*

(i) *Then for each $0 \leq t \leq 1$, $\operatorname{Op}_t(a)$ can be defined as an operator mapping $\mathcal{S}'$ to $\mathcal{S}$; and furthermore*

$$\operatorname{Op}_t(a) : \mathcal{S}' \to \mathcal{S}$$

is continuous.

(ii) *We have*

$$\mathrm{Op}_t(a)^* = \mathrm{Op}_{1-t}(\bar{a}) \qquad (0 \le t \le 1); \tag{4.4}$$

and in particular the Weyl quantization of a real symbol is self-adjoint:

$$a^{\mathrm{w}}(x, hD)^* = a^{\mathrm{w}}(x, hD) \quad \text{if } a \text{ is real.} \tag{4.5}$$

Proof. (i) We have

$$\mathrm{Op}_t(a)u(x) = \int_{\mathbb{R}^n} K_t(x, y)u(y)\, dy$$

for the kernel

$$\begin{aligned} K_t(x, y) &:= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(tx + (1-t)y, \xi)\, d\xi \\ &= \mathcal{F}_h^{-1}(a(tx + (1-t)y, \cdot))(x - y). \end{aligned}$$

Thus $K_t \in \mathcal{S}$, and so

$$\mathrm{Op}_t(a)u(x) = u(K_t(x, \cdot))$$

maps $\mathcal{S}'$ continuously into $\mathcal{S}$.

(ii) The kernel of $\mathrm{Op}_t(a)^*$ is $K_t^*(x, y) := \overline{K}_t(y, x) = \overline{K}_{1-t}(x, y)$, which is the kernel of $\mathrm{Op}_{1-t}(\bar{a})$. □

We next observe that the formulas (4.1)–(4.3) make sense if a is merely a distribution:

THEOREM 4.2 (Distributional symbols). *If $a \in \mathcal{S}'$, then $\mathrm{Op}_t(a)$ can be defined as an operator mapping $\mathcal{S}$ to $\mathcal{S}'$; and furthermore*

$$\mathrm{Op}_t(a) : \mathcal{S} \to \mathcal{S}' \qquad (0 \le t \le 1)$$

is continuous.

Proof. The formula for the distributional kernel K_t of $\mathrm{Op}_t(a)$ shows that $K_t \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. Hence $\mathrm{Op}_t(a)$ is well defined as an operator from $\mathcal{S}$ to $\mathcal{S}'$: if $u, v \in \mathcal{S}$ then

$$(\mathrm{Op}_t(a)u)(v) := K_t(u \otimes v).$$

□

4.2 COMPOSITION, ASYMPTOTIC EXPANSIONS

We begin now a careful study of the properties of the quantized operators defined above, especially the Weyl quantization. Our particular goal in this section is showing that if a and b are symbols, then there exists a symbol $c = a\#b$ such that

$$a^{\mathrm{w}}(x, hD)b^{\mathrm{w}}(x, hD) = c^{\mathrm{w}}(x, hD).$$

4.2.1 Linear symbols. We begin with linear symbols of the form

$$l(x, \xi) := \langle x^*, x\rangle + \langle \xi^*, \xi\rangle \tag{4.6}$$

for$(x^*, \xi^*) \in \mathbb{R}^{2n}$. To simplify calculations later on, *we will often identify the linear symbol l with the point (x^*, ξ^*).*

LEMMA 4.3 (Quantizing linear symbols). *Let l be given by* (4.6)*. Then*

$$\mathrm{Op}_t(l) = \langle x^*, x\rangle + \langle \xi^*, hD\rangle \qquad (0 \leq t \leq 1).$$

NOTATION. In view of this result, we hereafter write

$$l(x, hD) = l^{\mathrm{w}}(x, hD) = \langle x^*, x\rangle + \langle \xi^*, hD\rangle. \tag{4.7}$$

Proof. Let $u \in \mathcal{S}$ and compute the derivative

$$\begin{aligned}
\frac{d}{dt}\mathrm{Op}_t(l)u &= \frac{1}{(2\pi h)^n}\frac{d}{dt}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}(\langle x^*, tx + (1-t)y\rangle \\
&\qquad\qquad + \langle \xi^*, \xi\rangle)u(y)\,dyd\xi \\
&= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}\langle x^*, x-y\rangle u(y)\,dyd\xi \\
&= \frac{h}{(2\pi h)^n}\int_{\mathbb{R}^n}\left\langle x^*, D_\xi \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}u(y)\,dy\right\rangle d\xi \\
&= \frac{h}{(2\pi h)^n}\int_{\mathbb{R}^n}\left\langle x^*, D_\xi(e^{\frac{i}{h}\langle x,\xi\rangle}\hat{u}(\xi))\right\rangle d\xi.
\end{aligned}$$

Since $\hat{u}(\xi) \to 0$ rapidly as $|\xi| \to \infty$, the last expression vanishes. Therefore $\mathrm{Op}_t(l)$ does not in fact depend upon t; and consequently for all $0 \leq t \leq 1$, $\mathrm{Op}_t(l)u = \mathrm{Op}_1(l)u = \langle x^*, x\rangle u + \langle \xi^*, hD\rangle u$. $\square$

Next we compute the Weyl quantization of $e^{\frac{i}{h}l}$.

THEOREM 4.4 (Quantizing exponentials of linear symbols).
(i) *For each linear symbol l we have the identity*

$$(e^{\frac{i}{h}l})^{\mathrm{w}}(x, hD) = e^{\frac{i}{h}l(x,hD)}, \tag{4.8}$$

where

$$e^{\frac{i}{h}l(x,hD)}u(x) := e^{\frac{i}{h}\langle x^*,x\rangle + \frac{i}{2h}\langle x^*,\xi^*\rangle}u(x + \xi^*). \tag{4.9}$$

(ii) *If* $l, m \in \mathbb{R}^{2n}$, *then*

$$e^{\frac{i}{h}l(x,hD)}e^{\frac{i}{h}m(x,hD)} = e^{\frac{i}{2h}\sigma(l,m)}e^{\frac{i}{h}(l+m)(x,hD)}. \tag{4.10}$$

Proof. 1. Consider for $u \in \mathcal{S}$ the PDE

$$\begin{cases} ih\partial_t v + l(x, hD)v = 0 & (t \in \mathbb{R}) \\ v(0) = u & (t = 0). \end{cases}$$

Its unique solution is denoted

$$v(x,t) = e^{\frac{it}{h}l(x,hD)}u,$$

this formula defining the operators $e^{\frac{it}{h}l(x,hD)}$ for $t \in \mathbb{R}$. But we can check by a direct calculation using (4.7) that

$$v(x,t) = e^{\frac{it}{h}\langle x^*,x\rangle + \frac{it^2}{2h}\langle x^*,\xi^*\rangle}u(x + t\xi^*);$$

and therefore (4.9) holds.

2. Furthermore,

$$\begin{aligned}
(e^{\frac{il}{h}})^{\mathrm{w}}u &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}e^{\frac{i}{h}\left(\langle\xi^*,\xi\rangle + \langle x^*,\frac{x+y}{2}\rangle\right)}u(y)\,dyd\xi \\
&= \frac{e^{\frac{i}{2h}\langle x^*,x\rangle}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y+\xi^*,\xi\rangle}\left(e^{\frac{i}{2h}\langle x^*,y\rangle}u(y)\right)\,dyd\xi \\
&= \frac{e^{\frac{i}{2h}\langle x^*,x\rangle}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}\left(e^{\frac{i}{2h}\langle x^*,y+\xi^*\rangle}u(y+\xi^*)\right)\,dyd\xi \\
&= e^{\frac{i}{h}\langle x^*,x\rangle + \frac{i}{2h}\langle x^*,\xi^*\rangle}u(x+\xi^*),
\end{aligned}$$

since

$$\delta_{\{y=x\}} = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}d\xi \quad \text{in } \mathcal{S}',$$

according to (3.23). This proves (4.8).

3. Suppose $l(x,\xi) = \langle x_1^*, x\rangle + \langle \xi_1^*, \xi\rangle$ and $m(x,\xi) = \langle x_2^*, x\rangle + \langle \xi_2^*, \xi\rangle$. According to (4.9),

$$e^{\frac{i}{h}m(x,hD)}u(x) = e^{\frac{i}{h}\langle x_2^*,x\rangle + \frac{i}{2h}\langle x_2^*,\xi_2^*\rangle}u(x+\xi_2^*);$$

and consequently

$$\begin{aligned}
&e^{\frac{i}{h}l(x,hD)}e^{\frac{i}{h}m(x,hD)}u(x) = \\
&\qquad e^{\frac{i}{h}\langle x_1^*,x\rangle + \frac{i}{2h}\langle x_1^*,\xi_1^*\rangle}e^{\frac{i}{h}\langle x_2^*,x+\xi_1^*\rangle + \frac{i}{2h}\langle x_2^*,\xi_2^*\rangle}u(x+\xi_1^*+\xi_2^*).
\end{aligned}$$

Furthermore, (4.9) implies also that

$$e^{\frac{i}{h}(l+m)(x,hD)}u(x) = e^{\frac{i}{h}\langle x_1^*+x_2^*,x\rangle + \frac{i}{2h}\langle x_1^*+x_2^*,\xi_1^*+\xi_2^*\rangle}u(x+\xi_1^*+\xi_2^*).$$

Using the formula above, we therefore compute

$$e^{\frac{i}{h}(l+m)(x,hD)}u(x) = e^{\frac{i}{2h}(\langle x_1^*,\xi_2^*\rangle - \langle x_2^*,\xi_1^*\rangle)}e^{\frac{i}{h}l(x,hD)}e^{\frac{i}{h}m(x,hD)}u(x).$$

This proves (4.10), since $\sigma(l,m) = \langle \xi_1^*, x_2^*\rangle - \langle x_1^*, \xi_2^*\rangle$. □

4.2.2 Exponentials of quadratics. We record in the section some useful integral representation formulas for the quantization of certain quadratic exponentials. We will later apply the stationary phase expansions from Theorems 3.12 and 3.16 to these expressions.

THEOREM 4.5 (Quantizing quadratic exponentials).
(i) *Let Q denote a nonsingular, symmetric, $n \times n$ matrix. Then if $u \in \mathcal{S}(\mathbb{R}^n)$,*

$$(4.11)\quad e^{\frac{ih}{2}\langle QD,D\rangle}u(x) = \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi h)^{\frac{n}{2}}}e^{\frac{i\pi}{4}\operatorname{sgn} Q}\int_{\mathbb{R}^n} e^{-\frac{i}{2h}\langle Q^{-1}y,y\rangle}u(x+y)\,dy.$$

(ii) *In particular, if $u \in \mathcal{S}(\mathbb{R}^{2n})$, $u = u(x,y)$, then*

$$(4.12)\quad e^{ih\langle D_x,D_y\rangle}u(x,y) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x_1,y_1\rangle}u(x+x_1,y+y_1)\,dx_1 dy_1.$$

(iii) *Suppose that $u \in \mathcal{S}(\mathbb{R}^{4n})$, $u = u(z,w)$. Then*

$$(4.13)\quad e^{ih\sigma(D_z,D_z)}u(z,w) = \frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{h}\sigma(z_1,w_1)}u(z+z_1,w+w_1)\,dz_1 dw_1.$$

Proof. 1. Observe first that

$$\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle w,\xi\rangle}e^{\frac{i}{h}\langle Q\xi,\xi\rangle}d\xi = \mathcal{F}_h^{-1}(e^{\frac{i}{h}\langle Q\xi,\xi\rangle})(w) = \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi h)^{\frac{n}{2}}}e^{\frac{i\pi}{4}\operatorname{sgn} Q}e^{-\frac{i}{2h}\langle Q^{-1}w,w\rangle}.$$

Therefore

$$\begin{aligned} e^{\frac{ih}{2}\langle QD,D\rangle}u(x) &= e^{\frac{i}{h}\langle QhD,hD\rangle}u(x) \\ &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}e^{\frac{i}{h}\langle Q\xi,\xi\rangle}u(y)\,dyd\xi \\ &= \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi h)^{\frac{n}{2}}}e^{\frac{i\pi}{4}\operatorname{sgn} Q}\int_{\mathbb{R}^n} e^{-\frac{i}{2h}\langle Q^{-1}(x-y),x-y\rangle}u(y)\,dy \\ &= \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi h)^{\frac{n}{2}}}e^{\frac{i\pi}{4}\operatorname{sgn} Q}\int_{\mathbb{R}^n} e^{-\frac{i}{2h}\langle Q^{-1}y,y\rangle}u(x+y)\,dy. \end{aligned}$$

2. Assertion (4.12) is a special case of (4.11), had by replacing n by $2n$ and taking

$$Q := \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix}.$$

See the proof of Theorem 3.16,(i). It is a useful exercise to give a direct derivation.

3. Similarly, assertion (4.13) is a special case of (4.11) obtained by replacing n by $4n$ and taking

$$Q := \begin{pmatrix} O & -J \\ J & O \end{pmatrix}.$$

See the proof of Theorem 3.16,(ii). □

4.2.3 Composing symbols.

Next we establish the fundamental formula $a^{\mathrm{w}}b^{\mathrm{w}} = (a\#b)^{\mathrm{w}}$, along with a recipe for computing the new symbol $a\#b$:

THEOREM 4.6 (Composition for Weyl quantization).
(i) *Suppose that* $a, b \in \mathcal{S}$. *Then*

$$a^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD) = (a\#b)^{\mathrm{w}}(x,hD) \tag{4.14}$$

for the symbol

$$a\#b(x,\xi) := e^{\frac{ih}{2}\sigma(D_x,D_\xi,D_y,D_\eta)}\big(a(x,\xi)b(y,\eta)\big)\Big|_{\substack{y=x\\ \eta=\xi}}. \tag{4.15}$$

(ii) *We have the integral representation formula*

$$a\#b(x,\xi) = \frac{1}{(\pi h)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{-\frac{2i}{h}\sigma(w_1,w_2)}a(z+w_1)b(z+w_2)\,dw_1dw_2, \tag{4.16}$$

where $z = (x,\xi)$.

Proof. 1. For $l \in \mathbb{R}^{2n}$, define

$$\hat{a}(l) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{i}{h} l(x,\xi)} a(x,\xi)\, dx d\xi;$$

then

$$a(x,\xi) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h} l(x,\xi)} \hat{a}(l)\, dl.$$

Therefore Theorem 4.4,(i) implies

$$a^{\mathrm{w}}(x,hD) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}(l) e^{\frac{i}{h} l(x,hD)}\, dl, \tag{4.17}$$

and likewise

$$b^{\mathrm{w}}(x,hD) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \hat{b}(m) e^{\frac{i}{h} m(x,hD)} dm.$$

Theorem 4.4,(ii) lets us next compute

$$\begin{aligned} &a^{\mathrm{w}}(x,hD) b^{\mathrm{w}}(x,hD) \\ &\qquad = \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}(l)\hat{b}(m) e^{\frac{i}{h} l(x,hD)} e^{\frac{i}{h} m(x,hD)}\, dm dl \\ &\qquad = \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}(l)\hat{b}(m) e^{\frac{i}{2h}\sigma(l,m)} e^{\frac{i}{h}(l+m)(x,hD)}\, dl dm \\ &\qquad = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \hat{c}(r) e^{\frac{i}{h} r(x,hD)} dr \end{aligned}$$

for

$$\hat{c}(r) := \frac{1}{(2\pi h)^{2n}} \int_{\{l+m=r\}} \hat{a}(l)\hat{b}(m) e^{\frac{i\sigma(l,m)}{2h}} dl. \tag{4.18}$$

To get this, we changed variables by setting $r = m + l$.

2. We will show that $\hat{c}$ defined by (4.18) is the Fourier transform of the symbol c defined by the right hand side of (4.15). We first simplify notation by writing $z = (x,\xi)$, $w = (y,\eta)$. Then

$$c(z) = e^{\frac{ih}{2}\sigma(D_z,D_w)} a(z)b(w)|_{w=z} = e^{\frac{i}{2h}\sigma(hD_z,hD_w)} a(z)b(w)|_{w=z}$$

and

$$\begin{aligned} a(z) &= \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h} l(z)} \hat{a}(l)\, dl, \\ b(w) &= \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h} m(w)} \hat{b}(m)\, dm. \end{aligned}$$

Furthermore, a direct calculation, the details of which we leave to the reader, demonstrates that

$$e^{\frac{i}{2h}\sigma(hD_z,hD_w)} e^{\frac{i}{h}(l(z)+m(w))} = e^{\frac{i}{h}(l(z)+m(w)) + \frac{i}{2h}\sigma(l,m)}.$$

Consequently

$$\begin{aligned}c(z) &= \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{2h}\sigma(hD_z, hD_w)} e^{\frac{i}{h}(l(z)+m(w))} \Big|_{z=w} \hat{a}(l)\hat{b}(m)\, dldm \\ &= \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(l(z)+m(z))+\frac{i}{2h}\sigma(l,m)} \hat{a}(l)\hat{b}(m)\, dldm.\end{aligned}$$

The semiclassical Fourier transform of c is therefore

$$\frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left(\frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(l+m-r)(z)} dz \right) e^{\frac{i}{2h}\sigma(l,m)} \hat{a}(l)\hat{b}(m)\, dldm.$$

According to (3.23), the term inside the parentheses is $\delta_{\{l+m=r\}}$ in $\mathcal{S}'$. Thus the foregoing equals

$$\frac{1}{(2\pi h)^{2n}} \int_{\{l+m=r\}} e^{\frac{i}{2h}\sigma(l,m)} \hat{a}(l)\hat{b}(m)\, dl = \hat{c}(r),$$

in view of (4.18).

3. Formula (4.16) follows from Theorem 4.5,(iii), with $\frac{h}{2}$ replacing h. □

4.2.4 Asymptotics. We next apply stationary phase to derive a useful asymptotic expansion of $a\#b$:

THEOREM 4.7 (Semiclassical expansions). *Assume $a, b \in \mathcal{S}$.*

(i) *We have for $N = 0, 1, \dots,$*

(4.19) $a\#b(x,\xi) =$

$$\sum_{k=0}^{N} \frac{h^k}{k!} \left(\frac{i}{2} \sigma(D_x, D_\xi, D_y, D_\eta) \right)^k (a(x,\xi) b(y,\eta)) \Big|_{\substack{y=x \\ \eta=\xi}} + O(h^{N+1})$$

as $h \to 0$, the error taken in $\mathcal{S}$.

(ii) *In particular,*

$$a\#b = ab + \frac{h}{2i}\{a,b\} + O(h^2); \tag{4.20}$$

and

$$[a^{\mathrm{w}}, b^{\mathrm{w}}] = \frac{h}{i}\{a,b\}^{\mathrm{w}} + O(h^2). \tag{4.21}$$

(iii) *If* $\operatorname{spt}(a) \cap \operatorname{spt}(b) = \emptyset$, *then*

$$a\#b = O(h^\infty).$$

Proof. 1. To prove (4.19), we apply the stationary phase Theorem 3.16,(ii), with $\frac{h}{2}$ replacing h and $-\sigma$ replacing σ, to the integral formula (4.16).

2. Next, compute

$$\begin{aligned} a\#b &= ab + \frac{ih}{2}\sigma(D_x, D_\xi, D_y, D_\eta)a(x,\xi)b(y,\eta)\Big|_{\substack{y=x\\ \eta=\xi}} + O(h^2) \\ &= ab + \frac{ih}{2}(\langle D_\xi a, D_y b\rangle - \langle D_x a, D_\eta b\rangle)\Big|_{\substack{y=x\\ \eta=\xi}} + O(h^2) \\ &= ab + \frac{h}{2i}(\langle \partial_\xi a, \partial_x b\rangle - \langle \partial_x a, \partial_\xi b\rangle) + O(h^2) \\ &= ab + \frac{h}{2i}\{a,b\} + O(h^2). \end{aligned}$$

Consequently,

$$\begin{aligned} [a^{\mathrm{w}}, b^{\mathrm{w}}] &= a^{\mathrm{w}}b^{\mathrm{w}} - b^{\mathrm{w}}a^{\mathrm{w}} = (a\#b - b\#a)^{\mathrm{w}} \\ &= \left(ab + \frac{h}{2i}\{a,b\} - \left(ba + \frac{h}{2i}\{b,a\}\right) + O(h^2)\right)^{\mathrm{w}} \\ &= \frac{h}{i}\{a,b\}^{\mathrm{w}} + O(h^2). \end{aligned}$$

3. If $\mathrm{spt}(a) \cap \mathrm{spt}(b) = \emptyset$, each term in the expansion (4.19) vanishes. □

4.2.5 Standard quantization. Next we replace Weyl ($t = \frac{1}{2}$) by standard ($t = 1$) quantization in our formulas. The proofs are simpler.

THEOREM 4.8 (Composition for standard quantization).
(i) *Let $a, b \in \mathcal{S}$. Then*

$$a(x, hD)b(x, hD) = c(x, hD)$$

for the symbol

$$c(x,\xi) = e^{ih\langle D_\xi, D_y\rangle}(a(x,\xi)b(y,\eta))\Big|_{\substack{y=x\\ \eta=\xi}}. \tag{4.22}$$

(ii) *We have the integral representation formula*

$$c(x,\xi) := \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x_1, \xi_1\rangle} a(x, \xi+\xi_1)b(x+x_1, \xi)\, dx_1 d\xi_1. \tag{4.23}$$

(iii) *For each* $N = 0, 1, \dots,$

$$c(x,\xi) = \sum_{k=0}^{N} \frac{h^k}{k!} \left(i\langle D_\xi, D_y\rangle\right)^k \left(a(x,\xi)b(y,\eta)\right)\Big|_{\substack{y=x\\ \eta=\xi}} + O(h^{N+1}) \tag{4.24}$$

as $h \to 0$*, the error taken in* $\mathcal{S}$*.*

Proof. Let $u \in \mathcal{S}$. Then

$$\begin{aligned} &a(x,hD)b(x,hD)u(x) \\ &\quad = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\langle x,\eta\rangle + \langle y,\xi-\eta\rangle)} a(x,\eta)b(y,\xi)\hat{u}(\xi)\, dy d\eta d\xi \\ &\quad = \frac{1}{(2\pi h)^{n}} \int_{\mathbb{R}^n} c(x,\xi) e^{\frac{i}{h}\langle x,\xi\rangle}\hat{u}(\xi)\, d\xi, \\ &\quad = c(x,hD)u(x) \end{aligned}$$

for

$$c(x,\xi) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x-y,\xi-\eta\rangle} a(x,\eta)b(y,\xi)\, dy d\eta.$$

Change variables by putting $x_1 = y - x, \xi_1 = \eta - \xi$, to rewrite c in the form (4.23). Then (4.22) is a consequence of Theorem 4.5,(ii).

Finally, the stationary phase Theorem 3.16,(i) provides the asymptotic expansion (4.24) □

THEOREM 4.9 (Adjoints for standard quantization). *If* $a \in \mathcal{S}$*, then*

$$a(x,hD)^* = b(x,hD),$$

for

$$b(x,\xi) := e^{ih\langle D_x, D_\xi\rangle} \bar{a}(x,\xi). \tag{4.25}$$

Proof. 1. We first observe that, as in the proof of Theorem 4.4,

$$\mathrm{Op}_t\left(e^{\frac{i}{h}l(x,\xi)}\right)u(x) = e^{\frac{i}{h}\langle x,x^*\rangle + \frac{i}{h}(1-t)\langle x^*,\xi^*\rangle} u(x+\xi^*).$$

It follows that

$$\mathrm{Op}_t\left(e^{\frac{i}{h}l(x,\xi)}\right) = e^{\frac{i}{h}(s-t)\langle x^*,\xi^*\rangle}\mathrm{Op}_s\left(e^{\frac{i}{h}l(x,\xi)}\right). \tag{4.26}$$

Next we record an interesting conversion formula, namely that if

$$A = \mathrm{Op}_t(a_t) \qquad (0 \le t \le 1),$$

then

$$a_t(x,\xi) = e^{i(t-s)h\langle D_x, D_\xi\rangle} a_s(x,\xi). \tag{4.27}$$

To see this, notice that the decomposition formula (4.17) implies

$$\mathrm{Op}_t(a_t) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}_t(l)\mathrm{Op}_t(e^{\frac{i}{h}l})\, dl.$$

We apply (4.26) to derive (4.27).

2. Next recall (4.4):

$$a(x, hD)^* = \mathrm{Op}_1(a)^* = \mathrm{Op}_0(\bar{a}).$$

We now invoke (4.27), to write

$$\mathrm{Op}_0(\bar{a}) = \mathrm{Op}_1(b),$$

the symbol b defined by (4.25). □

4.3 GENERAL SYMBOL CLASSES

We next extend our calculus to symbols $a = a(x, \xi, h)$ which depend on the parameter h and which can grow, along with their derivatives, as $|x|, |\xi| \to \infty$.

4.3.1 More definitions.

DEFINITION. A function $m : \mathbb{R}^{2n} \to (0, \infty)$ is called an *order function* if there exist constants C, N such that

$$m(w) \leq C\langle z - w\rangle^N m(z) \tag{4.28}$$

for all $w, z \in \mathbb{R}^n$.

Observe that if m_1, m_2 are order functions, so is $m_1 m_2$.

EXAMPLES. Standard examples are $m(z) \equiv 1$ and $m(z) = \langle z\rangle = (1 + |z|^2)^{1/2}$. □

DEFINITIONS.

(i) Given an order function m on $\mathbb{R}^{2n}$, we define the corresponding class of symbols:

$$\begin{aligned} S(m) := \{a \in C^\infty \mid &\text{ for each multiindex } \alpha \\ &\text{there exists a constant } C_\alpha \text{ so that } |\partial^\alpha a| \leq C_\alpha m\}. \end{aligned}$$

(ii) We as well define

$$S^k(m) := \{a \in C^\infty \mid |\partial^\alpha a| \leq C_\alpha h^{-k} m \;\text{ for all multiindices } \alpha\}$$

and

$$S^k_\delta(m) := \{a \in C^\infty \mid |\partial^\alpha a| \leq C_\alpha h^{-\delta|\alpha| - k} m \;\text{ for all multiindices } \alpha\}.$$

The index k indicates how singular is the symbol a as $h \to 0$; the index δ allows for increasing singularity of the higher derivatives. Notice that the more *negative* k is, the more rapidly a and its derivatives vanish as $h \to 0$.

(iii) Write also

$$S^{-\infty}(m) := \{a \in C^\infty \mid \text{for each } \alpha \text{ and } N, \ \ |\partial^\alpha a| \le C_{\alpha,N} h^N m\}.$$

So if a is a symbol belonging to $S^{-\infty}(m)$, then a and all of its derivatives are $O(h^\infty)$ as $h \to 0$.

NOTATION. If the order function is the constant function $m \equiv 1$, we will usually not write it:

$$S^k := S^k(1), \ S^k_\delta := S^k_\delta(1).$$

We will also omit zero superscripts. Thus

$$S := \{a \in C^\infty(\mathbb{R}^{2n}) \mid |\partial^\alpha a| \le C_\alpha \ \text{ for all multiindices } \alpha\}$$
$$S_\delta := \{a \in C^\infty \mid |\partial^\alpha a| \le C_\alpha h^{-\delta|\alpha|} \ \text{ for all multiindices } \alpha\}.$$

REMARKS: rescaling in h.

(i) We will show in the next subsection that if $a \in S_\delta$, then the quantization formula

$$a^{\mathrm{w}}(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left(\tfrac{x+y}{2}, \xi\right) u(y) d\xi dy$$

makes sense for $u \in \mathcal{S}$. It is often convenient to rescale to the case $h = 1$, by changing to the new variables

$$\tilde{x} := h^{-\frac{1}{2}} x, \ \tilde{y} := h^{-\frac{1}{2}} y, \ \tilde{\xi} := h^{-\frac{1}{2}} \xi. \tag{4.29}$$

Then

$$\begin{aligned} a^{\mathrm{w}}&(x, hD)u(x) \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle} u(y)\, dy d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a_h\left(\tfrac{\tilde{x}+\tilde{y}}{2}, \tilde{\xi}\right) e^{i\langle \tilde{x}-\tilde{y},\tilde{\xi}\rangle} \tilde{u}(\tilde{y})\, d\tilde{y} d\tilde{\xi}; \end{aligned}$$

and therefore

$$a^{\mathrm{w}}(x, hD)u(x) = a_h^{\mathrm{w}}(\tilde{x}, D)\tilde{u}(\tilde{x}), \tag{4.30}$$

for

$$\tilde{u}(\tilde{x}) := u(x) = u(h^{\frac{1}{2}}\tilde{x}), \quad a_h(\tilde{x}, \tilde{\xi}) := a(x, \xi) = a(h^{\frac{1}{2}}\tilde{x}, h^{\frac{1}{2}}\tilde{\xi}). \tag{4.31}$$

(ii) Observe also that if $a \in S_\delta$, then

$$|\partial^\alpha a_h| = h^{\frac{|\alpha|}{2}} |\partial^\alpha a| \le C_\alpha h^{|\alpha|\left(\frac{1}{2}-\delta\right)} \tag{4.32}$$

for each multiindex α. If $\delta > \frac{1}{2}$, the last term is unbounded as $h \to 0$; and consequently we will henceforth always assume

$$0 \leq \delta \leq \tfrac{1}{2}.$$

We see also that the case

$$\delta = \tfrac{1}{2}$$

is critical, in that we do not then get decay as $h \to 0$ for the terms on the right hand side of (4.32) when $|\alpha| > 0$. $\square$

4.3.2 Quantization. Next we discuss the Weyl quantization of symbols in the class $S_\delta(m)$:

THEOREM 4.10 (Quantizing general symbols). *If $a \in S_\delta(m)$, then*

$$a^{\mathrm{w}}(x, hD) : \mathcal{S} \to \mathcal{S}.$$

Proof. 1. We take $h = 1$ for simplicity; so that

$$a^{\mathrm{w}}(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a\left(\tfrac{x+y}{2}, \xi\right) u(y)\, dy d\xi$$

for $u \in \mathcal{S}$. Observe next that $L_1 e^{i\langle x-y,\xi\rangle} = e^{i\langle x-y,\xi\rangle}$, where

$$L_1 := \frac{1 + \langle x-y, D_\xi\rangle}{1 + |x-y|^2};$$

and $L_2 e^{i\langle x-y,\xi\rangle} = e^{i\langle x-y,\xi\rangle}$ for

$$L_2 := \frac{1 - \langle \xi, D_y\rangle}{1 + |\xi|^2}.$$

We employ these operators and the usual integration by parts argument, to show $a^{\mathrm{w}}(x, D) : \mathcal{S} \to L^\infty$.

2. Furthermore,

$$x_j a^{\mathrm{w}}(x, D)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (D_{\xi_j} + y_j) e^{i\langle x-y,\xi\rangle} a u\, dy d\xi.$$

We integrate by parts, to conclude that $x^\alpha a^{\mathrm{w}}(x, D) : \mathcal{S} \to L^\infty$ for each multinomial x^α. Also, since

$$\left(e^{-\frac{i}{2}\langle D_x, D_\xi\rangle} a\right)(x, D)u = a^{\mathrm{w}}(x, D)u,$$

we have

$$
\begin{aligned}
D_{x_j} a^{\mathrm{w}}(x,D)u &= D_{x_j}\left(e^{-\frac{i}{2}\langle D_x, D_\xi\rangle} a\right)(x,D)u \\
&= D_{x_j}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{2}\langle D_y, D_\xi\rangle} a(y,\xi) e^{i\langle x-y,\xi\rangle} u(y)\,dyd\xi\right) \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{2}\langle D_y, D_\xi\rangle} a(y,\xi)(-D_{y_j} e^{i\langle x-y,\xi\rangle}) u(y)\,dyd\xi.
\end{aligned}
$$

Again integrate by parts, to deduce $D^\beta a^{\mathrm{w}}(x,D) : \mathcal{S} \to L^\infty$ for each partial derivative D^β.

Consequently, $D^\beta x^\alpha a^{\mathrm{w}}(x,D) : \mathcal{S} \to L^\infty$, for all multiindices α, β. It follows that $a^{\mathrm{w}}(x,D) : \mathcal{S} \to \mathcal{S}$. □

4.3.3 Asymptotic series. Next we consider infinite sums of terms in various symbol classes.

DEFINITION. Let $a \in S^{k_0}_\delta(m)$ and $a_j \in S^{k_j}_\delta(m)$, where $k_{j+1} < k_j$, $k_j \to -\infty$. We say that a is *asymptotic* to $\sum a_j$, and write

$$
a \sim \sum_{j=0}^{\infty} a_j,
$$

provided for each $N = 1, 2, \dots$

$$
a - \sum_{j=0}^{N-1} a_j \in S^{k_N}_\delta(m). \tag{4.33}
$$

INTERPRETATION. Observe that for each $h > 0$, the series $\sum_{j=0}^{\infty} a_j$ need not converge in any sense. We are requiring rather in (4.33) that for each N, the difference $a - \sum_{j=0}^{N-1} a_j$, and its derivatives, vanish at appropriate rates as $h \to 0$.

Perhaps surprisingly, we can always construct such an asymptotic sum of symbols:

THEOREM 4.11 (Borel's Theorem).
(i) *Assume $a_j \in S^{k_j}_\delta(m)$, where $k_{j+1} < k_j$, $k_j \to -\infty$. Then there exists a symbol $a \in S^{k_0}_\delta(m)$ such that*

$$
a \sim \sum_{j=0}^{\infty} a_j.
$$

(ii) *If also $\hat{a} \sim \sum_{j=0}^{\infty} a_j$, then*

$$
a - \hat{a} \in S^{-\infty}(m).
$$

Proof. 1. Choose a C^∞ function χ such that

$$0 \leq \chi \leq 1, \quad \chi \equiv 1 \text{ on } [0,1], \ \chi \equiv 0 \text{ on } [2,\infty).$$

We define

$$a := \sum_{j=0}^{\infty} a_j \chi(\lambda_j h), \tag{4.34}$$

where the sequence $\lambda_j \to \infty$ must be selected. Since $\lambda_j \to \infty$, there are for each $h > 0$ at most finitely many nonzero terms in the sum (4.34).

Now for each multiindex α, with $|\alpha| \leq j$, we have

$$\begin{aligned} |\partial^\alpha (a_j \chi(\lambda_j h))| &= |(\partial^\alpha a_j)\chi(\lambda_j h)| \\ &\leq C_{j,\alpha} h^{-k_j - \delta|\alpha|} m \chi(\lambda_j h) \\ &= C_{j,\alpha} h^{-k_j - \delta|\alpha|} \frac{\lambda_j h}{\lambda_j h} m \chi(\lambda_j h) \\ &\leq 2C_{j,\alpha} \frac{h^{-k_j - 1 - \delta|\alpha|}}{\lambda_j} m \\ &\leq h^{-k_j - 1 - |\alpha|\delta} 2^{-j} m \end{aligned} \tag{4.35}$$

if λ_j is selected sufficiently large. We can accomplish this for all j and multiindices α with $|\alpha| \leq j$. We may assume also $\lambda_{j+1} \geq \lambda_j$, for all j.

2. Then

$$a - \sum_{j=0}^{N} a_j = \sum_{j=N+1}^{\infty} a_j \chi(\lambda_j h) + \sum_{j=0}^{N} a_j (\chi(\lambda_j h) - 1).$$

Fix any multiindex α. Then taking $N \geq |\alpha|$, we have

$$\begin{aligned} \left| \partial^\alpha \left(a - \sum_{j=0}^{N} a_j \right) \right| &\leq \sum_{j=N+1}^{\infty} |(\partial^\alpha a_j)| \chi(\lambda_j h) \\ &\quad + \sum_{j=0}^{N} |\partial^\alpha a_j| (1 - \chi(\lambda_j h)) \\ &=: A + B. \end{aligned}$$

According to estimate (4.35),

$$A \leq \sum_{j=N+1}^{\infty} h^{-k_j - 1 - \delta|\alpha|} 2^{-j} m \leq m h^{-k_{N+1} - 1 - \delta|\alpha|}.$$

Also

$$B \leq \sum_{j=0}^{N} C_{\alpha,j} h^{-k_j - \delta|\alpha|} m(1 - \chi(\lambda_j h)).$$

Since $\chi \equiv 1$ on $[0,1]$, $B = 0$ if $0 < h \leq \lambda_N^{-1}$. If $\lambda_N^{-1} \leq h \leq 1$, we have $1 \leq \lambda_N h$ and hence

$$\begin{aligned} B &\leq m \sum_{j=0}^{N} C_{\alpha,j} h^{-|\alpha|\delta} \leq m \sum_{j=0}^{N} C_{\alpha,j} \lambda_N^{-k_N} h^{-k_N - \delta|\alpha|} \\ &= m \widetilde{C}_{\alpha,N} h^{-k_N - \delta|\alpha|}. \end{aligned}$$

Thus

$$\left| \partial^\alpha \left(a - \sum_{j=0}^{N} a_j \right) \right| \leq C_{\alpha,N} h^{-k_N - \delta|\alpha|} m$$

if $N \geq |\alpha|$. Therefore, for any N

$$\left| \partial^\alpha \left(a - \sum_{j=0}^{N-1} a_j \right) \right| \leq C_{\alpha,N} h^{-k_N - \delta|\alpha|} m.$$

□

4.3.4 Semiclassical expansions in S_δ. Next we need to reexamine some of our earlier asymptotic expansions, deriving improved estimates on the error terms:

THEOREM 4.12 (Semiclassical expansions in S_δ.). *Let Q be symmetric, nonsingular matrix.*

(i) *If $0 \leq \delta \leq \frac{1}{2}$, then*

$$e^{\frac{ih}{2}\langle QD,D\rangle} : S_\delta(m) \to S_\delta(m).$$

(ii) *If $0 \leq \delta < \frac{1}{2}$, we furthermore have for each symbol $a \in S_\delta(m)$ the expansion*

$$e^{\frac{ih}{2}\langle QD,D\rangle} a \sim \sum_{k=0}^{\infty} \frac{h^k}{k!} \left(i \frac{\langle QD, D\rangle}{2} \right)^k a \quad \text{in } S_\delta(m). \tag{4.36}$$

REMARK. Since we can always rescale to the case $h = 1$, there cannot exist an expansion like (4.36) for $\delta = 1/2$.

Proof. 1. First, let $0 \leq \delta < \frac{1}{2}$ and $a \in S_\delta(m)$. Recall from Theorem 4.5,(i) that

$$e^{\frac{ih}{2}\langle QD,D\rangle}a(z) = \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi h)^n}e^{\frac{i\pi}{4}\operatorname{sgn} Q}\int_{\mathbb{R}^{2n}} e^{\varphi(w)}a(z+w)\,dw$$

for the quadratic phase

$$\varphi(w) := -\frac{1}{2}\langle Q^{-1}w, w\rangle.$$

Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $\chi \equiv 1$ on $B(0,1)$, $\chi \equiv 0$ on $\mathbb{R}^n - B(0,2)$. Then

$$\begin{aligned} e^{\frac{ih}{2}\langle QD,D\rangle}a(z) &= \frac{C}{h^n}\int_{\mathbb{R}^{2n}} e^{\frac{i\varphi(w)}{h}}a(z-w)\,dw \\ &= \frac{C}{h^n}\int_{\mathbb{R}^{2n}} e^{\frac{i\varphi(w)}{h}}\chi(w)a(z-w)\,dw \\ &\quad + \frac{C}{h^n}\int_{\mathbb{R}^{2n}} e^{\frac{i\varphi(w)}{h}}(1-\chi(w))a(z-w)\,dw \\ &=: A + B, \end{aligned}$$

for the constant

$$C := \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi)^n}e^{\frac{i\pi}{4}\operatorname{sgn} Q}.$$

2. *Estimate of A.* Since $\chi(w)a(z-w)$ has compact support, the method of stationary phase, Theorem 4.5, gives

$$A \sim \sum_{k=0}^{\infty}\frac{h^k}{k!}(i\langle QD, D\rangle)^k a(z).$$

Furthermore, if $|w| \leq 2$, we have $m(z-w) \leq Cm(z)$. Consequently, $|A| \leq Cm(z)$; and similarly

$$|\partial^\alpha A| \leq C \sup_{0\leq\beta\leq\alpha}|\partial^\beta a| \leq Ch^{-|\alpha|\delta}m(z).$$

Hence $A \in S_\delta(m)$.

3. *Estimate of B.* Let

$$L := \frac{\langle\partial\varphi, hD\rangle}{|\partial\varphi|^2};$$

then $Le^{i\varphi/h} = e^{i\varphi/h}$ Furthermore, since $|\partial\varphi(w)| \geq \gamma|w|$ for some positive constant γ, the operator L has smooth coefficients on the support of $1-\chi$ and

$$\left|(L^*)^M((1-\chi)a)\right| \le C_M \frac{h^M}{\langle w\rangle^M} \sup_{|\alpha|\le M} |\partial^\alpha a(z-w)|$$

Consequently,

$$\begin{aligned} |B| &= \frac{C}{h^n}\left|\int_{\mathbb{R}^{2n}} \left(L^M e^{i\varphi/h}\right)(1-\chi(w))a(z-w)\,dw\right| \\ &= \frac{C}{h^n}\left|\int_{\mathbb{R}^{2n}} e^{i\varphi/h}(L^*)^M((1-\chi)a)\,dw\right| \\ &\le Ch^{M-n}\int_{\mathbb{R}^{2n}} \langle w\rangle^{-M} \sup_{|\alpha|\le M} |\partial^\alpha a(z-w)|\,dw \\ &\le Ch^{M-n-\delta M}\int_{\mathbb{R}^{2n}} \langle w\rangle^{N-M} m(z)\,dw \\ &= Ch^{M-n-\delta M} m(z), \end{aligned}$$

provided $M > 2n+N$. The number N is from the definition (4.28) of the order function m.

We similarly check also the higher derivatives, to conclude that $B \in S_\delta^{-\infty}(m)$.

4. Now assume $\delta = 1/2$. In this case we can rescale, by setting

$$\tilde{w} = wh^{-1/2}.$$

Then

$$e^{ih\langle QD,D\rangle}a(z) = C\int_{\mathbb{R}^n} e^{i\varphi(\tilde{w})}a(z-\tilde{w}h^{1/2})\,d\tilde{w}.$$

We use χ to break the integral into two pieces A and B, as above. □

THEOREM 4.13 (Symbol class of $a\#b$).

(i) *If $a \in S_\delta(m_1)$ and $b \in S_\delta(m_2)$, then*

$$a\#b \in S_\delta(m_1m_2) \tag{4.37}$$

and

$$a^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD) = (a\#b)^{\mathrm{w}}(x,hD)$$

in the sense of operators mapping $\mathcal{S}$ to $\mathcal{S}$.

(ii) *Furthermore,*

$$a\#b - ab \in S_\delta^{2\delta-1}(m_1m_2). \tag{4.38}$$

Proof. 1. Clearly

$$c(w,z) := a(w)b(z) \in S_\delta(m_1(w)m_2(z))$$

in $\mathbb{R}^{4n}$. If we put $D = (D_x, D_\xi, D_y, D_\eta)$ and

$$\langle QD, D\rangle = \sigma(D_x, D_\xi; D_y, D_\eta),$$

for $w = (x,\xi)$ and $z = (y,\eta)$, then according to Theorem 4.12, we have

$$e^{\frac{1}{2h}\langle QD,D\rangle} c \in S_\delta(m_1(w)m_2(z)).$$

Since (4.16) says

$$a\#b(z) = e^{\frac{1}{2h}\langle QD,D\rangle} c(z,z),$$

assertion (4.37) follows.

The second statement of assertion (i) follows from the density of $\mathcal{S}$ in $S_\delta(m)$.

2. We leave the verification of (4.38) as an exercise.

□

Finally we describe how to obtain the symbol from the operator, in the particularly nice case of the standard quantization:

THEOREM 4.14 (Constructing the symbol from the operator). *Suppose $a \in S_\delta(m)$. Then*

$$a(x,\xi) = e^{\frac{i}{h}\langle x,\xi\rangle} a(x,D)(e^{\frac{i}{h}\langle \cdot,\xi\rangle}). \tag{4.39}$$

Proof. For $a \in \mathcal{S}$ we verify this formula using the inverse Fourier transform:

$$\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\eta) e^{\frac{i}{h}\langle x,\eta-\xi\rangle} e^{-\frac{i}{h}\langle y,\eta-\xi\rangle} dy d\xi =$$

$$\int_{\mathbb{R}^n} a(x,\eta)\delta_0(\xi-\eta) e^{\frac{i}{h}\langle x,\eta-\xi\rangle}\, d\eta = a(x,\xi).$$

Approximation of a by elements of $\mathcal{S}$ concludes the proof. □

4.4 OPERATORS ON L^2.

So far our symbol calculus has built operators acting on either the Schwartz space $\mathcal{S}$ of smooth functions or its dual space $\mathcal{S}'$. But for applications we would like to handle functions in more convenient spaces, most notably L^2. Our next goal is therefore showing that if $a \in S_\delta$ for some $0 \le \delta \le \frac{1}{2}$, then $a^{\mathrm{w}}(x, hD)$ extends to become a bounded linear operator acting upon L^2.

For the time being, we take

$$h = 1.$$

Preliminaries. We select $\chi \in C_c^\infty(\mathbb{R}^{2n})$ such that

$$0 \leq \chi \leq 1, \quad \chi \equiv 0 \ \text{ on } \mathbb{R}^{2n} - B(0,2),$$

and

$$\sum_{\alpha \in \mathbb{Z}^{2n}} \chi_\alpha \equiv 1,$$

where $\chi_\alpha := \chi(\cdot - \alpha)$ denotes χ shifted by the lattice point $\alpha \in \mathbb{Z}^{2n}$. Write

$$a_\alpha := \chi_\alpha a;$$

then

$$a = \sum_{\alpha \in \mathbb{Z}^{2n}} a_\alpha.$$

We also define

$$b_{\alpha\beta} := \bar{a}_\alpha \# a_\beta \quad (\alpha, \beta \in \mathbb{Z}^{2n}). \tag{4.40}$$

THEOREM 4.15 (Decay of mixed terms).

(i) *For each N and each multiindex γ, we have the estimate*

$$|\partial^\gamma b_{\alpha\beta}(z)| \leq C_{\gamma,N} \langle \alpha - \beta \rangle^{-N} \langle z - \frac{\alpha + \beta}{2} \rangle^{-N} \tag{4.41}$$

for $z = (x, \xi) \in \mathbb{R}^{2n}$.

(ii) *For each N, there exists a constant C_N such that*

$$\| b^{\mathrm{w}}_{\alpha\beta}(x, D) \|_{L^2 \to L^2} \leq C_N \langle \alpha - \beta \rangle^{-N} \tag{4.42}$$

for all $\alpha, \beta \in \mathbb{Z}^{2n}$.

Proof. 1. We can rewrite formula (4.16) to read

$$b_{\alpha\beta}(z) = \frac{1}{\pi^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{i\varphi(w_1, w_2)} \bar{a}_\alpha(z - w_1) a_\beta(z - w_2)\, dw_1 dw_2,$$

for $\varphi(w_1, w_2) = -2\sigma(w_1, w_2)$.

Select $\zeta : \mathbb{R}^{4n} \to \mathbb{R}$ such that

$$0 \leq \zeta \leq 1, \ \zeta \equiv 1 \text{ on } \ B(0,1), \ \zeta \equiv 0 \text{ on } \mathbb{R}^{4n} - B(0,2).$$

Then

$$\begin{aligned} b_{\alpha\beta}(z) &= \frac{1}{\pi^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{i\varphi} \zeta(w) \bar{a}_\alpha(z - w_1) a_\beta(z - w_2)\, dw_1 dw_2 \\ &\quad + \frac{1}{\pi^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{i\varphi} (1 - \zeta(w)) \bar{a}_\alpha(z - w_1) a_\beta(z - w_2)\, dw_1 dw_2 \\ &=: A + B. \end{aligned}$$

2. *Estimate of A.* We have

$$|A| \le C \iint_{\{|w| \le 2\}} |\bar{a}_\alpha(z - w_1)||a_\beta(z - w_2)|\, dw_1 dw_2,$$

for $w = (w_1, w_2)$. The integrand equals

$$\chi(z - w_1 - \alpha)\chi(z - w_2 - \beta)|a(z - w_1)||a(z - w_2)|$$

and thus vanishes, unless

$$|z - w_1 - \alpha| \le 2 \text{ and } |z - w_2 - \beta| \le 2.$$

But then

$$|\alpha - \beta| \le 4 + |w_1| + |w_2| \le 8$$

and

$$\left| z - \frac{\alpha + \beta}{2} \right| \le 4 + |w_1| + |w_2| \le 8.$$

Hence

$$|A| \le C_N \langle \alpha - \beta \rangle^{-N} \langle z - \frac{\alpha + \beta}{2} \rangle^{-N}$$

for any N. Similarly, for each multiindex γ we can estimate

$$|\partial^\gamma A| \le C_{N,\gamma} \langle \alpha - \beta \rangle^{-N} \langle z - \frac{\alpha + \beta}{2} \rangle^{-N}. \tag{4.43}$$

3. *Estimate of B.* We have

$$|\partial \varphi(w)| = 2|w|$$

and $Le^{i\varphi} = e^{i\varphi}$, for

$$L := \frac{\langle \partial \varphi, D \rangle}{|\partial \varphi|^2}.$$

Since the integrand of B vanishes unless $|w| \ge 1$, the usual argument based on integration by parts shows that

$$|B| \le C_M \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \langle w \rangle^{-M} \bar{c}_\alpha(z - w_1) c_\beta(z - w_2)\, dw_1 dw_2$$

for appropriate functions c_α, c_β, with spt $c_\alpha \subseteq B(\alpha, 2)$, spt $c_\beta \subseteq B(\beta, 2)$. Thus the integrand vanishes unless

$$\frac{1}{c}\langle w\rangle \leq \langle \alpha-\beta\rangle, \langle z-\frac{\alpha+\beta}{2}\rangle \leq C\langle w\rangle.$$

Hence

$$\begin{aligned} |B| &\leq C_M\langle \alpha-\beta\rangle^{-N}\langle z-\frac{\alpha+\beta}{2}\rangle^{-N}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\langle w\rangle^{2N-M}\,dw_1dw_2 \\ &\leq C_M\langle \alpha-\beta\rangle^{-N}\langle z-\frac{\alpha+\beta}{2}\rangle^{-N} \end{aligned}$$

if M is large enough. Likewise,

$$|\partial^\gamma B| \leq C_{N,\gamma}\langle \alpha-\beta\rangle^{-N}\langle z-\frac{\alpha+\beta}{2}\rangle^{-N}. \tag{4.44}$$

4. Recall next that

$$a^{\mathrm{w}}(x,D) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}}\hat{a}(l)e^{il(x,D)}\,dl$$

and that, owing to (4.9), $e^{il(x,D)}$ is a unitary operator on L^2. Consequently

$$\|a^{\mathrm{w}}(x,D)\|_{L^2\to L^2} \leq C\int_{\mathbb{R}^{2n}}|\hat{a}(l)|\,dl.$$

Therefore we can estimate

$$\begin{aligned} \|b^{\mathrm{w}}_{\alpha\beta}(x,D)\|_{L^2\to L^2} &\leq C\|\hat{b}_{\alpha\beta}\|_{L^1} \leq C\|\langle\xi\rangle^{2n+1}\hat{b}_{\alpha\beta}\|_{L^\infty} \\ &\leq C\sup_{|\gamma|\leq 2n+1}\|\widehat{D^\gamma b_{\alpha\beta}}\|_{L^\infty} \\ &\leq C\sup_{|\gamma|\leq 2n+1}\|D^\gamma b_{\alpha\beta}\|_{L^1} \\ &\leq C\sup_{|\gamma|\leq 2n+1}\|\langle z\rangle^{2n+1}D^\gamma b_{\alpha\beta}\|_{L^1} \\ &\leq C\langle\alpha-\beta\rangle^{-N}, \end{aligned}$$

according to (4.43), (4.44). □

THEOREM 4.16 (Boundedness on L^2). *If the symbol a belongs to S_δ for some $0 \leq \delta \leq 1/2$, then*

$$a^{\mathrm{w}}(x,D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is bounded, with the estimate

$$\|a^{\mathrm{w}}(x,D)\|_{L^2\to L^2} \leq C\sum_{|\alpha|\leq 2n+1}\sup_{\mathbb{R}^n}|\partial^\alpha a|. \tag{4.45}$$

Furthermore,

$$\|a^{\mathrm{w}}(x,hD)u\|_{L^2\to L^2} \le C \sum_{|\alpha|\le 2n+1} h^{|\alpha|(\frac{1}{2}-\delta)} \sup_{\mathbb{R}^n} |\partial^\alpha a|. \tag{4.46}$$

Proof. 1. We have $b^{\mathrm{w}}_{\alpha\beta}(x,D) = A^*_\alpha A_\beta$, where $A_\alpha := a^{\mathrm{w}}_\alpha(x,D)$. Thus Theorem 4.15,(ii) asserts

$$\|A^*_\alpha A_\beta\|_{L^2\to L^2} \le C\langle \alpha-\beta\rangle^{-N}.$$

Therefore

$$\sup_\alpha \sum_\beta \|A_\alpha A^*_\beta\|^{1/2} \le C \sum_\beta \langle\alpha-\beta\rangle^{-N/2} \le C;$$

and similarly

$$\sup_\alpha \sum_\beta \|A^*_\alpha A_\beta\|^{1/2} \le C.$$

Since

$$a^{\mathrm{w}}(x,D) = \sum_\alpha A_\alpha,$$

we can apply the Cotlar–Stein Theorem C.5.

2. Estimate (4.46) follows from a rescaling, the details of which for $\delta = 0$ we will later provide in the proof of Theorem 5.1. □

As a first application, we record the useful

THEOREM 4.17 (Composition and multiplication). *Suppose that $a,b \in S_\delta$ for $0 \le \delta < \frac{1}{2}$.*

Then

$$\|a^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD) - (ab)^{\mathrm{w}}(x,hD)\|_{L^2\to L^2} = O(h^{1-2\delta}) \tag{4.47}$$

as $h \to 0$.

Proof. 1. In light of (4.38), we have

$$a\#b - ab \in S^{2\delta-1}_\delta.$$

Hence Theorem 4.16 implies

$$a^{\mathrm{w}}b^{\mathrm{w}} - (ab)^{\mathrm{w}} = (a\#b-ab)^{\mathrm{w}} = O(h^{1-2\delta}).$$

□

For the borderline case $\delta = \frac{1}{2}$, we have this assertion:

THEOREM 4.18 (Disjoint supports). *Suppose that* $a, b \in S_{\frac{1}{2}}$*, and and*

$$\operatorname{dist}(\operatorname{spt}(a), \operatorname{spt}(b)) \geq \gamma > 0,$$

for some constant γ*. Assume also that* $\operatorname{spt}(a) \subset K$ *where the compact set* K *and the constant* γ *are independent of* h*. Then*

$$\|a^{\mathrm{w}}(x, hD) b^{\mathrm{w}}(x, hD)\|_{L^2 \to L^2} = O(h^{\infty}). \tag{4.48}$$

Proof. Remember from (4.16) that

$$a \# b(z) = \frac{1}{(h\pi)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\varphi(w_1, w_2)} a(z - w_1) b(z - w_2) \, dw_1 dw_2,$$

for for $z = (x, \xi)$ and $\varphi(w_1, w_2) = -2\sigma(w_1, w_2)$.

We proceed as in the proof of Theorem 4.15: $|\partial \varphi| = 2|w|$ and thus the operator

$$L := \frac{\langle \partial \varphi, D \rangle}{|\partial \varphi|^2}$$

has smooth coefficients on the support of $a(z - w_1)b(z - w_2)$. From our assumption that $a, b \in S_{\frac{1}{2}}$, we see that

$$(L^*)^M (a(z - w_1) b(z - w_2)) = O(h^{\frac{M}{2}} \langle w \rangle^{-M}).$$

The uniform bound on the support shows that $a \# b \in S^{-\infty}$. Its Weyl quantization is therefore bounded on L^2, with norm of order $O(h^{\infty})$.
□

4.5 INVERSES

At this stage we have constructed in appropriate generality the quantizations $a^{\mathrm{w}}(x, hD)$ of various symbols a. We turn therefore to the practical problem of understanding how the algebraic and analytic behavior of the function a dictates properties of the corresponding quantized operators.

In this section we suppose that $a : \mathbb{R}^{2n} \to \mathbb{C}$ is nonvanishing; so that the function a is pointwise invertible. Can we draw the same conclusion about $a^{\mathrm{w}}(x, hD)$?

DEFINITION. We say the symbol a is *elliptic* if there exists a constant $\gamma > 0$ such that

$$|a| \geq \gamma > 0 \quad \text{on } \mathbb{R}^{2n}.$$

THEOREM 4.19 (Inverses for elliptic symbols). *Assume that $a \in S_\delta$ for $0 \le \delta < \frac{1}{2}$ and that a is elliptic.*

Then for some constant $h_0 > 0$,

$$a^{\mathrm{w}}(x, hD)^{-1}$$

exists as a bounded linear operator on $L^2(\mathbb{R}^n)$, provided $0 < h \le h_0$.

Proof. Let $b := \frac{1}{a}$, $b \in S_\delta$. Then (4.38) gives

$$a\#b = 1 + r_1, \text{ with } r_1 \in S_\delta^{2\delta-1}.$$

Likewise

$$b\#a = 1 + r_2, \text{ with } r_2 \in S_\delta^{2\delta-1}.$$

Hence if $A := a^{\mathrm{w}}(x, hD)$, $B := b^{\mathrm{w}}(x, hD)$, $R_1 := r_1^{\mathrm{w}}(x, hD)$ and $R_2 := r_2^{\mathrm{w}}(x, hD)$, we have

$$\begin{aligned} AB &= I + R_1 \\ BA &= I + R_2, \end{aligned}$$

with

$$\|R_1\|_{L^2 \to L^2}, \|R_2\|_{L^2 \to L^2} = O(h^{1-2\delta}) \le \tfrac{1}{2}$$

if $0 < h \le h_0$.

Thus $A = a^{\mathrm{w}}(x, hD)$ has an approximate left inverse and an approximate right inverse. Applying then Theorem C.3, we deduce that A^{-1} exists. □

4.6 GÅRDING INEQUALITIES

We continue studying how properties of the symbol a translate into properties of the corresponding quantized operators. In this section we suppose that a is real-valued and nonnegative, and ask the consequences for $a^{\mathrm{w}}(x, hD)$.

THEOREM 4.20 (Easy Gårding inequality). *Assume a is a real-valued symbol in S and*

$$a \ge \gamma > 0 \quad \text{on } \mathbb{R}^{2n}. \tag{4.49}$$

Then for each $\epsilon > 0$ there exists $h_0 = h_0(\epsilon) > 0$ such that

$$\langle a^{\mathrm{w}}(x, hD)u, u\rangle \ge (\gamma - \epsilon)\|u\|_{L^2}^2 \tag{4.50}$$

for all $0 < h \le h_0$, $u \in L^2(\mathbb{R}^n)$.

Proof. We will show that

$$(a-\lambda)^{-1} \in S \quad \text{if } \lambda < \gamma - \epsilon. \tag{4.51}$$

Indeed if $b := (a-\lambda)^{-1}$, then

$$(a-\lambda)\#b = 1 + \frac{h}{2i}\{a-\lambda, b\} + O(h^2) = 1 + O(h^2),$$

the bracket term vanishing since b is a function of $a - \lambda$. Therefore

$$(a^{\mathrm{w}}(x,hD) - \lambda)b^{\mathrm{w}}(x,hD) = I + O(h^2)_{L^2 \to L^2},$$

and so $b^{\mathrm{w}}(x,hD)$ is an approximate right inverse of $a^{\mathrm{w}}(x,hD) - \lambda$. Likewise $b^{\mathrm{w}}(x,hD)$ is an approximate left inverse.

Hence Theorem C.3 implies $a^{\mathrm{w}}(x,hD) - \lambda$ is invertible for each $\lambda < \gamma - \epsilon$. Consequently,

$$\operatorname{spec}(a^{\mathrm{w}}(x,hD)) \subset [\gamma - \epsilon, \infty).$$

According then to Theorem C.6,

$$\langle a^{\mathrm{w}}(x,hD)u, u\rangle \geq (\gamma - \epsilon)\|u\|_{L^2}^2$$

for all $u \in L^2$. □

We next improve the preceding estimate:

THEOREM 4.21 (Sharp Gårding inequality). *Let $a = a(x,\xi)$ be a symbol in S and suppose that*

$$a \geq 0 \quad \text{on } \mathbb{R}^{2n}. \tag{4.52}$$

Then there exist constants $h_0 > 0$, $C \geq 0$ such that

$$\langle a^{\mathrm{w}}(x,hD)u, u\rangle \geq -Ch\|u\|_{L^2}^2 \tag{4.53}$$

for all $0 < h \leq h_0$ and $u \in L^2(\mathbb{R}^n)$.

REMARK. The estimate (4.53) is in fact true for each quantization $\mathrm{Op}_t(a)$ $(0 \leq t \leq 1)$. And for the Weyl quantization, the stronger *Fefferman–Phong inequality* holds:

$$\langle a^{\mathrm{w}}(x,hD)u, u\rangle \geq -Ch^2\|u\|_{L^2}^2$$

for $0 < h \leq h_0$, $u \in L^2$.

We will need

LEMMA 4.22 (Gradient estimate). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2, with*

$$|\partial^2 f| \le A.$$

Suppose also $f \ge 0$. Then

$$|\partial f| \le (2Af)^{1/2}.$$

Proof. By Taylor's Theorem,

$$f(x+y) = f(x) + \langle \partial f(x), y\rangle + \int_0^1 (1-t)\langle \partial^2 f(x+ty)y, y\rangle \, dt.$$

Let $y = -\lambda \partial f(x)$, $\lambda > 0$ to be selected. Then since $f \ge 0$, we have

$$\begin{aligned} \lambda |\partial f(x)|^2 &\le f(x) + \lambda^2 \int_0^1 (1-t)\langle \partial^2 f(x - \lambda t \partial f(x))\partial f(x), \partial f(x)\rangle \, dt \\ &\le f(x) + \frac{\lambda^2}{2} A |\partial f(x)|^2. \end{aligned}$$

Put $\lambda = \frac{1}{A}$, to conclude $|\partial f(x)|^2 \le 2Af(x)$. □

Proof of Theorem 4.21 1. The primary goal is to show that if

$$\lambda = \mu h \tag{4.54}$$

and μ is fixed sufficiently large, then

$$h(a+\lambda)^{-1} \in S_{1/2}\left(\tfrac{1}{\mu}\right), \tag{4.55}$$

with estimates independent of μ.

To begin the proof of (4.55) we consider for any multiindex $\alpha = (\alpha_1, \dots, \alpha_{2n})$ the partial derivative ∂^α in the variables x and ξ.

We claim that $\partial^\alpha (a+\lambda)^{-1}$ has the form

$$\partial^\alpha (a+\lambda)^{-1} = (a+\lambda)^{-1} \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha = \beta^1 + \dots + \beta^k \\ |\beta^j| \ge 1}} C_{\beta^1, \dots, \beta^k} \prod_{j=1}^k \left((a+\lambda)^{-1} \partial^{\beta^j} a \right), \tag{4.56}$$

for appropriate constants $C_{\beta^1,\dots,\beta^k}$. To see this, observe that when we compute $\partial^\alpha (a+\lambda)^{-1}$ a typical term involves k differentiations of $(a+\lambda)^{-1}$ with the remaining derivatives falling on a. In obtaining (4.56) we for each $k \le |\alpha|$ partition α into multiindices $\beta^1, \dots, \beta^k$, each of which corresponds to one derivative falling on $(a+\lambda)^{-1}$ and the remaining derivatives falling on a.

2. Now Lemma 4.22 implies for $|\beta^j| = 1$ that

$$|\partial^{\beta_j} a|(a+\lambda)^{-1} \leq C\lambda^{-1/2} \tag{4.57}$$

since $\lambda^{1/2}|\partial a| \leq C\lambda^{1/2}a^{1/2} \leq C(\lambda + a)$. Furthermore,

$$|\partial^{\beta_j} a|(a+\lambda)^{-1} \leq C\lambda^{-1} \tag{4.58}$$

if $|\beta_j| \geq 2$, since $a \in S$.

Consequently, for each partition $\alpha = \beta^1 + \cdots + \beta^k$ and $0 < \lambda \leq 1$:

$$\left|\prod_{j=1}^k (a+\lambda)^{-1}\partial^{\beta_j} a\right| \leq C \prod_{|\beta_j|\geq 2} \lambda^{-1} \prod_{|\beta_j|=1} \lambda^{-1/2} \leq C \prod_{j=1}^k \lambda^{-\frac{|\beta_j|}{2}} = C\lambda^{-\frac{|\alpha|}{2}}.$$

Therefore

$$|\partial^\alpha (a+\lambda)^{-1}| \leq C_\alpha (a+\lambda)^{-1}\lambda^{-\frac{|\alpha|}{2}}. \tag{4.59}$$

But since $\lambda = \mu h$, this implies

$$(a+\lambda)^{-1} \in S_{1/2}\left(\tfrac{1}{\mu h}\right);$$

that is,

$$h(a+\lambda)^{-1} \in S_{1/2}\left(\tfrac{1}{\mu}\right),$$

with estimates independent of μ.

3. Since $a + \lambda \in S \subseteq S_{\frac{1}{2}}$, we can define $(a+\lambda)\#b$, for $b = (a+\lambda)^{-1}$. Using Taylor's formula, we compute

$$\begin{aligned}
&(a+\lambda)\#b(z)\\
&= e^{\frac{ih}{2}\sigma(D_z,D_w)}(a(z)+\lambda)b(w)\Big|_{w=z}\\
&= 1 + \int_0^1 (1-t)e^{\frac{ith}{2}\sigma(D_z,D_w)}\left(\frac{ih}{2}\sigma(D_z,D_w)\right)^2 (a(z)+\lambda)b(w)|_{w=z}\,dt\\
&=: 1 + r(z),
\end{aligned}$$

where we have noted that $\{a+\lambda, (a+\lambda)^{-1}\} = 0$.

Now according to (4.59), $hb \in S_{1/2}(1/\mu)$ and so $h^2\partial^\alpha b \in S_{1/2}(1/\mu)$ for $|\alpha| = 2$. An application of $e^{\frac{ih}{2}\sigma(D_z,D_w)}$ preserves the symbol class $S_{1/2}(1/\mu)$. Consequently,

$$\|r^{\mathrm{w}}(x,hD)\|_{L^2\to L^2} \leq \frac{C}{\mu} \leq \frac{1}{2},$$

if μ is now fixed large enough. Thus $b^{\mathrm{w}}(x,hD)$ is an approximate right inverse of $a^{\mathrm{w}}(x,hD)+\lambda$, and is similarly an approximate left inverse.

So $(a^{\mathrm{w}}(x, hD) + \lambda)^{-1}$ exists. Likewise $(a^{\mathrm{w}}(x, hD) + \gamma + \lambda)^{-1}$ exists for all $\gamma \geq 0$. Therefore

$$\operatorname{spec}(a^{\mathrm{w}}(x, hD)) \subseteq [-\lambda, \infty).$$

According then to Theorem C.6,

$$\langle a^{\mathrm{w}}(x, hD)u, u\rangle \geq -\lambda \|u\|_{L^2}^2$$

for all $u \in L^2$. Since $\lambda = \mu h$, this inequality finishes the proof. $\square$

5. Semiclassical defect measures

One way to understand limits as $h \to 0$ of a collection of functions $\{u(h)\}_{0<h\le h_0}$ bounded in L^2 is to construct corresponding *semiclassical defect measures* μ, which record the limiting behavior of certain quadratic forms acting on $u(h)$. If in addition these functions solve certain operator equations or PDE, we can deduce various properties of the measure μ and thereby indirectly recover information about asymptotoics as $h \to 0$ of the functions $u(h)$.

5.1 CONSTRUCTION, EXAMPLES

In the first two sections of this chapter, we consider a collection of functions $\{u(h)\}_{0<h\le h_0}$ that is bounded in $L^2(\mathbb{R}^n)$:

$$\sup_{0<h\le h_0} \|u(h)\|_{L^2} < \infty. \tag{5.1}$$

For the time being, we do not assume that $u(h)$ solves any PDE.

THEOREM 5.1 (An operator norm bound). *Suppose $a \in S$. Then*

$$\|a^{\mathrm{w}}(x, hD)\|_{L^2\to L^2} \le C \sup_{\mathbb{R}^{2n}} |a| + O(h^{\frac{1}{2}}) \tag{5.2}$$

as $h \to 0$.

Proof. We showed earlier in Theorem 4.16 that if $a \in S$ and $h = 1$, then

$$\|a^{\mathrm{w}}(x, D)\|_{L^2\to L^2} \le C \sup_{|\alpha|\le 2n+1} |\partial^\alpha a|. \tag{5.3}$$

Suppose now $a \in S$ and $u \in \mathcal{S}$. We rescale by taking

$$\tilde{x} := h^{-\frac{1}{2}}x,\ \tilde{y} := h^{-\frac{1}{2}}y,\ \tilde{\xi} := h^{-\frac{1}{2}}\xi$$

and

$$\tilde{u}(\tilde{x}) := h^{\frac{n}{4}}u(x) = h^{\frac{n}{4}}u(h^{\frac{1}{2}}\tilde{x}).$$

(This is a different rescaling of u from that introduced earlier in (4.31), the advantage being that $u \mapsto \tilde{u}$ is now a unitary transformation of L^2: $\|u\|_{L^2} = \|\tilde{u}\|_{L^2}$.)

Then

$$
\begin{aligned}
& a^{\mathrm{w}}(x,hD)u(x) \\
& \qquad = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2},\xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle}u(y)\,dyd\xi \\
& \qquad = \frac{h^{-\frac{n}{4}}}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a_h\left(\tfrac{\tilde{x}+\tilde{y}}{2},\tilde{\xi}\right) e^{i\langle \tilde{x}-\tilde{y},\tilde{\xi}\rangle}\tilde{u}(\tilde{y})\,d\tilde{y}d\tilde{\xi} \\
& \qquad = h^{-\frac{n}{4}} a_h^{\mathrm{w}}(\tilde{x},D)\tilde{u}(\tilde{x}),
\end{aligned}
\tag{5.4}
$$

for

$$a_h(\tilde{x},\tilde{\xi}) := a(x,\xi) = a(h^{\frac{1}{2}}\tilde{x}, h^{\frac{1}{2}}\tilde{\xi}).$$

Hence, noting that $dx = h^{\frac{n}{2}}d\tilde{x}$, we deduce from (5.4) and (5.3) that

$$
\begin{aligned}
\|a^{\mathrm{w}}(x,hD)u\|_{L^2} &= \|a_h^{\mathrm{w}}(\tilde{x},D)\tilde{u}\|_{L^2} \\
&\leq \|a_h^{\mathrm{w}}\|_{L^2\to L^2}\|\tilde{u}\|_{L^2} \\
&\leq C \sup_{|\alpha|\leq 2n+1} |\partial^\alpha a_h| \|u\|_{L^2} \\
&\leq C \sup_{|\alpha|\leq 2n+1} h^{\frac{|\alpha|}{2}} |\partial^\alpha a| \|u\|_{L^2}.
\end{aligned}
$$

This implies (5.2). □

THEOREM 5.2 (Existence of defect measure). *There exists a Radon measure μ on $\mathbb{R}^{2n}$ and a sequence $h_j \to 0$ such that*

$$\langle a^{\mathrm{w}}(x,h_jD)u(h_j), u(h_j)\rangle \to \int_{\mathbb{R}^{2n}} a(x,\xi)\,d\mu \tag{5.5}$$

for each symbol $a \in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$.

DEFINITION. We call μ a *microlocal defect measure* associated with the family $\{u(h)\}_{0<h\leq h_0}$.

Proof. 1. Choose $\{a_k\}_{k=0}^\infty \subset C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$ to be dense in $C_{\mathrm{c}}(\mathbb{R}^{2n})$. Select a sequence $h_j^1 \to 0$ such that

$$\langle a_1^{\mathrm{w}}(x,h_j^1D)u(h_j^1), u(h_j^1)\rangle \to \alpha_1.$$

Choose next a further subsequence $\{h_j^2\} \subseteq \{h_j^1\}$ such that

$$\langle a_2^{\mathrm{w}}(x,h_j^2D)u(h_j^2), u(h_j^2)\rangle \to \alpha_2.$$

Continue, at the k^{th} step extracting a subsequence $\{h_j^k\} \subseteq \{h_j^{k-1}\}$ such that

$$\langle a_k^{\mathrm{w}}(x,h_j^kD)u(h_j^k), u(h_j^k)\rangle \to \alpha_k.$$

By a standard diagonal argument, we see that the sequence $h_j := h_j^j$ converges to 0, with

$$\langle a_k^{\mathrm{w}}(x, h_j D)u(h_j), u(h_j)\rangle \to \alpha_k$$

for all $k = 1, \dots$.

2. Define $\Phi(a_k) := \alpha_k$. Owing to Theorem 5.1, we see for each k that

$$\begin{aligned}|\Phi(a_k)| = |\alpha_k| &= \lim_{h_j \to \infty} |\langle a_k^{\mathrm{w}} u(h_j), u(h_j)\rangle| \\ &\leq C \limsup_{h_j \to \infty} \|a_k^{\mathrm{w}}\|_{L^2 \to L^2} \leq C \sup_{\mathbb{R}^{2n}} |a_k|.\end{aligned}$$

The mapping Φ is bounded, linear and densely defined, and therefore uniquely extends to a bounded linear functional on $C_{\mathrm{c}}(\mathbb{R}^{2n})$, with the estimate

$$|\Phi(a)| \leq C \sup_{\mathbb{R}^{2n}} |a|$$

for all $a \in C_{\mathrm{c}}(\mathbb{R}^{2n})$. The Riesz Representation Theorem therefore implies the existence of a (possibly complex-valued) Radon measure on $\mathbb{R}^{2n}$ such that

$$\Phi(a) = \int_{\mathbb{R}^{2n}} a(x, \xi)\, d\mu.$$

□

REMARK. Theorem 5.2 is also valid if we replace the Weyl quantization $a^{\mathrm{w}} = \mathrm{Op}_{1/2}(a)$ by $\mathrm{Op}_t(a)$ for any $0 \leq t \leq 1$, since the error is then $O(h)$. □

THEOREM 5.3 (Positivity). *The measure μ is real and nonnegative:*

$$\mu \geq 0. \tag{5.6}$$

Proof. We must show that $a \geq 0$ implies

$$\int_{\mathbb{R}^{2n}} a\, d\mu \geq 0.$$

Now when $a \geq 0$, the sharp Gårding inequality, Theorem 4.21, implies

$$a^{\mathrm{w}}(x, hD) \geq -Ch;$$

that is,

$$\langle a^{\mathrm{w}}(x, hD)u(h), u(h)\rangle \geq -Ch\|u(h)\|_{L^2}^2$$

for sufficiently small $h > 0$. Let $h = h_j \to 0$, to deduce

$$\int_{\mathbb{R}^{2n}} a\, d\mu = \lim_{h_j \to \infty} \langle a^{\mathrm{w}}(x, h_j D) u(h_j), u(h_j) \rangle \geq 0.$$

$\square$

EXAMPLE 1: Coherent states. Fix a point (x_0, ξ_0) and define the corresponding *coherent state*

$$u(h)(x) := (\pi h)^{-\frac{n}{4}} e^{\frac{i}{h}\langle x - x_0, \xi_0 \rangle - \frac{1}{2h}|x - x_0|^2},$$

where we have normalized so that $\|u(h)\|_{L^2} = 1$. Then there exists precisely one associated semiclassical defect measure, namely

$$\mu := \delta_{(x_0, \xi_0)}.$$

To confirm this statement, take $t = 1$ in the quantization and calculate

$$\begin{aligned}
\langle a(x, hD) u(h), u(h) \rangle \\
&= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i}{h}\langle x - y, \xi \rangle} u(h)(y) \overline{u(h)}(x)\, dy d\xi dx \\
&= \frac{2^{\frac{n}{2}}}{(2\pi h)^{\frac{3n}{2}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i}{h}(\langle x - y, \xi \rangle + \langle y - x_0, \xi_0 \rangle - \langle x - x_0, \xi_0 \rangle)} \\
&\qquad\qquad e^{-\frac{1}{2h}(|y - x_0|^2 + |x - x_0|^2)}\, dy d\xi dx \\
&= \frac{2^{\frac{n}{2}}}{(2\pi h)^{\frac{3n}{2}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i}{h}\langle x - y, \xi - \xi_0 \rangle} \\
&\qquad\qquad e^{-\frac{1}{2h}(|y - x_0|^2 + |x - x_0|^2)}\, dy d\xi dx.
\end{aligned}$$

For each fixed x and ξ, the integral in y is

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x - y, \xi - \xi_0 \rangle} e^{-\frac{1}{2h}|y - x_0|^2}\, dy &= e^{\frac{i}{h}\langle x - x_0, \xi - \xi_0 \rangle} \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle y, \xi - \xi_0 \rangle} e^{-\frac{1}{2h}|y|^2}\, dy \\
&= e^{\frac{i}{h}\langle x - x_0, \xi - \xi_0 \rangle} \mathcal{F}\left(e^{-\frac{1}{2h}|y|^2}\right)\left(\tfrac{\xi - \xi_0}{h}\right) \\
&= (2\pi h)^{\frac{n}{2}} e^{\frac{i}{h}\langle x - x_0, \xi - \xi_0 \rangle} e^{-\frac{1}{2h}|\xi - \xi_0|^2},
\end{aligned}$$

where we used formula (3.2) for the last equality. Therefore

$$\begin{aligned}
\langle a(x, hD) u(h), u(h) \rangle \\
&= \frac{2^{\frac{n}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i}{h}\langle x - x_0, \xi - \xi_0 \rangle} e^{-\frac{1}{2h}(|x - x_0|^2 + |\xi - \xi_0|^2)}\, dx d\xi \\
&= a(x_0, \xi_0) \frac{2^{\frac{n}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x - x_0, \xi - \xi_0 \rangle} e^{-\frac{1}{2h}(|x - x_0|^2 + |\xi - \xi_0|^2)}\, dx d\xi + o(1) \\
&= C a(x_0, \xi_0) + o(1),
\end{aligned}$$

for the constant

$$C := \frac{2^{\frac{n}{2}}}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-\frac{1}{2}(|x|^2+|\xi|^2)}\, dx d\xi.$$

Taking $a \equiv 1$ and recalling that $\|u(h)\|_{L^2} = 1$, we deduce that $C = 1$. □

EXAMPLE 2: Stationary phase and defect measures. For our next example, take

$$u(h)(x) := e^{\frac{i\varphi(x)}{h}} b(x),$$

where $\varphi, b \in C^\infty$ and $\|b\|_{L^2} = 1$. Then

$$\langle a(x, hD)u(h), u(h)\rangle = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x,\xi) e^{\frac{i}{h}(\langle x-y,\xi\rangle + \varphi(y) - \varphi(x))} b(y)\overline{b(x)}\, dy d\xi dx.$$

We assume $a \in C_c^\infty(\mathbb{R}^{2n})$ and apply stationary phase. For a given value of x, define

$$\phi(y,\xi) := \langle x - y, \xi\rangle + \varphi(y) - \varphi(x).$$

Then

$$\partial_y \phi = \partial\varphi(y) - \xi,\ \partial_\xi \phi = x - y;$$

and the Hessian matrix of ϕ is

$$\partial^2 \phi = \begin{pmatrix} \partial^2\varphi & -I \\ -I & O \end{pmatrix}.$$

The signature of a matrix is integer valued, and consequently is invariant if we move along a curve of nonsingular matrices. Since

$$\operatorname{sgn} \begin{pmatrix} O & -I \\ -I & O \end{pmatrix} = 0,$$

it follows that

$$\operatorname{sgn} \begin{pmatrix} t\partial^2\varphi & -I \\ -I & O \end{pmatrix} = 0$$

for $0 \le t \le 1$; and therefore

$$\operatorname{sgn}(\partial^2\phi) = 0.$$

In addition, $|\det \partial^2\phi| = 1$. Thus as $h \to 0$ the stationary phase asymptotic expression (3.43) implies

$$\langle a^{\mathrm{w}}(x, hD)u(h), u(h)\rangle \to \int_{\mathbb{R}^n} a(x, \partial\varphi(x))|b(x)|^2\, dx = \int_{\mathbb{R}^{2n}} a(x,\xi)\, d\mu$$

for the semiclassical defect measure

$$\mu := |b(x)|^2 \delta_{\{\xi = \partial\varphi(x)\}}\, \mathcal{L}^n,$$

$\mathcal{L}^n$ denoting n-dimensional Lebesgue measure in the x-variables. $\square$

5.2 DEFECT MEASURES AND PDE

We now assume more about the family $\{u(h)\}_{0<h\leq h_0}$, namely that each function $u(h)$ is an approximate solution of a equation involving the operator $P(h) = p^{\mathrm{w}}(x, hD)$ for some real symbol $p \in S(\langle \xi \rangle^m)$ satisfying

$$|p| \geq \gamma \langle \xi \rangle^m \quad \textit{if} \;\; |\xi| \geq C \tag{5.7}$$

for constants $C, \gamma > 0$.

First, let us suppose $P(h)u(h)$ vanishes up to an $o(1)$ error term and see what we can conclude about a corresponding semiclassical defect measure μ.

THEOREM 5.4 (Support of defect measure). *Suppose that* $u(h)$

$$\begin{cases} \|P(h)u(h)\|_{L^2} = o(1) \quad \text{as } h \to 0, \\ \|u(h)\|_{L^2} = 1. \end{cases} \tag{5.8}$$

Then if μ *is any microlocal defect measure associated with* $\{u(h)\}_{0<h\leq h_0}$,

$$\operatorname{spt} \mu \subseteq p^{-1}(0). \tag{5.9}$$

Interpretation. We sometimes call $p^{-1}(0)$ the *characteristic variety* or *zero energy surface* of the symbol p. We understand (5.9) as saying that in the semiclassical limit $h \to 0$, all of the "mass" of the approximate solution $u(h)$ coalesces in phase space onto this set.

Proof. Select $a \in C_c^\infty(\mathbb{R}^{2n})$ such that $\operatorname{spt}(a) \cap p^{-1}(0) = 0$. We must show

$$\int_{\mathbb{R}^{2n}} a \, d\mu = 0.$$

To do so, first select $q \in S(\langle \xi \rangle^m)$ such that $\operatorname{spt}(a) \cap \operatorname{spt}(q) = \emptyset$ and

$$|p + iq| \geq \delta \langle \xi \rangle^m > 0 \quad \text{on } \mathbb{R}^{2n}$$

for some $\delta > 0$. We can for instance choose a function $q \in C^\infty$ that is equal to one on $p^{-1}(0)$, and then modify it near the compact support of a.

Write $Q(h) := q^{\mathrm{w}}(x, hD)$. Then Theorem 4.19 ensures us that for small enough h the operator $\langle hD \rangle^{-m}(P(h) + iQ(h))$ is invertible on L^2.

Next, put $A(h) := a^{\mathrm{w}}(x, hD)$. We observe that

$$\frac{ap}{p+iq} - a = -i\frac{aq}{p+iq}$$

Since a and q have disjoint support, Theorems 4.17 and 4.18 imply

$$\|A(h)(P(h) + iQ(h))^{-1}P(h) - A(h)\|_{L^2 \to L^2} = O(h).$$

Therefore (5.8) implies

$$\|A(h)u(h)\|_{L^2} = o(1);$$

and thus

$$\langle A(h)u(h), u(h)\rangle \to 0.$$

But also

$$\langle A(h_j)u(h_j), u(h_j)\rangle = \langle a^{\mathrm{w}}(x, h_j D)u(h_j), u(h_j)\rangle \to \int_{\mathbb{R}^{2n}} a\, d\mu.$$

□

Now we make the stronger assumption that the error term in (5.8) is $o(h)$.

THEOREM 5.5 (Flow invariance). *Assume*

$$\begin{cases} \|P(h)u(h)\|_{L^2} = o(h) & \text{as } h \to 0, \\ \|u(h)\|_{L^2} = 1. \end{cases} \tag{5.10}$$

Then

$$\int_{\mathbb{R}^{2n}} \{p, a\}\, d\mu = 0 \tag{5.11}$$

for all $a \in C_c^\infty(\mathbb{R}^{2n})$.

Interpretation. Let φ_t be the flow generated by the Hamiltonian vector field H_p. Then

$$\frac{d}{dt}\int_{\mathbb{R}^{2n}} \varphi_t^* a\, d\mu = \int_{\mathbb{R}^{2n}} (H_p a)(\varphi_t)\, d\mu = \int_{\mathbb{R}^{2n}} \{p, a\}\, d\mu.$$

Conseqently (5.11) asserts that *the semiclassical defect measure* μ *is flow-invariant.*

The proof below illustrates one of the basic principles mentioned in Chapter 1, that *an assertion about Hamiltonian dynamics involving the Poisson bracket corresponds to a commutator argument at the quantum level.*

Proof. Since p is real, $P(h) = p^{\mathrm{w}}(x, hD)$ is self-adjoint on L^2. Select a as above and write $A(h) = a^{\mathrm{w}}(x, hD)$. Recall that $A(h) = A(h)^*$. Then

$$\begin{aligned}\langle [P(h), A(h)]u(h), u(h)\rangle &= \langle (P(h)A(h) - A(h)P(h))u(h), u(h)\rangle \\ &= \langle A(h)u(h), P(h)u(h)\rangle \\ &\qquad -\langle P(h)u(h), A(h)u(h)\rangle \\ &= o(h)\end{aligned}$$

as $h \to 0$. On the other hand,

$$[P(h), A(h)] = \frac{h}{i}\{p, a\}^{\mathrm{w}}(x, hD) + O(h^2)_{L^2\to L^2}.$$

Hence

$$\langle [P(h), A(h)]u(h), u(h)\rangle = \frac{h}{i}\langle \{p, a\}^{\mathrm{w}}u(h), u(h)\rangle + o(h).$$

Divide by $h > 0$ and let $h = h_j \to 0$:

$$\int_{\mathbb{R}^{2n}} \{p, a\}\, d\mu = 0.$$

Note that even though p may not have compact support, $\{p, a\}$ does. □

We have similar statements if we replace $\mathbb{R}^n \times \mathbb{R}^n$ by the torus $\mathbb{T}^n \times \mathbb{R}^n$. We will need this observation for the following application.

5.3 APPLICATION: DAMPED WAVE EQUATION

A damped wave equation. In this section $\mathbb{T}^n$ denotes the flat n-dimensional torus. We consider now the initial-value problem

$$\text{(5.12)} \qquad \begin{cases} (\partial_t^2 + a\partial_t - \Delta)u = 0 & \text{on } \mathbb{T}^n \times \{t > 0\} \\ u = 0,\ u_t = f & \text{on } \mathbb{T}^n \times \{t = 0\}, \end{cases}$$

in which the smooth function $a = a(x)$ is nonnegative, and thus represents a damping mechanism, as we will see.

DEFINITION. The *energy* at time t is

$$E(t) := \frac{1}{2}\int_{\mathbb{T}^n} (\partial_t u)^2 + |\partial_x u|^2\, dx.$$

LEMMA 5.6 (Elementary energy estimates).
(i) *If* $a \equiv 0$, $t \mapsto E(t)$ *is constant.*
(ii) *If* $a \geq 0$, $t \mapsto E(t)$ *is nonincreasing.*

Proof. These assertions follow easily from the calculation

$$\begin{aligned} E'(t) &= \int_{\mathbb{T}^n} \partial_t u \partial_t^2 u + \langle \partial_x u, \partial_{xt}^2 u \rangle \, dx \\ &= \int_{\mathbb{T}^n} \partial_t u (\partial_t^2 u - \Delta u) \, dx = -\int_{\mathbb{T}^n} a(\partial_t u)^2 \, dx \leq 0. \end{aligned}$$

□

Our eventual goal is showing that if the support of the damping term a is large enough, then we have exponential energy decay for our solution of the wave equation (5.12). Here is the key assumption:

DYNAMICAL HYPOTHESIS.

$$\text{(5.13)} \qquad \begin{cases} \text{There exists a time } T > 0 \text{ such that any} \\ \text{trajectory of the Hamiltonian vector field of} \\ p(x,\xi) = |\xi|^2, \text{ starting at time 0 with } |\xi| = 1, \\ \text{intersects the set } \{a > 0\} \text{ by the time } T. \end{cases}$$

Equivalently, for each initial point $z = (x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n$, with $|\xi| = 1$, we have

$$\langle a \rangle_T := \frac{1}{T} \int_0^T a(x + t\xi) \, dt > 0.$$

Motivation. Since the damping term a in general depends upon x, we cannot use Fourier transform (or Fourier series) in x to solve (5.12). Instead we define $u \equiv 0$ for $t < 0$ and take the Fourier transform in t:

$$\hat{u}(x, \tau) := \int_0^\infty e^{-it\tau} u(x,t) \, dt \qquad (\tau \in \mathbb{R}).$$

Then

$$\begin{aligned} \Delta \hat{u} &= \int_0^\infty e^{-it\tau} \Delta u \, dt = \int_0^\infty e^{-it\tau} (\partial_t^2 u + a \partial_t u) \, dt \\ &= \int_0^\infty ((i\tau)^2 + ai\tau) e^{-it\tau} u \ dt - f = (-\tau^2 + ai\tau)\hat{u} - f. \end{aligned}$$

Consequently,

$$\text{(5.14)} \qquad P(\tau)\hat{u} := (-\Delta + i\tau a - \tau^2)\hat{u} = f.$$

Now take τ to be complex, with Re $\tau \geq 0$, and define

$$\text{(5.15)} \qquad P(z, h) := -h^2 \Delta + i\sqrt{z} h a - z$$

for the rescaled variable

$$\text{(5.16)} \qquad z = \tau^2 h^2.$$

Then (5.14) reads

$$P(z,h)\hat{u} = h^2 f;$$

and so, if $P(z,h)$ is invertible,

$$\hat{u} = h^2 P(z,h)^{-1} f. \tag{5.17}$$

We therefore need to study the inverse of $P(z,h)$.

THEOREM 5.7 (Resolvent bounds). *Under the dynamical assumption* (5.13)*, there exist constants* $\alpha, C, h_0 > 0$ *such that*

$$\|P(z,h)^{-1}\|_{L^2 \to L^2} \leq \frac{C}{h} \tag{5.18}$$

for

$$|\mathrm{Im}\ z| \leq \alpha h,\ |z-1| \leq \alpha,\ 0 < h \leq h_0. \tag{5.19}$$

Proof. 1. It is enough to show that there exists a constant C such that

$$\|u\|_{L^2} \leq \frac{C}{h} \|P(z,h)u\|_{L^2}$$

for all $u \in L^2$, provided z and h satisfy (5.19).

We argue by contradiction. If the assertion were false, then for $m = 1, 2, \dots$ there would exist $z_m \in \mathbb{C}$, $0 < h_m \leq 1/m$ and functions u_m in L^2 such that

$$\|P(z_m,h_m)u_m\|_{L^2} \leq \frac{h_m}{m}\|u_m\|_{L^2}, \quad |\mathrm{Im}\ z_m| \leq \frac{h_m}{m}, \quad |z_m - 1| \leq \frac{1}{m}.$$

We may assume $\|u_m\|_{L^2} = 1$. Then

$$P(z_m,h_m)u_m = o(h_m). \tag{5.20}$$

Also,

$$z_m \to 1,\ \mathrm{Im}(z_m) = o(h_m). \tag{5.21}$$

2. Let μ be a microlocal defect measure associated with $\{u_m\}_{m=1}^\infty$. Then Theorem 5.4 implies for the symbol $p := |\xi|^2 - 1$ that

$$\mathrm{spt}(\mu) \subseteq p^{-1}(0) = \{|\xi|^2 = 1\}.$$

But $\langle u_m, u_m \rangle = 1$, and so

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} d\mu = 1. \tag{5.22}$$

We will derive a contradiction to this.

3. Hereafter write $P_m := P(z_m, h_m)$. Then

$$\begin{aligned} P_m &= -h_m^2\Delta + i\sqrt{z_m}h_m a - z_m, \\ P_m^* &= -h_m^2\Delta - i\sqrt{\bar{z}_m}h_m a - \bar{z}_m; \end{aligned}$$

and therefore

$$P_m - P_m^* = i(\sqrt{z_m} + \sqrt{\bar{z}_m})h_m a - z_m + \bar{z}_m = 2ih_m a + o(h_m), \tag{5.23}$$

since (5.21) implies that $\sqrt{z_m} + \sqrt{\bar{z}_m} = 2 + o(1)$ and that $-z_m + \bar{z}_m = -2i\mathrm{Im}(z_m) = o(h_m)$.

Now select $b \in C_c^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ and set $B_m := b^{\mathrm{w}}(x, h_m D)$. Then $B_m = B_m^*$. Using (5.20) and (5.23), we calculate that

$$\begin{aligned} o(h_m) = 2i\ \mathrm{Im}\langle B_m P_m u_m, u_m\rangle &= \langle B_m P_m u_m, u_m\rangle - \langle u, B_m P_m u_m\rangle \\ &= \langle (B_m P_m - P_m^* B_m)u_m, u_m\rangle \\ &= \langle [B_m, P_m]u_m, u_m\rangle \\ &\quad + \langle (P_m - P_m^*)B_m u_m, u_m\rangle \\ &= \frac{h_m}{i}\langle \{b, p\}^{\mathrm{w}} u_m, u_m\rangle \\ &\quad + 2h_m i\langle (ab)^{\mathrm{w}} u_m, u_m\rangle + o(h_m). \end{aligned}$$

Divide by h_m and let $h_m \to 0$, through a subsequence if necessary, to discover that

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \{p, b\} + 2ab\, d\mu = 0. \tag{5.24}$$

We will build a function b so that $\{p, b\} + 2ab > 0$ on $\mathrm{spt}(\mu)$. This will imply $\int_{\mathbb{T}^n \times \mathbb{R}^n} d\mu = 0$, a contradiction to (5.22).

4. For $(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n$, with $|\xi| = 1$, define

$$c(x, \xi) := \frac{1}{T}\int_0^T (T - t)a(x + \xi t)\, dt,$$

where T is the time from the dynamical hypothesis (5.13). Hence

$$\begin{aligned} \langle \xi, \partial_x c\rangle &= \frac{1}{T}\int_0^T (T - t)\langle \xi, \partial a(x + \xi t)\rangle\, dt \\ &= \frac{1}{T}\int_0^T (T - t)\frac{d}{dt}a(x + \xi t)\, dt \\ &= \frac{1}{T}\int_0^T a(x + \xi t)\, dt - a(x) \\ &= \langle a\rangle_T - a. \end{aligned}$$

Let

$$b := e^c \chi(p),$$

where $\chi \in C_c^\infty(\mathbb{R})$ is equal to 1 near 0. Then

$$\langle \xi, \partial_x b\rangle = e^c \langle \xi, \partial_x c\rangle \chi(p) = e^c \langle a\rangle_T \chi(p) - a e^c \chi(p)$$

since $H_p(\chi(p)) = 0$. Consequently

$$\{p, b\} + 2ab = 2\langle \xi, \partial_x b\rangle + 2ab = 2e^c \langle a\rangle_T \chi(p) > 0 \quad \text{on } p^{-1}(0),$$

as desired. □

THEOREM 5.8 (Exponential energy decay). *Assume the dynamic hypothesis* (5.13) *and suppose u solves the wave equation with damping* (5.12).

Then there exists constants $C, \beta > 0$ such that

$$E(t) \le C e^{-\beta t} \|f\|_{L^2} \quad \textit{for all times } t > 0. \tag{5.25}$$

Motivation. The following calculations are based upon this idea: to get decay estimates of g on the positive real axis, we estimate $\hat{g}$ in a complex strip $|\mathrm{Im}\ z| \le \alpha$. Then if $\beta < \alpha$,

$$\widehat{e^{\beta t} g}(\tau) = \int_{-\infty}^{\infty} e^{\beta t} g(t) e^{-it\tau}\, dt = \int_{-\infty}^{\infty} g(t) e^{-it(\tau + i\beta)}\, dt = \hat{g}(\tau + i\beta).$$

Hence our L^2 estimate of $\hat{g}(\cdot + i\beta)$ will imply exponential decay of $g(t)$ for $t \to \infty$.

Proof. 1. Recall from (5.15), (5.16) that

$$P(\tau) = h^{-2} P(z, h) \quad \text{for } \tau^2 = h^{-2} z.$$

First we assert that there exists $\gamma > 0$ such that

$$\|P(\tau)^{-1}\|_{L^2 \to H^1} \le C \quad \text{for } |\mathrm{Im}\ \tau| \le \gamma, |\tau| > 1/\gamma. \tag{5.26}$$

To prove (5.26), we note that provided the inequalities (5.19) hold, then

$$\|h^2 \Delta P(z, h)^{-1} u\|_{L^2} = \|(i\sqrt{z} h a - z) P(z, h)^{-1} u - u\|_{L^2} \le \frac{C}{h} \|u\|_{L^2},$$

the last inequality holding according to Theorem 5.7. Thus

$$\|h^2 P(z, h)^{-1} u\|_{H^2} \le \frac{C}{h} \|u\|_{L^2}. \tag{5.27}$$

Recall next that $z = h^2 \tau^2$. Write $\tau = \lambda + i\mu$, for $\lambda > 0$, and set $h = \lambda^{-1}$; so that

$$z = h^2(\lambda^2 - \mu^2) + i(h^2 2\lambda\mu).$$

Thus $|\text{Im } z| \leq \alpha h$ and $|z-1| \leq \alpha$ provided if $|\mu| \leq \gamma$ and $|\lambda| > 1/\gamma$ for some sufficiently small γ, and so the inequalities (5.19) hold. Hence (5.27) implies

$$\|P(\tau)^{-1}u\|_{H^2} \leq \frac{C}{|\tau|}\|u\|_{L^2} \tag{5.28}$$

for $|\mu| \leq \gamma$ and $|\lambda| > 1/\gamma$.

Also

$$\|P(\tau)^{-1}u\|_{L^2} \leq \frac{C}{|\tau|}\|u\|_{L^2}.$$

Interpolating between the last two inequalities demonstrates that

$$\|P(\tau)^{-1}u\|_{H^1} \leq C\|u\|_{L^2}$$

for $|\text{Im } \tau| \leq \gamma$ and $|\tau| > 1/\gamma$.

This proves (5.26) except for a bounded range of τ's. To see the estimate for all τ, outside of a neighborhood of 0, we simply need to exclude the possibility of a real non-zero τ satisfying

$$(-\Delta - \tau^2 + i\tau a)u = 0 \tag{5.29}$$

for some $u \neq 0$. Multiplying by $\bar{u}$, integrating, and taking the imaginary part shows that

$$\int_{\mathbb{T}^n} a|u|^2\,dx = 0.$$

Since $a \geq 0$, this implies that $u \equiv 0$ on $\operatorname{spt} a$. Hence $(-\Delta - \tau^2)u = 0$. But this is impossible owing to unique continuation results which we will prove in Section 7.2, since $\operatorname{spt} a$ has a nonempty interior. The Fredholm alternative now guarantees that $P(\tau)^{-1}$ has no pole on the real axis.

2. Since $P(\tau)^{-1}$ is meromorphic, we conclude that in $|\text{Im}\tau| \leq \alpha$ the only possible pole occurs at $\tau = 0$. In particular for $0 < \beta < \alpha$ we have

$$\sup_{\tau\in\mathbb{R}} \|P(\tau + i\beta)^{-1}\|_{L^2\to H^1} \leq C_\beta. \tag{5.30}$$

3. Next select $\chi : \mathbb{R} \to \mathbb{R}$, $\chi = \chi(t)$, such that

$$0 \leq \chi \leq 1,\ \chi \equiv 1 \text{ on } [1,\infty), \chi \equiv 0 \text{ on } (-\infty, 0).$$

Then if $u_1 := \chi u$, we have

$$(\partial_t^2 + a\partial_t - \Delta)u_1 = g_1, \tag{5.31}$$

for

$$g_1 := \chi'' u + 2\chi'\partial_t u + a(x)\chi' u. \tag{5.32}$$

Note that $u_1(t) = 0$ for $t \leq 0$, and observe also that the support of g_1 lies within $\mathbb{T}^n \times [0,1]$. Furthermore, using energy estimates in Lemma 5.6, we see that

$$\begin{aligned} \|g_1\|_{L^2(\mathbb{R}_+;L^2)} & \\ &\leq C\left(\|u\|_{L^2((0,1);L^2)} + \|\partial_t u\|_{L^2((0,1);L^2)}\right) \leq C\|f\|_{L^2}. \end{aligned} \tag{5.33}$$

Now take the Fourier transform of (5.31) in time:

$$P(\tau)\hat{u}_1(\tau) = \hat{g}_1(\tau).$$

Then

$$\hat{u}_1(\tau) = P(\tau)^{-1}\hat{g}_1(\tau), \tag{5.34}$$

where, in principle, we allow the left hand side to have a pole at $\tau = 0$.

4. We now deduce exponential decay. Noting that u_1 is supported in $t > 0$, we use Plancherel's theorem to compute

$$\begin{aligned} \|e^{\beta t}u_1\|_{L^2(\mathbb{R}_+;H^1)} &= (2\pi)^{-\frac{1}{2}}\|\widehat{e^{\beta t}u_1}\|_{L^2(\mathbb{R};H^1)} \\ &= (2\pi)^{-\frac{1}{2}}\|\hat{u}_1(\cdot + i\beta)\|_{L^2(\mathbb{R};H^1)} \\ &= (2\pi)^{-\frac{1}{2}}\|P(\cdot + i\beta)^{-1}\hat{g}_1(\cdot + i\beta)\|_{L^2(\mathbb{R};H^1)} \\ &\leq C\|\hat{g}_1(\cdot + i\beta)\|_{L^2(\mathbb{R};H^1)} \end{aligned}$$

Since g_1 is compactly supported in t we also see that

$$\hat{g}_1(\cdot + i\beta) = \widehat{e^{\beta t}g_1}(\cdot);$$

and hence

$$\begin{aligned} \|e^{\beta t}u_1\|_{L^2(\mathbb{R}_+;H^1)} &\leq C\|\hat{g}_1\|_{L^2(\mathbb{R};L^2)} \\ &\leq C\|g_1\|_{L^2(\mathbb{R}_+;L^2)} \leq C\|f\|_{L^2}. \end{aligned}$$

Since $u_1 = \chi u$, it follows that

$$\|e^{\beta t}u\|_{L^2((1,\infty);H^1)} \leq C\|f\|_{L^2}. \tag{5.35}$$

5. Finally, fix $T > 2$ and

$$\chi_T := \chi(t - T + 1),$$

where χ is as in Step 2. Let $u_2 = \chi_T u$. Then

$$(\partial_t^2 + a\partial_t - \Delta)u_2 = g_2, \tag{5.36}$$

for

$$g_2 := \chi_T'' u + 2\chi_T'\partial_t u + a\chi_T' u. \tag{5.37}$$

Therefore $\operatorname{spt} g \subseteq \mathbb{T}^n \times (T-1, T)$.

Define

$$E_2(t) := \frac{1}{2}\int_{\mathbb{T}^n} (\partial_t u_2)^2 + |\partial_x u_2|^2 \, dx.$$

Modifying the calculations in the proof of Lemma 5.6, we use (5.36) and (5.37) to compute

$$\begin{aligned}
E_2'(t) &= \int_{\mathbb{T}^n} \partial_t u_2 \partial_t^2 u_2 + \langle \partial_x u_2, \partial_{xt}^2 u_2 \rangle \, dx \\
&= \int_{\mathbb{T}^n} \partial_t u_2 (\partial_t^2 u_2 - \Delta u_2) \, dx \\
&= -\int_{\mathbb{T}^n} a(\partial_t u_2)^2 \, dx + \int_{\mathbb{T}^n} \partial_t u_2 g_2 \, dx \\
&\leq C \int_{\mathbb{T}^n} |\partial_t u_2| (|\partial_t u| + |u|) \, dx \\
&\leq C E_2(t) + C \int_{\mathbb{T}^n} u^2 + (\partial_t u)^2 \, dx.
\end{aligned}$$

Since $E_2(T-1) = 0$ and $E_2(T) = E(T)$, Gronwall's inequality implies that

$$E(T) \leq C \left(\|u\|^2_{L^2((T-1,T);L^2)} + \|\partial_t u\|^2_{L^2((T-1,T);L^2)} \right). \tag{5.38}$$

6. We need to control the right hand term in (5.38). For this, select $\chi : \mathbb{R} \to \mathbb{R}$, such that

$$\begin{cases} 0 \leq \chi \leq 1, \\ \chi \equiv 0 \text{ for } t \leq T-2 \text{ and } t \geq T+1, \\ \chi \equiv 1 \text{ for } T-1 \leq t \leq T. \end{cases}$$

We multiply the wave equation (5.12) by $\chi^2 u$ and integrate by parts, to find

$$\begin{aligned}
0 &= \int_{T-2}^{T+1} \int_{\mathbb{T}^n} \chi^2 u (\partial_t^2 u + a\partial_t u - \Delta u) \, dx dt \\
&= \int_{T-2}^{T+1} \int_{\mathbb{T}^n} -\chi^2 (\partial_t u)^2 - 2\chi\chi' u \partial_t u + \chi^2 a u \partial_t u + \chi^2 |\partial_x u|^2 \, dx dt.
\end{aligned}$$

From this identity we derive the estimate

$$\|\partial_t u\|_{L^2((T-1,T);L^2)} \leq C \|u\|_{L^2((T-2,T+1);H^1)}.$$

This, (5.38) and (5.35) therefore imply

$$E(T) \leq C \|u\|^2_{L^2((T-2,T+1);H^1)} \leq C e^{-\beta T} \|f\|_{L^2},$$

as asserted. □

Our methods extend with no difficulty if $\mathbb{T}^n$ is replaced by a general compact Riemannian manifold: see Appendix E.

6. Eigenvalues and eigenfunctions

In this chapter we are given the potential $V : \mathbb{R}^n \to \mathbb{R}$, and investigate how the symbol

$$p(x, \xi) = |\xi|^2 + V(x) \tag{6.1}$$

provides interesting information about the corresponding operator

$$P(h) := P(x, hD) = -h^2\Delta + V. \tag{6.2}$$

We will focus mostly upon learning how p controls the aysmptotic distribution of the eigenvalues of $P(h)$ in the semiclassical limit $h \to 0$.

6.1 THE HARMONIC OSCILLATOR

Our plan is to consider first the simplest case, when the potential is quadratic; and to simplify even more, we begin in one dimension. So suppose that $n = 1$, $h = 1$ and $V(x) = x^2$. Thus we start with the *one-dimensional quantum harmonic oscillator*, meaning the operator

$$P_0 := -\partial^2 + x^2.$$

6.1.1 Eigenvalues and eigenfunctions of P_0. We can as follows employ certain auxiliary first-order differential operators to compute explicitly the eigenvalues and eigenfunctions for P_0.

NOTATION. Let us write

$$A_+ := D_x + ix, \ \ A_- := D_x - ix,$$

where $D_x = \frac{1}{i}\partial_x$, and call A_+ the *creation operator* and A_- the *annihilation operator*. (This terminology is from particle physics.)

LEMMA 6.1 (Properties of $A_\pm$). *The creation and annihilation operators satisfy these identities:*

$$A_+^* = A_-, \ \ A_-^* = A_+,$$

$$P_0 = A_+A_- + 1 = A_-A_+ - 1.$$

Proof..It is easy to check that $D_x^* = D_x$ and $(ix)^* = -ix$. Furthermore,

$$\begin{aligned} A_+A_-u &= (D_x + ix)(D_x - ix)u \\ &= \left(\frac{1}{i}\partial_x + ix\right)\left(\frac{1}{i}u_x - ixu\right) \\ &= -u_{xx} - (xu)_x + xu_x + x^2u \\ &= -u_{xx} - u - xu_x + xu_x + x^2u \\ &= P_0u - u; \end{aligned}$$

and similarly,

$$\begin{aligned} A_-A_+u &= (D_x - ix)(D_x + ix)u \\ &= \left(\frac{1}{i}\partial_x - ix\right)\left(\frac{1}{i}u_x + ixu\right) \\ &= -u_{xx} + (xu)_x - xu_x + x^2u \\ &= P_0u + u. \end{aligned}$$

□

We can now use $A_\pm$ to find all the eigenvalues and eigenfunctions of P_0:

THEOREM 6.2 (Eigenvalues and eigenfunctions).

(i) *We have*

$$\langle P_0u, u\rangle \geq \|u\|_{L^2}^2$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. *That is,*

$$P_0 \geq 1.$$

(ii) *The function*

$$v_0 =: e^{-\frac{x^2}{2}}$$

is an eigenfunction corresponding to the smallest eigenvalue 1.

(iii) *Set*

$$v_n := A_+^n v_0$$

for $n = 1, 2, \ldots$. *Then*

$$P_0 v_n = (2n+1)v_n. \tag{6.3}$$

(iv) *Define the normalized eigenfunctions*

$$u_n := \frac{v_n}{\|v_n\|_{L^2}}.$$

Then

$$u_n(x) = H_n(x)e^{-\frac{x^2}{2}} \tag{6.4}$$

where $H_n(x) = c_n x^n + \cdots + c_0 \quad (c_n \neq 0)$ *is a polynomial of degree* n.

(v) *We have*

$$\langle u_n, u_m \rangle = \delta_{nm};$$

and furthermore, the collection of eigenfunctions $\{u_n\}_{n=0}^{\infty}$ *is complete in* $L^2(\mathbb{R}^n)$.

The functions H_n mentioned in assertion (iv) are the *Hermite polynomials.*

Proof. 1. We note that

$$[D_x, x]u = \frac{1}{i}(xu)_x - \frac{x}{i}u_x = \frac{u}{i},$$

and consequently $i[D_x, x] = 1$. Therefore

$$\begin{aligned} \|u\|_{L^2}^2 &= \langle i[D_x, x]u, u\rangle \leq 2\|xu\|_{L^2}\|D_x u\|_{L^2} \\ &\leq \|xu\|_{L^2}^2 + \|D_x u\|_{L^2}^2 = \langle P_0 u, u\rangle. \end{aligned}$$

Next, observe

$$A_- v_0 = \frac{1}{i}\left(e^{-\frac{x^2}{2}}\right)_x - ixe^{-\frac{x^2}{2}} = 0;$$

so that $P_0 v_0 = (A_+ A_- + 1)v_0 = v_0$.

2. We can further calculate that

$$\begin{aligned} P_0 v_n &= (A_+A_- + 1)A_+ v_{n-1} \\ &= A_+(A_-A_+ - 1)v_{n-1} + 2A_+ v_{n-1} \\ &= A_+ P_0 v_{n-1} + 2A_+ v_{n-1} \\ &= (2n-1)A_+ v_{n-1} + 2A_+ v_{n-1} \quad \text{(by induction)} \\ &= (2n+1)v_n. \end{aligned}$$

The form (6.4) of v_n, u_n follows by induction.

3. Note also that

$$\begin{aligned} [A_-, A_+] &= A_-A_+ - A_+A_- \\ &= (P_0 + 1) - (P_0 - 1) = 2. \end{aligned}$$

Hence if $m > n$,

$$\begin{aligned} \langle v_n, v_m\rangle &= \langle A_+^n v_0, A_+^m v_0\rangle \\ &= \langle A_-^m A_+^n v_0, v_0\rangle \quad (\text{since } A_- = A_+^*) \\ &= \langle A_-^{m-1}(A_+A_- + 2)A_+^{n-1} v_0, v_0\rangle. \end{aligned}$$

After finitely many steps, the foregoing equals

$$\langle (\dots)A_- v_0, v_0\rangle = 0,$$

since $A_- v_0 = 0$.

4. Lastly, we demonstrate that the collection of eigenfunctions we have found spans L^2. Suppose $\langle u_n, g\rangle = 0$ for $n = 0, 1, 2, \dots$; we must show $g \equiv 0$.

Now since $H_n(x) = c_n x^n + \dots$, with $c_n \neq 0$, we have

$$\int_{-\infty}^{\infty} g(x) e^{-\frac{x^2}{2}} p(x)\, dx = 0$$

for each polynomial p. Hence

$$\int_{-\infty}^{\infty} g(x) e^{-\frac{x^2}{2}} e^{-ix\xi}\, dx = \int_{-\infty}^{\infty} g(x) e^{-\frac{x^2}{2}} \sum_{k=0}^{\infty} \frac{(-ix\xi)^k}{k!}\, dx;$$

and so $\mathcal{F}\left(g e^{-\frac{x^2}{2}}\right) \equiv 0$. This implies $g e^{-\frac{x^2}{2}} \equiv 0$ and consequently $g \equiv 0$. □

6.1.2 Higher dimensions, rescaling. Suppose now $n > 1$, and write

$$P_0 := -\Delta + |x|^2;$$

this is the *n-dimensional quantum harmonic oscillator*. We define also

$$u_\alpha(x) := \prod_{j=1}^{n} u_{\alpha_j}(x_j) = \prod_{j=1}^{n} H_{\alpha_j}(x_j) e^{-\frac{|x|^2}{2}}$$

for each multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$. Then

$$P_0 u_\alpha = (-\Delta + |x|^2) u_\alpha = (2|\alpha| + n) u_\alpha,$$

for $|\alpha| = \alpha_1 + \dots + \alpha_n$. Hence u_α is an eigenfunction of P_0 corresponding to the eigenvalue $2|\alpha| + n$.

We next restore the parameter $h > 0$ by setting

$$P_0(h) := -h^2 \Delta + |x|^2, \tag{6.5}$$

$$u_\alpha(h)(x) := h^{-\frac{n}{4}} \prod_{j=1}^{n} H_{\alpha_j}\left(\frac{x_j}{\sqrt{h}}\right) e^{-\frac{|x|^2}{2h}}, \tag{6.6}$$

and

$$E_\alpha(h) := (2|\alpha| + n) h. \tag{6.7}$$

Then

$$P_0(h) u_\alpha(h) = E_\alpha(h) u_\alpha(h);$$

and upon reindexing, we can write these eigenfunction equations as

$$P_0(h)u_j(h) = E_j(h)u_j(h) \qquad (j = 1, \dots). \tag{6.8}$$

6.1.3 Asymptotic distribution of eigenvalues. With these explicit formulas in hand, we can study the behavior in the semiclassical limit of the eigenvalues $E(h)$ of the harmonic oscillator:

THEOREM 6.3 (Weyl's law for harmonic oscillator). *Assume that* $0 \le a < b < \infty$. *Then*

$$\begin{aligned}(6.9) \quad & \#\{E(h) \mid a \le E(h) \le b\} \\ & \qquad = \frac{1}{(2\pi h)^n}(|\{a \le |\xi|^2 + |x|^2 \le b\}| + o(1)).\end{aligned}$$

as $h \to 0$.

Proof. We may assume that $a = 0$. Since $E(h) = (2|\alpha| + n)h$ for some multiindex α according to (6.7), we have

$$\begin{aligned}\#\{E(h) \mid 0 \le E(h) \le b\} &= \#\left\{\alpha \mid 0 \le 2|\alpha| + n \le \frac{b}{h}\right\} \\ &= \#\left\{\alpha \mid \alpha_1 + \dots + \alpha_n \le R\right\},\end{aligned}$$

for $R := \frac{b-nh}{2h}$. Therefore

$$\begin{aligned}\#\{E(h) & \mid 0 \le E(h) \le b\} \\ &= |\{x \mid x_i \ge 0,\ x_1 + \dots + x_n \le R\}| + o(R^n) \\ &= \frac{1}{n!}R^n + o(R^n) \quad \text{as } R \to \infty \\ &= \frac{1}{n!}\left(\frac{b}{2h}\right)^n + o(h^{-n}) \quad \text{as } h \to 0.\end{aligned}$$

We used in this calculation the fact that the volume of the simplex $\{x \mid x_i \ge 0,\ x_1 + \dots + x_n \le 1\}$ is $(n!)^{-1}$. Next we note that $|\{|\xi|^2 + |x|^2 \le b\}| = \alpha(2n)b^n$, where $\alpha(k) := \pi^{\frac{k}{2}}(\Gamma(\frac{k}{2} + 1))^{-1}$ is the volume of the unit ball in $\mathbb{R}^k$. Setting $k = 2n$, we compute that $\alpha(2n) = \pi^n(n!)^{-1}$. Hence

$$\begin{aligned}\#\{E(h) \mid 0 \le E(h) \le b\} &= \frac{1}{n!}\left(\frac{b}{2h}\right)^n + o(h^{-n}) \\ &= \frac{1}{(2\pi h)^n}|\{|\xi|^2 + |x|^2 \le b\}| + o(h^{-n}).\end{aligned}$$

□

6.2 SYMBOLS AND EIGENFUNCTIONS

For this section, we return to the general symbol (6.1) and the quantized operator (6.2). We assume that the potential $V : \mathbb{R}^n \to \mathbb{R}$ is smooth, and satisfies the growth conditions:

$$|\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^k, \quad V(x) \geq C\langle x \rangle^k \quad \text{for } |x| \geq R, \tag{6.10}$$

for appropriate constants $k, C, C_\alpha, R > 0$.

Our plan in the next section is to employ our detailed knowledge about the eigenvalues of the harmonic oscillator (6.5) to estimate the asymptotics of the eigenvalues of $P(h)$. This section develops some useful techniques that will aid us in this task.

6.2.1 Concentration in phase space. First, we make the important observation that in the semiclassical limit the eigenfunctions $u(h)$ "are concentrated in phase space" on the energy surface $\{|\xi|^2 + V(x) = E\}$.

THEOREM 6.4 (h^∞ estimates). *Suppose that $u(h) \in L^2(\mathbb{R}^n)$ solves*

$$P(h)u(h) = E(h)u(h). \tag{6.11}$$

Assume as well that $a \in S$ is a symbol satisfying

$$\{|\xi|^2 + V(x) = E\} \cap \operatorname{spt}(a) = \emptyset.$$

Then if

$$|E(h) - E| < \delta$$

for some sufficiently small $\delta > 0$, we have the estimate

$$\|a^{\mathrm{w}}(x, hD)u(h)\|_{L^2} = O(h^\infty)\|u(h)\|_{L^2}. \tag{6.12}$$

Proof. 1. The set $K := \{|\xi|^2 + V(x) = E\} \subset \mathbb{R}^{2n}$ has compact. Hence there exists $\chi \in C_c^\infty(\mathbb{R}^{2n})$ such that

$$0 \leq \chi \leq 1, \ \chi \equiv 1 \text{ on } K, \ \chi \equiv 0 \text{ on } \operatorname{spt}(a).$$

Define the symbol

$$b := |\xi|^2 + V(x) - E(h) + i\chi = p - E(h) + i\chi$$

and the order function

$$m := \langle \xi \rangle^2 + \langle x \rangle^k.$$

Therefore if $|E(h) - E|$ is small enough,

$$|b| \geq \gamma m \quad \text{on } \mathbb{R}^{2n}$$

for some constant $\gamma > 0$. Consequently $b \in S(m)$, with $b^{-1} \in S(m^{-1})$.

2. Thus there exist $c \in S(m^{-1})$, $r_1, r_2 \in S$ such that

$$\begin{cases} b^{\mathrm{w}}(x,hD)c^{\mathrm{w}}(x,hD) = I + r_1^{\mathrm{w}}(x,hD) \\ c^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD) = I + r_2^{\mathrm{w}}(x,hD). \end{cases}$$

where $r_1^{\mathrm{w}}(x,hD), r_2^{\mathrm{w}}(x,hD)$ are $O(h^\infty)$. Then

$$a^{\mathrm{w}}(x,hD)c^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD) = a^{\mathrm{w}}(x,hD) + O(h^\infty), \tag{6.13}$$

and

$$b^{\mathrm{w}}(x,hD) = P(h) - E(h) + i\chi^{\mathrm{w}}(x,hD). \tag{6.14}$$

Furthermore

$$a^{\mathrm{w}}(x,hD)c^{\mathrm{w}}(x,hD)\chi^{\mathrm{w}}(x,hD) = O(h^\infty),$$

since $\mathrm{spt}(a) \cap \mathrm{spt}(\chi) = \emptyset$. Since $P(h)u(h) = E(h)u(h)$, (6.13) and (6.14) imply that

$$\begin{aligned} a^{\mathrm{w}}(x,hD)u(h) = a^{\mathrm{w}}(x,hD)c^{\mathrm{w}}(x,hD)(P(h)-E(h)+i\chi^{\mathrm{w}})u(h)+O(h^\infty) \\ = O(h^\infty). \end{aligned}$$

□

For the next result, we temporarily return to the case of the quantum harmonic oscillator, developing some sharper estimates:

THEOREM 6.5 (Improved estimates for the harmonic oscillator). *Suppose that $u(h) \in L^2(\mathbb{R}^n)$ is an eigenfuction of the harmonic oscillator:*

$$P_0(h)u(h) = E(h)u(h). \tag{6.15}$$

Assume also that $a \in C_c^\infty$.

Then there exists $R > 0$, depending only on the support of a, such that for $E(h) > R$,

$$\|a^{\mathrm{w}}(x,hD)u(h)\|_{L^2} = O\left(\left(\frac{h}{E(h)}\right)^\infty\right)\|u(h)\|_{L^2}. \tag{6.16}$$

The precise form of the right hand side of (6.16) will later let us handle eigenvalues $E(h) \to \infty$.

Proof. 1. We rescale the harmonic oscillator so that we can work near a fixed energy level E. Set

$$y := \frac{x}{E^{\frac{1}{2}}}, \quad \tilde{h} := \frac{h}{E}, \quad E(\tilde{h}) := \frac{E(h)}{E},$$

where we choose E so that $|E(h) - E| \leq E/4$. Then put

$$P_0(h) := -h^2\Delta_x + |x|^2, \quad P_0(\tilde{h}) := -\tilde{h}^2\Delta_y + |y|^2;$$

whence

$$P_0(h) - E(h) = E(P(\tilde{h}) - \widetilde{E}(\tilde{h})).$$

We next introduce the unitary transformation

$$Uu(y) := E^{\frac{n}{2}}u(E^{\frac{1}{2}}y).$$

Then

$$UP_0(h)U^{-1} = EP_0(\tilde{h});$$

and more generally

$$Ub^{\mathrm{w}}(x, hD)U^{-1} = \widetilde{b}^{\mathrm{w}}(y, \tilde{h}D), \quad \widetilde{b}(y, \eta) := b(E^{\frac{1}{2}}y, E^{\frac{1}{2}}\eta).$$

We will denote the symbol classes defined using $\tilde{h}$ by the symbol $\widetilde{S}_\delta$.

2. We now apply Theorem 6.4, to eigenfuctions of $P_0(\tilde{h})$. If

$$(P_0(\tilde{h}) - E(\tilde{h}))\widetilde{u}(\tilde{h}) = 0, \quad |E(\tilde{h}) - 1| < \delta,$$

and $\widetilde{b}(y, \eta) \in \widetilde{S}$ has its support contained in

$$\{|y|^2 + |\eta|^2 \leq 1/2\},$$

then

$$\|\widetilde{b}^{\mathrm{w}}(y, \tilde{h}D)\widetilde{u}(\tilde{h})\|_{L^2} = O(\tilde{h}^\infty)\|\widetilde{u}(\tilde{h})\|_{L^2}.$$

Translated to the original h and x as above, this assertion provides us with the bound

$$\|b^{\mathrm{w}}(x, hD)u(h)\|_{L^2} = O((h/E)^\infty)\|u(h)\|_{L^2}, \tag{6.17}$$

for

$$b(x, \xi) = \widetilde{b}(E^{-1/2}x, E^{-1/2}\xi) \in S.$$

Note that $\mathrm{spt}(b) \subset \{|x|^2 + |\xi|^2 \leq E/2\}$.

3. In view of (6.17), we only need to show that for

$$a \in C^\infty(\mathbb{R}^{2n}), \quad \mathrm{spt}(a) \subset \{|x|^2 + |\xi|^2 \leq 1/4\},$$

we have

$$\|(a^{\mathrm{w}}(x, hD)(1 - b^{\mathrm{w}}(x, hD))\|_{L^2 \to L^2} = O((h/E)^\infty),$$

for E large enough, where b is as in (6.17). That is the same as showing

$$\|\widetilde{a}^{\mathrm{w}}(y, \tilde{h}D)(1 - \widetilde{b}^{\mathrm{w}}(y, \tilde{h}D))\|_{L^2 \to L^2} = O(\tilde{h}^\infty), \tag{6.18}$$

for

$$\widetilde{a}(y, \eta) = a(E^{\frac{1}{2}}y, E^{\frac{1}{2}}\eta).$$

We first observe that $E = h/\tilde{h} < 1/\tilde{h}$ and hence

$$\widetilde{a} \in \widetilde{S}_{\frac{1}{2}}.$$

Since the support of a is compact, we see that for E large enough,

$$\operatorname{dist}(\operatorname{spt}(\widetilde{a}), \operatorname{spt}(1 - \widetilde{b})) \geq 1/C > 0,$$

uniformly in $\tilde{h}$. The estimate (6.18) is now a consequence of Theorem 4.18. □

6.2.2 Projections. We next study how projections onto the span of various eigenfunctions of the harmonic oscillator $P_0(h)$ are related to our symbol calculus.

THEOREM 6.6 (Projections and symbols). *Suppose for the symbol $a \in S$ that*

$$\operatorname{spt}(a) \subset \{|\xi|^2 + |x|^2 < R\}.$$

Let

$$\Pi := \textit{projection in } L^2 \textit{ onto}$$
$$\textit{span}\{u(h) \mid P_0(h)u(h) = E(h)u(h) \textit{ for } E(h) \leq R\}.$$

Then

$$\|a^{\mathrm{w}}(x, hD)(I - \Pi)\|_{L^2 \to L^2} = O(h^\infty) \tag{6.19}$$

and

$$\|(I - \Pi)a^{\mathrm{w}}(x, hD)\|_{L^2 \to L^2} = O(h^\infty). \tag{6.20}$$

Proof. First of all, observe

$$(I - \Pi) = \sum_{E_j(h) > R} u_j(h) \otimes u_j(h),$$

meaning that

$$(I - \Pi)u = \sum_{E_j(h) > R} \langle u_j(h), u \rangle u_j(h).$$

Therefore

$$a^{\mathrm{w}}(x, hD)(I - \Pi) = \sum_{E_j(h) > R} (a^{\mathrm{w}}(x, hD)u_j(h)) \otimes u_j(h);$$

and so

$$\|a^{\mathrm{w}}(x, hD)(I - \Pi)\|_{L^2 \to L^2} \leq \left(\sum_{E_j(h) > R} \|a^{\mathrm{w}}(x, hD)u_j(h)\|_{L^2}^2 \right)^{\frac{1}{2}}. \tag{6.21}$$

Next, observe that Weyl's Law for the harmonic oscillator, Theorem 6.3, implies that

$$E_j(h) \geq \gamma j^{\frac{1}{n}} h$$

for some constant $\gamma > 0$. According then to Theorem 6.5, for each $M < N$ we have

$$\begin{aligned} \|a^{\mathrm{w}}(x, hD) u_j(h)\|_{L^2} &\leq C_N \left(\frac{h}{E_j(h)} \right)^N \\ &\leq C h^M \left(\frac{h}{E_j(h)} \right)^{N-M} \\ &\leq C h^M j^{-\frac{N-M}{n}}. \end{aligned}$$

Consequently, if we fix $N - M > n$, the sum on the right hand side of (6.21) is less than or equal to Ch^M. This proves (6.19), and the proof of (6.20) is similar. □

6.3 SPECTRUM AND RESOLVENTS

We next show that the spectrum of $P(h)$ consists entirely of eigenvalues.

THEOREM 6.7 (Resolvents and spectrum). *There exists a constant $h_0 > 0$ such that if $0 < h \leq h_0$, then the resolvent*

$$(P(h) - z)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a meromorphic function of z with only simple, real poles.

In particular, the spectrum of $P(h)$ is discrete.

Proof. 1. Let $|z| \leq E$, where E is fixed; and as before let $P_0(h) = -h^2\Delta + |x|^2$ be the harmonic oscillator. As in Theorem 6.6 define

$$\begin{aligned} \Pi := \text{projection in } L^2 \text{ onto} \\ \operatorname{span}\{u \mid P_0(h)u = E(h)u \text{ for } E(h) \leq R + 1\}. \end{aligned}$$

Suppose now $\operatorname{spt}(a) \subset \{|x|^2 + |\xi|^2 \leq R\}$. Owing to Theorem 6.6, we have

$$\|a^{\mathrm{w}}(x, hD) - a^{\mathrm{w}}(x, hD)\Pi\|_{L^2 \to L^2} = O(h^\infty).$$

and

$$\|a^{\mathrm{w}}(x, hD) - \Pi a^{\mathrm{w}}(x, hD)\|_{L^2 \to L^2} = O(h^\infty).$$

2. Fix $R > 0$ so large that

$$\{|\xi|^2 + V(x) \leq E\} \subset \{|x|^2 + |\xi|^2 < R\}.$$

Select $\chi \in C^\infty(\mathbb{R}^{2n})$ with $\mathrm{spt}(\chi) \subset \{|x|^2 + |\xi|^2 \le R\}$ so that

$$|\xi|^2 + V(x) - z + \chi \ge \gamma m$$

for $m = \langle \xi \rangle^2 + \langle x \rangle^k$ and all $|z| \le E$. Then $\chi = \Pi\chi\Pi + O(h^\infty)$. Recall that the symbolic calculus guarantees that $P(h) - z + \chi$ is invertible, if h is small enough. Consequently, so is $P(h) - z + \Pi\chi\Pi$, since the two operators differ by an $O(h^\infty)$ term.

3. Now write

$$P(h) - z = P(h) - z + \Pi\chi\Pi - \Pi\chi\Pi$$

Consequently

$$P(h) - z \;\; = \;\; (P(h) - z + \Pi\chi\Pi)(I - (P(h) - z + \Pi\chi\Pi)^{-1}\Pi\chi\Pi)$$

Note that $\Pi\chi\Pi$ is an operator of finite rank. So Theorem D.4 asserts that the family of operators

$$(I - (P(h) - z + \Pi\chi\Pi)^{-1}\Pi\chi\Pi)^{-1}$$

is meromorphic in z. It follows that $(P(h) - z)^{-1}$ is meromorphic on L^2. The poles are the eigenvalues, and the self-adjointness of $P(h)$ implies these eigenvalues are real and simple. $\square$

REMARK: Theorem 6.7 can be obtained more directly by using the Spectral Theorem and compactness of $(P(h) + i)^{-1}$.

6.4 WEYL'S LAW

We are now ready for the main result of this chapter:

THEOREM 6.8 (Weyl's Law). *Suppose that V satisfies the conditions* (6.10) *and that $E(h)$ are the eigenvalues of $P(h) = -h^2\Delta + V(x)$.*

Then for each $a < b$, we have

(6.22)
$$\#\{E(h) \mid a \le E(h) \le b\} = \frac{1}{(2\pi h)^n}(|\{a \le |\xi|^2 + V(x) \le b\}| + o(1)).$$

as $h \to 0$.

Proof. 1. Let

$$N(\lambda) = \#\{E(h) \mid E(h) \le \lambda\}.$$

Select $\chi \in C_c^\infty(\mathbb{R}^{2n})$ so that

$$\chi \equiv 1 \text{ on } \{p \le \lambda + \epsilon\}, \;\; \chi \equiv 0 \text{ on } \{p \ge \lambda + 2\epsilon\}.$$

Then

$$a := p + (\lambda + \epsilon)\chi - \lambda \geq \gamma_\epsilon m,$$

for $m = \langle \xi \rangle^2 + \langle x \rangle^m$ and some constant $\gamma_\epsilon > 0$. Hence a is elliptic; and so for small $h > 0$, $a^{\mathrm{w}}(x, hD)$ is invertible.

2. *Claim #1*: We have

$$\langle (P(h) + (\lambda + \epsilon)\chi^{\mathrm{w}}(x, hD) - \lambda)u, u \rangle \geq \gamma \|u\|_{L^2}^2 \tag{6.23}$$

for some $\gamma > 0$.

To see this, take $b \in S(m^{1/2})$ so that $b^2 = a$. Then $b^2 = b \# b + r_0$, where $r_0 \in S^{-1}(m)$. We also recall from Theorem 4.19, or rather its proof, that $b^{\mathrm{w}}(x, hD)^{-1}$ exists and

$$b^{\mathrm{w}}(x, hD)^{-1} r_0^{\mathrm{w}}(x, hD) b^{\mathrm{w}}(x, hD)^{-1} = O(1)_{L^2 \to L^2}.$$

Thus

$$\begin{aligned} a^{\mathrm{w}}(x, hD) &= b^{\mathrm{w}}(x, hD) b^{\mathrm{w}}(x, hD) + r_0^{\mathrm{w}}(x, hD) \\ &= b^{\mathrm{w}}(x, hD) \\ &\qquad (1 + b^{\mathrm{w}}(x, hD)^{-1} r_0^{\mathrm{w}}(x, hD) b^{\mathrm{w}}(x, hD)^{-1}) b^{\mathrm{w}}(x, hD) \\ &= b^{\mathrm{w}}(x, hD)(1 + O(h)_{L^2 \to L^2}) b^{\mathrm{w}}(x, hD). \end{aligned}$$

Hence for sufficiently small $h > 0$,

$$\begin{aligned} \langle (P(h) + (\lambda + \epsilon)\chi^{\mathrm{w}} - \lambda)u, u \rangle &= \langle a^{\mathrm{w}}(x, hD)u, u \rangle \\ &\geq \|b^{\mathrm{w}}(x, hD)u\|_{L^2}^2 (1 - O(h)) \\ &\geq \gamma \|u\|_{L^2}^2, \end{aligned}$$

for some $\gamma > 0$, since $b^{\mathrm{w}}(x, hD)^{-1}$ exists. This proves (6.23).

3. *Claim #2*: For each $\delta > 0$, there exists a bounded linear operator Q such that

$$\chi^{\mathrm{w}}(x, hD) = Q + O(h^\infty)_{L^2 \to L^2} \tag{6.24}$$

and

$$\operatorname{rank}(Q) \leq \frac{1}{(2\pi h)^n} (|\{p \leq \lambda + 2\epsilon\}| + \delta). \tag{6.25}$$

To prove this, cover the set $\{p \leq \lambda + 2\epsilon\}$ with balls

$$B_j := B((x_j, \xi_j), r_j) \quad (j = 1, \cdots, N)$$

such that

$$\sum_{j=1}^N |B_j| \leq |\{p \leq \lambda + 2\epsilon\}| + \frac{\delta}{2}.$$

Define the "shifted" harmonic oscillator

$$P_j(h) := |hD_x - \xi_j|^2 + |x - x_j|^2;$$

and set

$$\begin{aligned}\Pi \;\; &:= \;\; \text{orthogonal projection in } L^2 \text{ onto } V\text{, the span of}\\ &\qquad \{u \mid P_j(h)u = E_j(h)u,\ E_j(h) \le r_j,\ j = 1, \dots, N\}.\end{aligned}$$

We now claim that

$$(I - \Pi)\chi^{\mathrm{w}}(x, hD) = O(h^\infty)_{L^2 \to L^2}. \tag{6.26}$$

To see this, let $\chi = \sum_{j=1}^{N} \chi_j$, where $\operatorname{spt} \chi_j \subset\subset B((x_j, \xi_j), r_j)$, and put

$$\begin{aligned}\Pi_j \;\; &:= \;\; \text{orthogonal projection in } L^2 \text{ onto the span of}\\ &\qquad \{u \mid P_j(h)u = E_j(h)u,\ E_j(h) \le r_j\}.\end{aligned}$$

Theorem 6.6 shows that $(I - \Pi_j)\chi_j^{\mathrm{w}}(x, hD) = O(h^\infty)$. We note that $\Pi\Pi_j = \Pi_j$ and hence

$$\begin{aligned}(I - \Pi)\chi^{\mathrm{w}}(x, hD) &= \sum_{j=1}^{N} (I - \Pi)\chi_j^{\mathrm{w}}(x, hD)\\ &= \sum_{j=1}^{N} (I - \Pi)(I - \Pi_j)\chi_j^{\mathrm{w}}(x, hD)\\ &= O(h^\infty)_{L^2 \to L^2}.\end{aligned}$$

This proves (6.26).

It now follows that

$$\chi^{\mathrm{w}}(x, hD) = \Pi\chi^{\mathrm{w}}(x, hD) + (I - \Pi)\chi^{\mathrm{w}}(x, hD) = Q + O(h^\infty)$$

for

$$Q := \Pi\chi^{\mathrm{w}}(x, hD).$$

Clearly Q has finite rank, since

$$\begin{aligned}\operatorname{rank} Q \;\; &= \;\; \dim(\text{image of } Q) \le \dim(\text{image of } \Pi)\\ &\le \;\; \sum_{j=1}^{N} \#\{E_j(h) \mid E_j(h) \le r_j\}\\ &= \;\; \frac{1}{(2\pi h)^n}\left(\sum_{j=1}^{N} |B_j| + o(1)\right),\end{aligned}$$

according to Weyl's law for the harmonic oscillator, Theorem 6.3. Consequently

$$\operatorname{rank} Q \le \frac{1}{(2\pi h)^n}\left(|\{p \le \lambda + 2\epsilon\}| + \frac{\delta}{2} + o(1)\right). \tag{6.27}$$

This proves Claim #2.

4. We next employ Claims #1,2 and Theorem C.8. We have

$$\begin{aligned} \langle P(h)u,u\rangle &\geq (\lambda+\gamma)\|u\|_{L^2}^2-(\lambda+\epsilon)\langle Qu,u\rangle+\langle O(h^\infty)u,u\rangle \\ &\geq \lambda\|u\|_{L^2}^2-(\lambda+\epsilon)\langle Qu,u\rangle, \end{aligned}$$

where the rank of Q is bound by (6.27). Theorem C.8,(i) implies then that

$$N(\lambda)\leq \frac{1}{(2\pi h)^n}(|\{p\leq\lambda+2\epsilon\}|+\delta+o(1)).$$

This holds for all $\epsilon,\delta>0$, and so

$$N(\lambda)\leq \frac{1}{(2\pi h)^n}(|\{p\leq\lambda\}|+o(1)) \tag{6.28}$$

as $h\to 0$.

5. We must prove the opposite inequality.

Claim #3: Suppose $B_j=B((x_j,\xi_j),r_j)\subset\{p<\lambda\}$. Then if

$$P_j(h)u=E_j(h)u$$

and $E(h)\leq r_j$, we have

$$\langle P(h)u,u\rangle\leq(\lambda+\epsilon+O(h^\infty))\|u\|_{L^2}^2 \tag{6.29}$$

To prove this claim, select a symbol $a\in C_c^\infty(\mathbb{R}^{2n})$, with

$$a\equiv 1 \text{ on } \{p\leq\lambda\},\ \operatorname{spt}(a)\subset\{p\leq\lambda+\epsilon\}.$$

Let $c:=1-a$. Then $u-a^{\mathrm{w}}(x,hD)u=c^{\mathrm{w}}(x,hD)u=O(h^\infty)$ according to Theorem 6.6, since $\operatorname{spt}(1-a)\cap B_j=\emptyset$.

Define $b^{\mathrm{w}}:=P(h)a^{\mathrm{w}}(x,hD)$. Now $p\in S(m)$ and $a\in S(m^{-1})$. Thus $b=pa+O(h)\in S$ and so b^{w} is bounded in L^2. Observe also that $b\leq\lambda+\frac{\epsilon}{2}$, and so

$$b^{\mathrm{w}}(x,hD)\leq\lambda+\frac{3\epsilon}{4}.$$

Therefore

$$\langle P(h)a^{\mathrm{w}}(x,hD)u,u\rangle=\langle b^{\mathrm{w}}(x,hD)u,u\rangle\leq\left(\lambda+\frac{3\epsilon}{4}\right)\|u\|_{L^2}^2.$$

Since $a^{\mathrm{w}}(x,hD)u=u+O(h^\infty)$, we deduce

$$\langle P(h)u,u\rangle\leq(\lambda+\epsilon+O(h^\infty))\|u\|_{L^2}^2.$$

This proves Claim #3.

6. Now find disjoint balls $B_j\subset\{p<\lambda\}$ such that

$$|\{p<\lambda\}|\leq\sum_{j=1}^N|B_j|+\delta.$$

Let $V := \text{span}\{u \mid P_j(h)u = E_j(h)u, E_j(h) \leq r_j, j = 1, \ldots, N\}$. Owing to Claim #3,

$$\langle Pu, u \rangle \leq (\lambda + \delta)\|u\|_{L^2}^2$$

for all $u \in V$. Also, Theorem 6.3 implies

$$\begin{aligned} \dim V &\geq \sum_{j=1}^{N} \#\{E_j(h) \leq r_j\} \\ &= \frac{1}{(2\pi h)^n}\left(\sum_{j=1}^{N} |B_j| + o(1)\right) \\ &\geq \frac{1}{(2\pi h)^n}(|\{p < \lambda\}| - \delta + o(1)). \end{aligned}$$

According then to Theorem C.8,(ii),

$$N(\lambda) \geq \frac{1}{(2\pi h)^n}(|\{p < \lambda\}| - \delta + o(1)).$$

□

7. Exponential estimates for eigenfunctions

This chapter continues our study of semiclassical behavior of eigenfunctions:

$$P(h)u(h) = E(h)u(h) \tag{7.1}$$

for the operator

$$P(h) = -h^2\Delta + V(x)$$

and corresponding symbol

$$p(x,\xi) = |\xi|^2 + V(x).$$

We first demonstrate that if $E(h)$ is close to the energy level E, then $u(h)$ exponentially small within the *classically forbidden region*

$$V^{-1}(E,\infty) = \{x \in \mathbb{R}^n \mid V(x) > E\}.$$

Then we show, conversely, that in any open set the L^2 norm of $u(h)$ is bounded from below by a quantity exponentially small in h. We conclude with a discussion of the order of vanishing of eigenfunctions in the semiclassical limit.

7.1 CLASSICALLY FORBIDDEN REGIONS

We begin with some definitions and general facts.

DEFINITION. Let $U \subset \mathbb{R}^n$ be an open set. The *semiclassical Sobolev norms* are defined as

$$\|u\|_{H_h^k(U)} := \left(\sum_{|\alpha|\le k} \int_U |(hD)^\alpha u|^2 \, dx \right)^{1/2}$$

for $u \in C^\infty(U)$, $k = 0, 1, \dots$.

These differ from the standard Sobolev norms by the introduction of appropriate powers of h.

LEMMA 7.1 (Semiclassical elliptic estimates). *Let $W \subset\subset U$ be open sets. Then there exists a constant C such that*

$$\|u\|_{H_h^2(W)} \le C(\|P(h)u\|_{L^2(U)} + \|u\|_{L^2(U)}) \tag{7.2}$$

for all $u \in C^\infty(U)$.

Proof. 1. Let $\chi \in C_c^\infty(U)$, $\chi \equiv 1$ on W. We multiply $P(h)u$ by $\chi^2\bar{u}$ and integrate by parts:

$$\int_U h^2\langle \partial(\chi^2\bar{u}), \partial u\rangle + (V-E)|u|^2\chi^2\,dx = \int_U P(h)u\bar{u}\chi^2\,dx.$$

Therefore

$$h^2\int_U \chi^2|\partial u|^2\,dx \le C\int_U |P(h)u|^2 + |u|^2\,dx;$$

and so

$$h^2\int_W |\partial u|^2\,dx \le C\int_U |P(h)u|^2 + |u|^2\,dx$$

2. Similarly, multiply $P(h)u$ by $\chi^2\Delta\bar{u}$ and integrate by parts, to deduce

$$h^4\int_W |\partial^2 u|^2\,dx \le C\int_U |P(h)u|^2 + |u|^2\,dx.$$

□

Before turning again to eigenfunctions, we present the following general estimates. Our primary tool will be properly designed conjugations of the operator $P(h)$.

DEFINITION Given $\varphi \in C^\infty(\mathbb{R}^n)$, we define the *conjugation* of $P(h)$ by $e^{\varphi/h}$:

$$P_\varphi(h) := e^{\varphi/h}Pe^{-\varphi/h}. \tag{7.3}$$

LEMMA 7.2 (Symbol of conjugation). *We have*

$$P_\varphi(h) = (p_\varphi)^{\mathrm{w}} + O(h) \tag{7.4}$$

for the symbol

$$p_\varphi(x,\xi) := |\xi + i\partial\varphi(x)|^2 + V(x). \tag{7.5}$$

Proof. We calculate for functions $u \in C^\infty(\mathbb{R}^n)$ that

$$\begin{aligned} P_\varphi(h)u &= e^{\varphi/h}(-h^2\Delta + V)(e^{-\varphi/h}u) \\ &= -h^2\Delta u + 2h\langle\partial\varphi, \partial u\rangle - |\partial\varphi|^2u + Vu + h\Delta\varphi u. \end{aligned}$$

The expression on the right is $(p_\varphi)^{\mathrm{w}}u + O(h)u$. □

THEOREM 7.3 (Exponential estimate from above). *Suppose that U is an open set such that*

$$U \subset\subset V^{-1}(E,\infty).$$

For each open set $W \supset\supset U$ and for each λ near E, there exist constants $h_0, \delta, C > 0$, such that

$$\|u\|_{L^2(U)} \leq Ce^{-\delta/h}\|u\|_{L^2(W)} + C\|(P(h) - \lambda)u\|_{L^2(W)} \tag{7.6}$$

for $u \in C_c^\infty(\mathbb{R}^n)$ and $0 < h \leq h_0$.

We call (7.6) an *Agmon-type estimate.*

Proof. 1. Select $\psi, \varphi \in C_c^\infty(W)$ such that $0 \leq \psi, \varphi \leq 1$, $\psi \equiv 1$ on U, and $\varphi \equiv 1$ on spt ψ. We may assume as well that $W \subset\subset V^{-1}(E, \infty)$.

As in Lemma 7.2, we observe that the symbol of

$$A(h) := e^{\delta\psi/h}(P(h) - \lambda)e^{-\delta\psi/h}$$

is

$$|\xi + i\delta\partial\psi|^2 + V - \lambda + O(h).$$

Now for λ close to E, $x \in W$ and δ sufficiently small, we have

$$(|\xi + i\delta\partial\psi|^2 + V - \lambda)^2 \geq \gamma > 0$$

for some positive constant γ. Then according to the sharp Gårding inequality, Theorem 4.21, we see that provided $\delta > 0$ is sufficiently small,

$$\varphi A(h)^* A(h)\varphi \geq \sigma^2\varphi^2 - O_{L^2\to L^2}(h),$$

for some constant $\sigma > 0$. Hence if h is small enough, we have

$$\varphi A(h)^* A(h)\varphi \geq \frac{1}{2}\sigma^2\varphi^2$$

in the sense of operators.

2. This implies that

$$\begin{aligned}\|e^{\delta\psi/h}\varphi u\|_{L^2} &\leq C\|A(h)(e^{\delta\psi/h}\varphi u)\|_{L^2} = C\|e^{\delta\psi/h}(P(h) - \lambda)\varphi u\|_{L^2} \\ &\leq C\|e^{\delta\psi/h}\varphi(P(h) - \lambda)u\|_{L^2} + C\|e^{\delta\psi/h}[P(h), \varphi]u\|_{L^2},\end{aligned}$$

for $u \in C_c^\infty(\mathbb{R}^n)$.

Next is the key observation that since $\varphi \equiv 1$ on spt ψ, we have $\psi \equiv 0$ on spt $[P(h), \varphi]u$. Thus Lemma 7.1 implies

$$\begin{aligned}\|e^{\delta\psi/h}[P(h), \varphi]u\|_{L^2} &= \|[P(h), \varphi]u\|_{L^2} \\ &\leq C\big(\|hD_x u\|_{L^2(W)} + \|u\|_{L^2(W)}\big) \\ &\leq C\|u\|_{L^2(W)} + C\|(P(h) - \lambda)u\|_{L^2(W)}.\end{aligned}$$

Combining these estimates, we conclude that

$$\begin{aligned}e^{\delta/h}\|u\|_{L^2(U)} &\leq \|e^{\delta\psi/h}u\|_{L^2} \\ &\leq C\|u\|_{L^2(W)} + C(e^{\delta/h} + 1)\|(P(h) - \lambda)u\|_{L^2(W)}.\end{aligned}$$

□

Specializing to eigenfunctions, we deduce

THEOREM 7.4 (Exponential decay estimates). *Suppose that* $U \subset\subset V^{-1}(E, \infty)$, *and that* $u(h) \in L^2(\mathbb{R}^n)$ *solves*

$$P(h)u(h) = E(h)u(h),$$

where

$$E(h) \to E \quad \text{as } h \to 0.$$

Then there exists a constant $\delta > 0$ *such that*

$$\|u(h)\|_{L^2(U)} \le e^{-\delta/h} \|u(h)\|_{L^2(\mathbb{R}^n)}. \tag{7.7}$$

7.2 TUNNELING

We continue to assume in this section $u = u(h)$ solves the eigenvalue problem (7.1).

In the previous section we showed that $u(h)$ is exponentially small in the physically forbidden region. In this section we will show that it can never be smaller than this: for small $h > 0$ and any bounded, open subset U of $\mathbb{R}^n$, we have the lower bound

$$\|u\|_{L^2(U)} \ge e^{-\frac{C}{h}} \|u\|_{L^2(\mathbb{R}^n)}.$$

This is a mathematical version of quantum mechanical "tunneling into the physically forbidden region".

DEFINITION. *Hörmander's hypoellipticity condition* is the requirement for the symbol p_φ, defined by (7.5), that

$$\text{if} \quad p_\varphi = 0, \quad \text{then} \quad i\{p_\varphi, \overline{p_\varphi}\} > 0. \tag{7.8}$$

Observe that for any complex function $q = q(x, \xi)$,

$$i\{q, \bar{q}\} = i\{\text{Re } q + i \text{ Im } q,\ \text{Re } q - i \text{ Im } q\} = 2\{\text{Re } q, \text{Im } q\}.$$

Hence the expression $i\{p_\varphi, \overline{p_\varphi}\}$ is real.

THEOREM 7.5 (L^2-estimate for $P_\varphi(h)$). *If Hörmander's condition* (7.8) *is valid within* $\overline{W} \subset\subset \mathbb{R}^n$, *then*

$$h^{1/2} \|u\|_{L^2(W)} \le C \|P_\varphi(h) u\|_{L^2(W)} \tag{7.9}$$

for all $u \in C_c^\infty(W)$, *provided* $0 < h \le h_0$ *for* $h_0 > 0$ *sufficiently small.*

Proof. We calculate

$$\begin{aligned}\|P_\varphi(h)u\|_{L^2}^2 &= \langle P_\varphi(h)u, P_\varphi(h)u\rangle = \langle P_\varphi^*(h)P_\varphi u, u\rangle \\ &= \langle P_\varphi(h)P_\varphi^*(h)u, u\rangle + \langle [P_\varphi^*(h), P_\varphi(h)]u, u\rangle \\ &= \|P_\varphi^*(h)u\|_{L^2} + \langle [P_\varphi^*(h), P_\varphi(h)]u, u\rangle.\end{aligned}$$

The idea will be to use the positivity of the second term on the right hand side wherever $P_\varphi^*(h)$ fails to be elliptic. More precisely, for any $M > 1$ and h small enough the calculation above gives

$$\begin{aligned}\|P_\varphi(h)u\|_{L^2}^2 &\geq Mh\|P_\varphi^*(h)u\|_{L^2} + \langle [P_\varphi^*(h), P_\varphi(h)]u, u\rangle \\ &= h\langle (M|p_\varphi|^2 + i\{p_\varphi, \bar{p}_\varphi\})^{\mathrm{w}}u, u\rangle - O(h^2)\|u\|_{H_h^2}^2,\end{aligned}$$

the last term resulting from estimates of the lower order terms in $\bar{p}_\varphi \# p_\varphi$ and the commutator. Hörmander's hypoellipticity condition (7.8) implies for M large enough that

$$M|p_\varphi(x,\xi)|^2 + i\{p_\varphi, \bar{p}_\varphi\}(x,\xi) \geq \gamma > 0.$$

for $x \in \bar{W}$. Then Theorem 4.20, the easy Gårding inequality, and Lemma 7.1 show us that

$$\|P_\varphi(h)u\|_{L^2}^2 \geq Ch\|u\|_{L^2}^2 - O(h^2)(\|P_\varphi(h)u\|_{L^2}^2 + \|u\|_{L^2}^2).$$

□

Next we carefully design a weight φ, to ensure that $P_\varphi(h)$ satisfies the hypothesis of Theorem 7.5.

LEMMA 7.6 (Constructing a weight). *Let $0 < r < R$. There exists a nonincreasing radial function $\varphi \in C^\infty(\mathbb{R}^n)$ such that Hörmander's hypoellipticity condition* (7.8) *holds on $B(0,R) - B(0,r)$.*

Proof. 1. Recall that

$$p_\varphi = |\xi + i\partial\varphi|^2 + V - E = |\xi|^2 + 2i\langle \xi, \partial\varphi\rangle - |\partial\varphi|^2 + V - E.$$

So $p_\varphi = 0$ implies both

$$|\xi|^2 - |\partial\varphi|^2 + V - E = 0 \tag{7.10}$$

and

$$\langle \xi, \partial\varphi\rangle = 0. \tag{7.11}$$

Furthermore,

$$\text{(7.12)}\qquad \begin{aligned} \frac{i}{2}\{p_\varphi, \overline{p_\varphi}\} &= \{\text{Re } p_\varphi, \text{Im } p_\varphi\} \\ &= \langle \partial_\xi(|\xi|^2 - |\partial\varphi|^2 + V - E), 2\partial_x\langle \xi, \partial\varphi\rangle\rangle \\ &\quad -\langle \partial_x(|\xi|^2 - |\partial\varphi|^2 + V - E), 2\partial_\xi\langle \xi, \partial\varphi\rangle\rangle \\ &= 4\langle \partial^2\varphi\xi, \xi\rangle + 4\langle \partial^2\varphi\partial\varphi, \partial\varphi\rangle - 2\langle \partial V, \partial\varphi\rangle. \end{aligned}$$

2. Assume now

$$\varphi = e^{\lambda\psi},$$

where $\lambda > 0$ will be selected and $\psi : \mathbb{R}^n \to \mathbb{R}$ is positive and radial, $\psi = \psi(|x|)$. Then

$$\partial\varphi = \lambda\partial\psi e^{\lambda\psi}$$

and

$$\partial^2\varphi = (\lambda^2\partial\psi \otimes \partial\psi + \lambda\partial^2\psi)e^{\lambda\psi}.$$

Hence

$$\langle \partial^2\varphi\xi, \xi\rangle = (\lambda^2\langle \partial\psi, \xi\rangle^2 + \lambda\langle \partial^2\psi\xi, \xi\rangle)e^{\lambda\psi} = \lambda\langle \partial^2\psi\xi, \xi\rangle e^{\lambda\psi},$$

since (7.11) implies $\langle \partial\psi, \xi\rangle = 0$. Also

$$\langle \partial^2\varphi\partial\varphi, \partial\varphi\rangle = \lambda^4|\partial\psi|^4 e^{3\lambda\psi} + \lambda^3\langle \partial^2\psi\partial\psi, \partial\psi\rangle e^{3\lambda\psi},$$

and

$$\langle \partial V, \partial\varphi\rangle = \lambda\langle \partial V, \partial\psi\rangle e^{\lambda\psi}.$$

According to (7.12), we have

$$\text{(7.13)}\qquad \begin{aligned} \frac{i}{2}\{p_\varphi, \overline{p_\varphi}\} &= 4\lambda\langle \partial^2\psi\xi, \xi\rangle e^{\lambda\psi} + 4\lambda^4|\partial\psi|^4 e^{3\lambda\psi} \\ &\quad + 4\lambda^3\langle \partial^2\psi\partial\psi, \partial\psi\rangle e^{3\lambda\psi} - 2\lambda\langle \partial V, \partial\psi\rangle e^{\lambda\psi}. \end{aligned}$$

3. Now take

$$\psi := \mu - |x|,$$

for a constant μ so large that $\psi \geq 1$ on $B(0, R)$. Then φ is radial and nonincreasing. Furthermore

$$|\partial\psi| = 1, \ |\partial^2\psi| \leq C \quad \text{ on } B(0, R) - B(0, r).$$

Owing to (7.10) we have

$$|\xi|^2 \leq C + |\partial\varphi|^2 \leq C + C\lambda^2 e^{2\lambda\psi} \quad \text{ on } B(0, R) - B(0, r).$$

Plugging these estimates into (7.13), we compute

$$\frac{i}{2}\{p_\varphi, \overline{p_\varphi}\} \geq 2\lambda^4 e^{3\lambda\psi} - C\lambda^3 e^{3\lambda\psi} - C \geq 1,$$

in $B(0, R) - B(0, r)$, if λ is selected large enough.

Lastly, we modify ψ within $B(0,r)$ to obtain a smooth fuction on $B(0,R)$. □

THEOREM 7.7 (Exponential estimate from below). *Let $a<b$ and suppose $U\subset\mathbb{R}^n$ is an open set. There exist constants C, $h_0>0$ such that if $u(h)$ solves*

$$P(h)u = E(h)u(h) \quad \text{in } \mathbb{R}^n$$

for $E(h)\in[a,b]$ and $0<h\le h_0$, then

$$\|u(h)\|_{L^2(U)} \ge e^{-\frac{C}{h}}\|u(h)\|_{L^2(\mathbb{R}^n)}. \tag{7.14}$$

We call (7.14) a *Carleman-type estimate.*

Proof. 1. We may assume without loss that $U=B(0,3r)$ for some $0<r<\frac{1}{3}$. Select $R>1$ so large that

$$p(x,\xi)-\lambda = |\xi|^2+V(x)-\lambda \ge |\xi|^2+\langle x\rangle^k$$

for $|x|\ge R$ and $a\le\lambda\le b$. Since $p-E(h)$ is therefore elliptic on $\mathbb{R}^n - B(0,R)$, we have the estimate

$$\|v\|_{L^2(\mathbb{R}^n-B(0,R))} \le C\|(P(h)-E(h))v\|_{L^2(\mathbb{R}^n-B(0,R))} \tag{7.15}$$

for all $v\in C_c^\infty(\mathbb{R}^n-B(0,R))$.

2. Select two radial functions $\chi_1,\chi_2:\mathbb{R}^n\to\mathbb{R}$ such that $0\le\chi_1\le1$ and

$$\begin{cases} \chi_1\equiv 0 & \text{on } B(0,r),\\ \chi_1\equiv 1 & \text{on } B(0,R+2)-B(0,2r),\\ \chi_1\equiv 0 & \text{on } \mathbb{R}^{2n}-B(0,R+3); \end{cases}$$

and $0\le\chi_2\le1$,

$$\begin{cases} \chi_2\equiv 0 & \text{on } B(0,R)\\ \chi_2\equiv 1 & \text{on } \mathbb{R}^{2n}-B(0,R+1). \end{cases}$$

Applying (7.15) to $v=\chi_2 u$ gives

$$\|\chi_2 u\|_{L^2} \le C\|(P(h)-E(h))(\chi_2 u)\|_{L^2} = C\|[P(h),\chi_2]u\|_{L^2}.$$

Now $[P(h),\chi_2]u = -h^2 u\Delta\chi_2 - 2h^2\langle\partial\chi_2,\partial u\rangle$, and consequently $[P(h),\chi_2]u$ is supported within $B(0,R+1)-B(0,R)$. Hence Lemma 7.1 implies

$$\begin{aligned}
\|[P(h),\chi_2]u\|_{L^2} &\leq Ch\|u\|_{H_h^1(B(0,R+1)-B(0,R))}\\
&\leq Ch(\|P(h)u\|_{L^2(B(0,R+2)-B(0,R-1))}\\
&\qquad\qquad + \|u\|_{L^2(B(0,R+2)-B(0,R-1))})\\
&\leq Ch\|u\|_{L^2(B(0,R+2)-B(0,R-1))}\\
&\leq Ch\|\chi_1 u\|_{L^2}
\end{aligned}$$

Therefore

$$\|\chi_2 u\|_{L^2} \leq Ch\|\chi_1 u\|_{L^2}. \tag{7.16}$$

3. Next apply Theorem 7.5:

$$h^{1/2}\|e^{\frac{\varphi}{h}}\chi_1 u\|_{L^2} \leq C\|e^{\frac{\varphi}{h}}(P(h)-E(h))(\chi_1 u)\|_{L^2} = C\|e^{\frac{\varphi}{h}}[P(h),\chi_1]u\|_{L^2}$$

Now $[P(h),\chi_1]$ is supported within the union of $B(0,2r)-B(0,r)$ and $B(0,R+3)-B(0,R+2)$. Since φ is nonincreasing, we therefore have

$$\begin{aligned}
\|e^{\frac{\varphi}{h}}[P(h),\chi_1]u\|_{L^2} \leq Che^{\frac{\varphi(R+2)}{h}}&\|\chi_2 u\|_{H_h^1(B(0,R+3)-B(0,R+2)))}\\
&+ Che^{\frac{\varphi(0)}{h}}\|u\|_{H_h^1(B(0,2r)}.
\end{aligned}$$

The right hand sides can be estimated by Lemma 7.1. This gives

$$\|e^{\frac{\varphi}{h}}\chi_1 u\|_{L^2} \leq Ch^{1/2}e^{\frac{\varphi(R+2)}{h}}\|\chi_2 u\|_{L^2} + Ch^{1/2}e^{\frac{\varphi(0)}{h}}\|u\|_{L^2(U)}. \tag{7.17}$$

4. Select a constant $A>0$ so that

$$\begin{cases}\varphi > A & \text{on } B(0,R+1)-B(0,R)\\ \varphi < A & \text{on } B(0,R+3)-B(0,R+2).\end{cases}$$

Multiply (7.16) by $e^{A/h}$ and add to (7.17):

$$\begin{aligned}
&\|e^{\frac{A}{h}}\chi_2 u\|_{L^2} + \|e^{\frac{\varphi}{h}}\chi_1 u\|_{L^2}\\
&\quad \leq Ch\|e^{\frac{\varphi}{h}}\chi_1 u\|_{L^2} + Ch^{1/2}\|e^{\frac{A}{h}}\chi_2 u\|_{L^2} + Ch^{1/2}e^{\frac{\varphi(0)}{h}}\|u\|_{L^2(U)}.
\end{aligned}$$

Take $0<h\leq h_0$, for h_0 sufficiently small, to deduce

$$\|e^{\frac{A}{h}}\chi_2 u\|_{L^2} + \|e^{\frac{\varphi}{h}}\chi_1 u\|_{L^2} \leq Ch^{1/2}e^{\frac{\varphi(0)}{h}}\|u\|_{L^2(U)}.$$

Since $\chi_1+\chi_2\geq 1$ on $\mathbb{R}^n - B(0,2r) = \mathbb{R}^n - U$, the Theorem follows. $\square$

7.3 ORDER OF VANISHING

Assume, as usual, that

$$P(h)u(h) = E(h)u(h), \tag{7.18}$$

where $E(h) \in [a, b]$. To simplify notation, we will in this subsection write u for $u(h)$.

We propose now to give an estimate for the order of vanishing of u, following a suggestion of N. Burq.

DEFINITION. We say u *vanishes to order* N at the point x_0 if

$$u(x) = O(|x - x_0|^N) \quad \text{as } x \to x_0.$$

We will consider potentials which are analytic in x and, to avoid technical difficulties, make a strong assumption on the growth of derivatives:

$$V(x) \geq \langle x \rangle^m / C_0 - C_0, \quad |\partial^\alpha V(x)| \leq C_0^{1+|\alpha|} |\alpha|^{|\alpha|} \langle x \rangle^m \tag{7.19}$$

for some $m > 0$ and all multiindices α.

We note that the second condition holds when V has a holomorphic extension bounded by $|z|^m$ into a conic neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$.

THEOREM 7.8 (Semiclassical estimate on vanishing order). *Suppose that $u \in L^2$ solves* (7.18) *for $a \leq E(h) \leq b$ and that V a real analytic potential satisfying* (7.19)*. Let K be compact subset of $\mathbb{R}^n$.*

Then there exists a constant C such that if u vanishes to order N at a point $x_0 \in K$, we have the estimate

$$N \leq C h^{-1}. \tag{7.20}$$

We need the following lemma to establish analyticity of the solution in a semiclassically quantitative way:

LEMMA 7.9 (Semiclassical derivative estimates). *If u satisfies the assumptions of Theorem 7.8, then there exists a constant C_1 such that for any positive integer k:*

$$\|u\|_{H_h^k(\mathbb{R}^n)} \leq C_1^k (1 + kh)^k \|u\|_{L^2(\mathbb{R}^n)}. \tag{7.21}$$

Proof. 1. By adding C_0 to V we can assume without loss that $V(x) \geq \langle x \rangle^m / C_0$. The Lemma will follow from the following stronger estimate, which we will prove by induction:

$$\begin{aligned} \|\langle x \rangle^{m/2} (hD)^\alpha u\|_{L^2} + \|(h\partial)(hD)^\alpha u\|_{L^2} \\ \leq C_2^{k+2} (1 + kh)^{k+1} \|u\|_{L^2}. \end{aligned} \tag{7.22}$$

for $|\alpha| = k$.

2. To prove this inequality, we observe first that our multiplying (7.18) by $\bar{u}$ and integrating by parts shows that estimate (7.22) holds for $|\alpha| = 0$

Next, note that

$$\begin{aligned}
&\|V^{\frac{1}{2}}(hD)^\alpha u\|_{L^2}^2 + \|(h\partial)(hD)^\alpha u\|_{L^2}^2 \\
&\qquad = \langle(-h^2\Delta + V - E(h))(hD)^\alpha u, (hD)^\alpha u\rangle + E(h)\|(hD)^\alpha u\|_{L^2}^2 \\
&\qquad = \langle V^{-\frac{1}{2}}[V, (hD)^\alpha]u, V^{\frac{1}{2}}(hD)^\alpha u\rangle + E(h)\|(hD)^\alpha u\|_{L^2}^2 \\
&\qquad \leq 2\|V^{-\frac{1}{2}}[V, (hD)^\alpha]u\|_{L^2}^2 + \frac{1}{2}\|V^{\frac{1}{2}}(hD)^\alpha u\|_{L^2}^2 + E(h)\|(hD)^\alpha u\|_{L^2}^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{2}\|V^{\frac{1}{2}}(hD)^\alpha u\|_{L^2}^2 + \|(h\partial)(hD)^\alpha u\|_{L^2}^2 \\
&\qquad\qquad \leq 2\|V^{-\frac{1}{2}}[V, (hD)^\alpha]u\|_{L^2}^2 + E(h)\|(hD)^\alpha u\|_{L^2}^2.
\end{aligned} \tag{7.23}$$

3. We can now expand the commutator, to deduce from (7.19) (with V replaced by $V + C_0$) the inequality:

$$\begin{aligned}
&\|V^{-\frac{1}{2}}[V, (hD)^\alpha]u\|_{L^2} \leq \\
&\qquad \sum_{l=0}^{k-1} \binom{k}{l} C_0^{k-l+1}(h(k-l))^{k-l} \sup_{|\beta|=l} \|\langle x\rangle^{m/2}(hD)^\beta u\|_{L^2}.
\end{aligned} \tag{7.24}$$

We proceed by induction, and so now assume that (7.22) is valid for $|\alpha| < k$. Now Stirling's formula implies

$$\binom{k}{l} \leq C\frac{k^k}{l^l(k-l)^{k-l}}.$$

Hence, in view of (7.23) and (7.24), it is enough to show that there exists a constant C_2 such that

$$\sum_{l=0}^{k-1} h^{k-l}\frac{k^k}{l^l}C_0^{k-l+1}C_2^{l+2}(1+lh)^{l+1} + C_2^{k+1}(1+hk)^k \leq C_2^{k+2}(1+hk)^{k+1}.$$

This estimate we can rewrite as

$$\begin{aligned}
&C_0\sum_{l=1}^{k-1}\left(\frac{C_0}{C_2}\right)^{k-l}(hl)^{-l}(1+hl)^l(1+hl) + C_2^{-1}(hk)^{-k}(1+hk)^k \\
&\qquad\qquad \leq (hk)^{-k}(1+hk)^k(1+hk).
\end{aligned}$$

Since we can choose C_2 to be large and since we can estimate the $(1+hl)$ factor in the sum by $(1+hk)$, this will follow once we show that for ϵ small enough,

$$\sum_{l=0}^{k-1} \epsilon^{k-l} a_l \leq a_k \quad \text{for} \quad a_l := (1+(hl)^{-1})^l.$$

This is true by induction if a_{k-1}/a_k is bounded:

$$\sum_{l=0}^{k-1} \epsilon^{k-l} a_l \leq 2\epsilon a_{k-1}.$$

In our case,

$$\begin{aligned} \frac{a_{k-1}}{a_k} &= \left(\frac{1+(h(k-1))^{-1}}{1+(hk)^{-1}} \right)^{k-1} (1+(hk)^{-1})^{-1} \\ &= \left(1 + \frac{1}{(k-1)(1+hk)} \right)^{k-1} (1+(hk)^{-1})^{-1} \\ &\leq \exp\left(\frac{1}{1+hk} \right) \frac{hk}{1+hk} < 1. \end{aligned}$$

□

Proof of Theorem 7.8: Assume now that $\|u\|_{L^2} = 1$ and that u vanishes to order N at a point $x_0 \in K$.

Then $D^\alpha u(x_0) = 0$ for $|\alpha| < N$ and Taylor's formula shows that

$$|u(x)| \leq \frac{\epsilon^N}{N!} \sup_{|\alpha|=N} \sup_{y \in \mathbb{R}^n} |D^\alpha u(y)| \quad \text{for } |x - x_0| < \epsilon. \tag{7.25}$$

The Sobolev inequality (Lemma 3.5) and Lemma 7.9 allow us to estimate the derivatives. If say $M = N + n$ and $|\alpha| = N$, then

$$\sup_{y \in \mathbb{R}^n} |D^\alpha u(y)| \leq \|u\|_{H^M} \leq h^{-M} \|u\|_{H_h^M} \leq h^{-M} C_1^M (1+hM)^M.$$

Inserting this into (7.25) and using Stirling's formula, we deduce that if for $|x - x_0| < \epsilon$, then

$$|u(x)| \leq \left(\frac{e\epsilon}{N}\right)^N \left(\frac{C}{h}\right)^M (1+hM)^M \leq \left(\frac{N}{e\epsilon}\right)^n \left(\frac{e\epsilon C}{Nh}\right)^M (1+hM)^M.$$

If we put $A := Mh$, then for ϵ small enough we have, with $K = C\epsilon^{-1}$ large,

$$\begin{aligned}|u(x)| &\leq (KAh^{-1})^n \left(\frac{1}{KA}\right)^{Ah^{-1}} (1+A)^{Ah^{-1}} \\ &= (KAh^{-1})^n (1+1/A)^{Ah^{-1}} \exp(-Ah^{-1}\log K).\end{aligned}$$

We can assume that A is large, as otherwise there is nothing to prove. Hence

$$|u(x)| \leq \exp(-\alpha A h^{-1}),$$

for $\alpha > 0$ and $|x - x_0| < \epsilon$. It follows that

$$\int_{\{|x-x_0|<\epsilon\}} |u(x)|^2 \, dx \leq C_1 e^{-2\alpha A/h},$$

uniformly in h. But according to Theorem 7.7,

$$\int_{\{|x-x_0|<\epsilon\}} |u(x)|^2 \, dx > e^{-C_2/h}.$$

Consequently $A = Mh = (N+n)h$ is bounded, and this means that $N \leq Ch^{-1}$, as claimed. □

EXAMPLE : Optimal order of vanishing. Theorem 7.8 is optimal in the semiclassical limit, meaning as regards the dependence on h in estimate (7.20).

We can see this by considering the harmonic oscillator in dimension $n = 2$. In polar coordinates (r, θ) the harmonic oscillator for $h = 1$ takes the form

$$P_0 = r^{-2}((rD_r)^2 + D_\theta^2 + r^4).$$

The eigenspace corresponding to the eigenvalue $2k + 2$ has dimension $k + 1$, corresponding to the number of multiindicies $\alpha = (\alpha_1, \alpha_2)$, with $|\alpha| = \alpha_1 + \alpha_2 = k$. Separating variables, we look for eigenfunctions of the form

$$u = u_{km}(r)e^{im\theta}.$$

Then

$$r^{-2}((rD_r)^2 + m^2 + r^4 - (2n+1)r^2)u_{km}(r) = 0.$$

Since the number of linearly independent eigenfunctions is $k+1$, there must be solution for some integer $m > k/2$. Near $r = 0$, we have the asymptotics $u_{km} \simeq r^{\pm m}$, and the case $u_{km} \simeq r^{-m}$ is impossible since $u \in L^2$. Therefore $u \simeq r^m$ has to vanish to order m.

Rescaling to the semiclassical case, we see that for the eigenvalue $E(h) = (2k+1)h \simeq 1$ we have an eigenfunction vanishing to order $\simeq 1/h$. □

8. Quantum ergodicity

In this chapter we are given a smooth potential V on a compact Riemannian manifold (M, g) and write

$$p(x,\xi) = |\xi|_g^2 + V(x) \tag{8.1}$$

for $(x, \xi) \in T^*M$, the cotangent space of M. As explained in Appendix E, the associated quantum operator is

$$P(h) = -h^2\Delta_g + V, \tag{8.2}$$

and the Hamiltonian flow generated by p is denoted

$$\varphi_t = \exp(tH_p) \qquad (t \in \mathbb{R}).$$

We address in this chapter quantum implications of ergodicity for the classical evolution $\{\varphi_t\}_{t\in\mathbb{R}}$. The proofs will rely on various advanced material presented in Appendix E.

8.1 CLASSICAL ERGODICITY

We hereafter select $a < b$, and assume that

$$|\partial p| \geq \gamma > 0 \text{ on } \{a \leq p \leq b\}. \tag{8.3}$$

According then to the Implicit Function Theorem, for each $a \leq c \leq b$, the set

$$\Sigma_c := p^{-1}(c)$$

is a smooth, $2n-1$ dimensional hypersurface in the cotangent space T^*M. We can interpret Σ_c as an energy surface.

NOTATION. For each $c \in [a, b]$, we denote by μ *Liouville measure* on the hypersurface $\Sigma_c = p^{-1}(c)$ corresponding to p. This measure is characterized by the formula

$$\iint_{p^{-1}[a,b]} f\, dx d\xi = \int_a^b \int_{\Sigma_c} f\, d\mu\, dc$$

for all $a < b$ and each continuous function $f : T^*M \to \mathbb{R}^n$.

DEFINITION. Let $m \in \Sigma_c$ and $f : T^*M \to \mathbb{C}$. For $T > 0$ we define the *time average*

$$\langle f\rangle_T := \frac{1}{T}\int_0^T f\circ\varphi_t(m)\, dt = ⨍_0^T f\circ\varphi_t(m)\, dt, \tag{8.4}$$

the slash through the second integral denoting an average. Note carefully that $\langle f\rangle_T = \langle f\rangle_T(m)$ depends upon the starting point m.

DEFINITION. We say the flow φ_t is *ergodic* on $p^{-1}[a,b]$ if for each $c \in [a,b]$,

$$\begin{cases} \text{if } E \subseteq \Sigma_c \text{ is flow invariant, then} \\ \text{either } \mu(E)=0 \text{ or else } \mu(E)=\mu(\Sigma_c). \end{cases} \tag{8.5}$$

In other words, we are requiring that each flow invariant subset of the energy level Σ_c have either zero measure or full measure.

THEOREM 8.1 (Mean Ergodic Theorem). *Suppose the flow is ergodic on* $\Sigma_c := p^{-1}(c)$. *Then for each* $f \in L^2(\Sigma_c,\mu)$ *we have*

$$\lim_{T\to\infty} \int_{\Sigma_c} \left(\langle f\rangle_T - \int_{\Sigma_c} f\,d\mu \right)^2 d\mu = 0. \tag{8.6}$$

REMARK. According to Birkhoff's Ergodic Theorem,

$$\langle f\rangle_T \to \int_{\Sigma_c} f\,d\mu \quad \text{ as } T\to\infty,$$

for μ–a.e. point m belonging to Σ_c, But we will only need the weaker statement of Theorem 8.1. □

Proof. 1. Define

$$A := \{ f \in L^2(\Sigma_c,\mu) \mid \ \varphi_t^* f = f \text{ for all times } t \},$$
$$B_0 := \{ H_p g \mid g \in C^\infty(\Sigma_c) \}, \quad B := \bar{B}_0.$$

We claim that

$$h \in B_0^\perp \ \text{ if and only if } \ h \in A. \tag{8.7}$$

To see this, first let $h \in A$ and $f = H_p g \in B_0$. Then

$$\begin{aligned} \int_{\Sigma_c} h\bar{f}\,d\mu &= \int_{\Sigma_c} h\overline{H_p g}\,d\mu = \frac{d}{dt}\int_{\Sigma_c} h\overline{\varphi_t^* g}\,d\mu|_{t=0} \\ &= \frac{d}{dt}\int_{\Sigma_c} \varphi_{-t}^* h\overline{g}\,d\mu|_{t=0} = \frac{d}{dt}\int_{\Sigma_c} h\overline{g}\,d\mu|_{t=0} = 0; \end{aligned}$$

and consequently $h \in B_0^\perp$.

Conversely, suppose $h \in B_0^\perp$. Then for any $g \in C^\infty$, we have

$$0 = \int_{\Sigma_c} h\,\overline{H_p \varphi_{-t}^* g}\,d\mu = \frac{d}{dt}\int_{\Sigma_c} h\,\overline{\varphi_{-t}^* g}\,d\mu = \frac{d}{dt}\int_{\Sigma_c} \varphi_t^* h\,\overline{g}\,d\mu.$$

Therefore for all times t and all functions g,

$$\int_{\Sigma_c} \varphi_t^* h \, \overline{g} \, d\mu = \int_{\Sigma_c} h \, \overline{g} \, d\mu.$$

Hence $\varphi_t^* h \equiv h$, and so $h \in A$.

2. It follows from (8.7) that we have the orthogonal decomposition

$$L^2(\Sigma_c, \mu) = A \oplus B.$$

Thus if we write $f = f_A + f_B$, for $f_A \in A$, $f_B \in B$, then

$$\langle f_A \rangle_T \equiv f_A$$

for all T.

Now suppose $f_B = H_p g \in B_0$. We can then compute

$$\begin{aligned} \int_{\Sigma_c} |\langle f_B \rangle_T|^2 \, d\mu &= \frac{1}{T^2} \int_{\Sigma_c} \left| \int_0^T (d/dt) \varphi_t^* g dt \right|^2 d\mu \\ &= \frac{1}{T^2} \int_{\Sigma_c} |\varphi_T^* g - g|^2 \, d\mu \\ &\leq \frac{4}{T^2} \int_{\Sigma_c} |g|^2 d\mu \longrightarrow 0, \end{aligned}$$

as $T \to \infty$. Since $f_B \in B := \overline{B}_0$, we have $\langle f_B \rangle_T \to 0$ in $L^2(\Sigma_c, d\mu)$.

3. The ergodicity hypothesis is equivalent to saying that A consists of constant functions. Indeed, for any $h \in A$, the set $h^{-1}([\alpha, \infty))$ is invariant under the flow, and hence has either full measure or measure zero. Since the functions in $L^2(\Sigma_c, d\mu)$ are defined up to sets of measure zero, h is equivalent to a constant function.

Lastly, observe that the orthogonal projection $f \mapsto f_A$ is just the space average with respect to μ. This proves (8.6). $\square$

8.2 EGOROV'S THEOREM

We next estimate the difference between the classical and quantum evolutions governed by our symbol $p(x, \xi) = |\xi|^2 + V(x)$.

NOTATION. (i) We write

$$e^{-\frac{itP(h)}{h}} \qquad (t \in \mathbb{R}) \tag{8.8}$$

for the unitary group on $L^2(M)$ generated by the self-adjoint operator $P(h)$.

Note that since $P(h)u_j(h) = E_j(h)u_j(h)$, we have

$$e^{-\frac{itP(h)}{h}} u_j(h) = e^{-\frac{itE_j}{h}} u_j(h) \qquad (t \in \mathbb{R}). \tag{8.9}$$

(ii) If A is a symbol in $\Psi^{-\infty}$, we also write

$$A_t := e^{\frac{itP(h)}{h}} A e^{-\frac{itP(h)}{h}} \qquad (t \in \mathbb{R}). \tag{8.10}$$

THEOREM 8.2 (Weak form of Egorov's Theorem). *Fix a time $T > 0$ and define for $0 \leq t \leq T$*

$$\tilde{A}_t := \mathrm{Op}(a_t), \tag{8.11}$$

where

$$a_t(x, \xi) := a(\varphi_t(x, \xi)). \tag{8.12}$$

Then

$$\|A_t - \tilde{A}_t\|_{L^2 \to L^2} = O(h) \quad \text{uniformly for } 0 \leq t \leq T. \tag{8.13}$$

Proof. We have

$$\frac{d}{dt} a_t = \{p, a_t\}.$$

Recall from Appendix E that σ denotes the symbol of an operator. Then, since $\sigma\left(\frac{i}{h}[P(h), B]\right) = \{p, \sigma(B)\}$, it follows that

$$\frac{d}{dt}\tilde{A}_t = \frac{i}{h}[P(h), \tilde{A}_t] + E_t, \tag{8.14}$$

with an error term $\|E_t\|_{L^2 \to L^2} = O(h)$. Hence

$$\begin{aligned} \frac{d}{dt}&\left(e^{-\frac{itP(h)}{h}} \tilde{A}_t e^{\frac{itP(h)}{h}}\right) \\ &= e^{-\frac{itP(h)}{h}} \left(\frac{d}{dt}\tilde{A}_t - \frac{i}{h}[P(h), \tilde{A}_t]\right) e^{\frac{itP(h)}{h}} \\ &= e^{-\frac{itP(h)}{h}} \left(\frac{i}{h}[P(h), \tilde{A}_t] + E_t - \frac{i}{h}[P(h), \tilde{A}_t]\right) e^{\frac{itP(h)}{h}} \\ &= e^{-\frac{itP(h)}{h}} E_t e^{\frac{itP(h)}{h}} = O(h). \end{aligned}$$

Integrating, we deduce

$$\|e^{-\frac{itP(h)}{h}} \tilde{A}_t e^{\frac{itP(h)}{h}} - A\|_{L^2 \to L^2} = O(h);$$

and so

$$\|\tilde{A}_t - A_t\|_{L^2 \to L^2} = \|\tilde{A}_t - e^{\frac{itP(h)}{h}} A e^{-\frac{itP(h)}{h}}\|_{L^2 \to L^2} = O(h),$$

uniformly for $0 \leq t \leq T$. ☐

8.3 WEYL'S THEOREM GENERALIZED

NOTATION. We hereafter consider the eigenvalue problems

$$P(h)u_j(h) = E_j(h)u_j(h) \qquad (j = 1, \dots).$$

To simplify notation a bit, we write $u_j = u_j(h)$ and $E_j = E_j(h)$. We assume as well the normalization

$$\|u_j\|_{L^2(M)} = 1. \tag{8.15}$$

The following result generalizes Theorem 6.8, showing that we can localize the asymptotics using a quantum observable:

THEOREM 8.3 (Weyl's Theorem generalized). *Let $B \in \Psi(M)$. Then*

$$(2\pi h)^n \sum_{a \leq E_j \leq b} \langle Bu_j, u_j \rangle \to \iint_{\{a \leq p \leq b\}} \sigma(B)\, dxd\xi. \tag{8.16}$$

REMARK. If $B = I$, whence $\sigma(B) \equiv 1$, (8.16) reads

$$(2\pi h)^n \#\{a \leq E_j \leq b\} \to \operatorname{Vol}(\{a \leq p \leq b\}).$$

This is the usual form of Weyl's Law, Theorem E.7. □

Proof. 1. We first assume that $B \in \Psi^{-\infty}$; so that the operator $B : L^2(M) \to L^2(M)$ is of trace class. According to Lidskii's Theorem C.9 from Appendix B, we have

$$\operatorname{tr}(B) = \frac{1}{(2\pi h)^n} \left(\int_M \int_{\mathbb{R}^n} \sigma(B)\, dxd\xi + O(h) \right). \tag{8.17}$$

2. Fix a small munber $\epsilon > 0$ and write $\Omega_\epsilon := p^{-1}(a - \epsilon, a + \epsilon) \cup p^{-1}(b - \epsilon, b + \epsilon)$. Select $\psi_\epsilon \in C_c^\infty$, $\varphi_\epsilon \in C^\infty$ so that

$$\begin{cases} \mathrm{WF}_h(\psi_\epsilon) \subset \{a < p < b\} \\ \mathrm{WF}_h(\varphi_\epsilon) \subset \{p < a\} \cup \{p > b\} \\ \mathrm{WF}_h(I - \varphi_\epsilon + \psi_\epsilon) \subset \Omega_\epsilon. \end{cases}$$

Define

$$\Pi := \text{projection onto the span of } \{u_j \mid a \leq E_j \leq b\}.$$

We claim that

$$\begin{cases} \psi_\epsilon \Pi = \psi_\epsilon + O(h^\infty) \\ \varphi_\epsilon \Pi = O(h^\infty). \end{cases} \tag{8.18}$$

The second assertion follows by an adaptation of the proof of Theorem E.7.

To establish the first part, we need to show that $\psi_\epsilon(I-\Pi) = O(h^\infty)$. We can find f satisfying the assumptions of Theorem 7.6 and such that $\psi_\epsilon(x,\xi)/f(p(x,\xi))$ is smooth. Using a symbolic construction we can find $T_\epsilon \in \Psi^{-\infty}$ with $\mathrm{WF}_h(T_\epsilon) = \mathrm{WF}_h(\psi_\epsilon)$, for which

$$\psi_\epsilon(I-\Pi) = T_\epsilon f(P)(I-\Pi) + O(h).$$

The first term on the right hand side can be rewritten as

$$\sum_{E_j(h)<a, E_j(h)>b} f(E_j(h)) T_\epsilon u_j \otimes \bar{u}_j.$$

The rough estimate (E.31) and the rapid decay of f show that for all M we have the bound

$$f(E_j(h)) \leq C_M (hj)^{-M}.$$

The proof of Theorem 6.4 shows also that $T_\epsilon u_j = O(h^\infty)$, uniformly in j. Hence

$$\|T_\epsilon f(P)(I-\Pi)\|_{L^2\to L^2} = O(h^\infty).$$

3. We now write

$$\begin{aligned}\sum_{a\leq E_j\leq b} \langle Bu_j, u_j\rangle &= \mathrm{tr}(\Pi B \Pi) \\ &= \mathrm{tr}(\Pi B(\psi_\epsilon + \varphi_\epsilon + (1-\varphi_\epsilon-\psi_\epsilon)\Pi).\end{aligned}$$

Using (8.18) we see that

$$(2\pi h)^n \mathrm{tr}(\Pi B \varphi_\epsilon \Pi) = O(h).$$

The Weyl Law given in Theorem E.7 implies

$$(2\pi h)^n \mathrm{tr}(\Pi B(1-\varphi_\epsilon-\psi_\epsilon)\Pi) = O(\epsilon),$$

since $1-\varphi_\epsilon-\psi_\epsilon \neq 0$ only on Ω_ϵ. Furthermore,

$$\begin{aligned}&(2\pi h)^n \mathrm{tr}(\Pi B \psi_\epsilon \Pi) \\ &\qquad = (2\pi h)^n \mathrm{tr}(\Pi B \psi_\epsilon) + O(h^\infty) \\ &\qquad = (2\pi h)^n \mathrm{tr}(((\psi_\epsilon + \varphi_\epsilon \\ &\qquad\qquad + (1-\varphi_\epsilon-\psi_\epsilon)\Pi B \psi_\epsilon) + O(h^\infty) \\ &\qquad = (2\pi h)^n \mathrm{tr}(\psi_\epsilon B \psi_\epsilon) + O(h^\infty) + O(\epsilon).\end{aligned}$$

Combining these calculations gives

$$\begin{aligned}(2\pi h)^n \sum_{a\leq E_j\leq b} \langle Bu_j, u_j\rangle &= (2\pi h)^n \mathrm{tr}(\psi_\epsilon B \psi_\epsilon) + O(h) + O(\epsilon) \\ &= \iint \sigma(\psi_\epsilon)^2 \sigma(B)\, dx d\xi + O(h) + O(\epsilon) \\ &\to \iint_{\{a\leq p\leq b\}} \sigma(B)\, dx d\xi\end{aligned}$$

as $h \to 0, \epsilon \to 0$.

4. Finally, to pass from $B \in \Psi^{-\infty}$ to an arbitrary $B \in \Psi$, we decompose the latter as

$$B = B_0 + B_1,$$

with $B_0 \in \Psi^{-\infty}$ and

$$\begin{aligned}&\mathrm{WF}_h(B_0) \subset \{a-2 < p < b+2\}, \\ &\mathrm{WF}_h(B_1) \cap \{a-1 < p < b+1\} = \emptyset.\end{aligned}$$

We have $B_1 u_j = O(h^\infty)$ for $a \leq E_j(h) \leq b$; and hence only the B_0 part contributes to the limit. □

8.4 A QUANTUM ERGODIC THEOREM

Assume now that $A \in \Psi(M)$ has the symbol $\sigma(A)$ with the property that

$$\alpha := \fint_{\Sigma_c} \sigma(A)\, d\mu \;\text{ is the same for all } c \in [a,b], \tag{8.19}$$

where the slash through the integral denotes the average. In other words, we are requiring that the averages of the symbol of A over each level surface $p^{-1}(c)$ are equal.

THEOREM 8.4 (Quantum ergodicity). *Assume the ergodic condition* (8.5) *and that* $A \in \Psi(M)$ *satisfies the condition* (8.19).

(i) *Then*

$$(2\pi h)^n \sum_{a\leq E_j\leq b} \left| \langle Au_j, u_j\rangle - \fint_{\{a\leq p\leq b\}} \sigma(A)\, dx d\xi \right|^2 \longrightarrow 0. \tag{8.20}$$

(ii) *In addition, there exists a family of subsets* $\Lambda(h) \subseteq \{a \leq E_j \leq b\}$ *such that*

$$\lim_{h\to 0} \frac{\#\Lambda(h)}{\#\{a \leq E_j \leq b\}} = 1; \tag{8.21}$$

and for each $A \in \Psi(M)$ satisfying (8.19)*, we have*

$$\langle Au_j, u_j\rangle \to ⨍_{\{a\leq p\leq b\}} \sigma(A)\,dxd\xi \quad \text{as } h \to 0 \tag{8.22}$$

for $E_j \in \Lambda(h)$.

Proof. 1. We first show that assertion (i) implies (ii). For this, let

$$B := A - \alpha I, \tag{8.23}$$

α defined by (8.19). Then $\int_{\{a\leq p\leq b\}} \sigma(B)\,dxd\xi = 0$. According to (8.20),

$$(2\pi h)^n \sum_{a\leq E_j\leq b} \langle Bu_j, u_j\rangle^2 =: \epsilon(h) \to 0.$$

Define

$$\Gamma(h) := \{a \leq E_j \leq b \mid \langle Bu_j, u_j\rangle^2 \geq \epsilon^{1/2}(h)\};$$

so that

$$(2\pi h)^n \#\Gamma(h) \leq \epsilon(h)^{1/2}.$$

Next, write

$$\Lambda(h) := \{a \leq E_j \leq b\} - \Gamma(h).$$

Then if $E_j \in \Lambda(h)$,

$$|\langle Bu_j, u_j\rangle| \leq \epsilon^{1/4}(h);$$

and so

$$|\langle Au_j, u_j\rangle - \alpha| \leq \epsilon^{1/4}(h).$$

Also,

$$\frac{\#\Lambda(h)}{\#\{a \leq E_j \leq b\}} = 1 - \frac{\#\Gamma(h)}{\#\{a \leq E_j < b\}}.$$

But according to Weyl's law,

$$\frac{\#\Gamma(h)}{\#\{a \leq E_j \leq b\}} = \frac{(2\pi h)^n \#\Gamma(h)}{\mathrm{Vol}(\{a \leq p \leq b\}) + o(1)} \leq C\epsilon(h)^{1/2} \to 0.$$

This proves (ii).

2. Next we establish assertion (i). Let B be again given by (8.23); so that in view of our hypothesis (8.19)

$$\int_{\Sigma_c} \sigma(B)\,d\mu = 0 \quad \text{for each } c \in [a, b]. \tag{8.24}$$

Define

$$\epsilon(h) := (2\pi h)^n \sum_{a\leq E_j\leq b} \langle Bu_j, u_j\rangle^2;$$

we must show $\epsilon(h) \to 0$.

Now

$$\langle Bu_j, u_j\rangle = \langle Be^{-\frac{itE_j}{h}}u_j, e^{-\frac{itE_j}{h}}u_j\rangle = \langle Be^{-\frac{itP(h)}{h}}u_j, e^{-\frac{itP(h)}{h}}u_j\rangle$$

according to (8.9). Consequently

$$\langle Bu_j, u_j\rangle = \langle e^{\frac{itP(h)}{h}}Be^{-\frac{itP(h)}{h}}u_j, u_j\rangle = \langle B_t u_j, u_j\rangle \tag{8.25}$$

in the notation of (8.10). This identity is valid for each time $t \in \mathbb{R}$. We can therefore average:

$$\langle Bu_j, u_j\rangle = \langle ⨍_0^T B_t\, dt\, u_j, u_j\rangle = \langle\langle B\rangle_T u_j, u_j\rangle, \tag{8.26}$$

for

$$\langle B\rangle_T := \frac{1}{T}\int_0^T B_t\, dt = ⨍_0^T B_t\, dt.$$

Now since $\|u_j\|^2 = 1$, (8.26) implies

$$\langle Bu_j, u_j\rangle^2 = \langle\langle B\rangle_T u_j, u_j\rangle^2 \le \|\langle B\rangle_T u_j\|^2 = \langle\langle B^*\rangle_T\langle B\rangle_T u_j, u_j\rangle.$$

Hence

$$\epsilon(h) \le (2\pi h)^n \sum_{a\le E_j\le b}\langle\langle B^*\rangle_T\langle B\rangle_T u_j, u_j\rangle \tag{8.27}$$

3. Theorem 8.2 tells us that

$$\langle B\rangle_T = \langle\tilde{B}\rangle_T + O_T(h), \quad \langle\tilde{B}\rangle_T := ⨍_0^T \tilde{B}_t\, dt,$$

where $\tilde{B}_t \in \Psi(M)$ and $\sigma(\tilde{B}_t) = \varphi_t^*\sigma(B)$. Hence

$$\sigma(\langle\tilde{B}\rangle_T) = ⨍_0^T \sigma(B)\circ\varphi_t\, dt + O_T(h) = \langle\sigma(B)\rangle_T + O_T(h).$$

as $h \to 0$.

Since modulo $O(h)$ errors we can replace $e^{itP(h)/h}Be^{-itP(h)/h}$ by $\tilde{B}_t$, Theorem 8.3 shows that

$$\begin{aligned}\limsup_{h\to 0}\epsilon(h) &\le \iint_{\{a\le p\le b\}}\sigma(\langle(\tilde{B}^*\rangle_T\langle\tilde{B}\rangle_T)\, dxd\xi \\ &= \iint_{\{a\le p\le b\}}|\sigma(\langle B\rangle_T))|^2\, dxd\xi,\end{aligned} \tag{8.28}$$

as the symbol map is multiplicative and the symbol of an adjoint is given by the complex conjugate.

4. We can now apply Theorem 8.1 with $f = \sigma(B)$, to conclude that

$$\int_{p^{-1}[a,b]}|\langle\sigma(B)\rangle_T|^2 dxd\xi \to 0,$$

as $T \to \infty$. Since the left hand side of (8.28) is independent of T, this calculation shows that the limit must in fact be zero. □

APPLICATION. The simplest and most striking application concerns the complete set of eigenfuctions of the Laplacian on a compact Riemannian manifold:

$$-\Delta_g u_j = \lambda_j u_j \qquad (j = 1, \dots),$$

normalized so that

$$\|u_j\|_{L^2(M)} = 1.$$

THEOREM 8.5 (Equidistribution of eigenfunctions). *There exists a sequence $j_k \to \infty$ of density one,*

$$\lim_{m\to\infty} \frac{\#\{k \mid j_k \leq m\}}{m} = 1,$$

such that for any $f \in C^\infty(M)$,

$$\int_M |u_{j_k}|^2 f \, d\mathrm{vol}_g \to \int_M f \, d\mathrm{vol}_g. \tag{8.29}$$

9. More on the symbol calculus

This chapter collects together various more advanced topics concerning the symbol calculus, discussing in particular a semiclassical version of Beals's characterization of pseudodifferential operators and invariance properties under changes of variable. Chapter 10 will provide further applications.

9.1 WAVEFRONT SETS, ESSENTIAL SUPPORT

We devote this section to a few concepts built around the semiclassical wavefront set of a collection of functions bounded in L^2:

DEFINITION. Let $u = \{u(h)\}_{0<h\leq h_0}$ be a family of functions bounded in $L^2(\mathbb{R}^n)$. We define the *semiclassical wavefront set*

$$\mathrm{WF}_h(u)$$

to the *complement* of the set of points $(x_0, \xi_0) \in \mathbb{R}^{2n}$ for which there exists a symbol $a \in S$ such that

$$a(x_0, \xi_0) \neq 0 \tag{9.1}$$

and

$$\|a^{\mathrm{w}}(x, hD)u(h)\|_{L^2} = O(h^\infty). \tag{9.2}$$

The definition of wavefront sets does not depend on the choice of coordinates. We note that this is a local property of the family $\{u(h)\}_{0<h\leq h_0}$ in phase space: see Theorem 9.2 below. The meaning of the wavefront set is elucidated by the following

THEOREM 9.1 (Localization and wavefront sets). *Suppose that*

$$(x_0, \xi_0) \notin \mathrm{WF}_h(u).$$

Then for any $b \in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$ with support sufficiently close to (x_0, ξ_0), we have

$$b^{\mathrm{w}}(x, hD)u(h) = O_{L^2}(h^\infty).$$

Proof. 1. Suppose $a \in S$, $a(x_0, \xi_0) \neq 0$. There exists $\chi \in C^\infty(\mathbb{R}^{2n})$ supported near (x_0, ξ_0) such that

$$|\chi(x,\xi)(a(x,\xi) - a(x_0,\xi_0)) + a(x_0,\xi_0)| \geq \gamma > 0$$

for $(x,\xi) \in \mathbb{R}^{2n}$. Then according to Theorem 4.19 there exists $c \in S$ such that

$$c^{\mathrm{w}}(x,hD)(\chi^{\mathrm{w}}(x,hD)a^{\mathrm{w}}(x,hD) + a(x_0,\xi_0)(I - \chi^{\mathrm{w}}(x,hD))) = I.$$

2. Now consider

$$\begin{aligned} b^{\mathrm{w}}(x,hD)u(h) =& b^{\mathrm{w}}(x,hD)c^{\mathrm{w}}(x,hD)\chi^{\mathrm{w}}(x,hD)a^{\mathrm{w}}(x,hD)u(h) \\ &+ b^{\mathrm{w}}(x,hD)a(x_0,\xi_0)(I - \chi^{\mathrm{w}}(x,hD)))u(h). \end{aligned}$$

If we choose a to be the symbol appearing in (9.2), then the first term on the right hand side is bounded by $O(h^\infty)$ in L^2. If the support of b is sufficiently close to (x_0, ξ_0) then $\mathrm{spt}(b) \cap \mathrm{spt}(1-\chi) = \emptyset$ and the second term has the same property, according to Theorem ????? □

Since compactness of the support is not preserved by changes of variables or by other operations such as composition, we introduce the more flexible notion of the essential support.

DEFINITION. Let $a = \{a(x,\xi,h)\}_{0<h\leq h_0}$ be an h-dependent family of symbols in $\mathcal{S}$. The *essential support* of a is the smallest closed set $K \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\text{(9.3)} \qquad \mathrm{spt}\chi \cap K = \emptyset \ \text{ implies } \ \chi a = O_S(h^\infty)$$

for each $\chi \in S$. We write

$$\text{ess-spt}(a)$$

for the essential support.

Recall that a symbol $b = b(x,\xi,h)$ satisfies $b = O_S(h^\infty)$ if

$$|\partial^\alpha b| \leq C_{\alpha,N} h^N$$

for all multiindices α and nonnegative integers N.

REMARK. We will see later, in the proof of Theorem 9.12, that this notion does not depend on the choice of coordinates. In particular, if γ is a diffeomorphism and $\mathrm{Op}(a_\gamma) = \mathrm{Op}((\gamma^{-1})^* \mathrm{Op}(a)\gamma^*)$, then

$$\text{(9.4)} \qquad \text{ess-spt}(a_\gamma) = \{(\gamma(x), (\partial\gamma(x)^T)^{-1}\xi) \mid (x,\xi) \in \text{ess-spt}(a)\}.$$

THEOREM 9.2 (Wavefront sets and pseudodifferential operators). *Suppose that* $a = \{a(x, \xi, h)\}_{0<h\leq h_0} \subset S(m)$ *for some order function* m*, and that* $u = \{u(h)\}_{0<h\leq h_0}$ *is bounded in* $L^2(\mathbb{R}^n)$*.*

Then

$$\mathrm{WF}_h(a^{\mathrm{w}} u) \subset \mathrm{WF}_h(u) \cap \text{ess-spt}(a) \tag{9.5}$$

for $a^{\mathrm{w}} u := \{a^{\mathrm{w}}(x, hD, h)u(h)\}_{0<h\leq h_0}$.

Proof: 1. We need to show that if $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$ or if $(x, \xi) \notin$ ess-spt(a), then $(x, \xi) \notin \mathrm{WF}_h(a^{\mathrm{w}} u)$.

2. Suppose first that $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$. Choose $b \in C_c^\infty(\mathbb{R}^{2n})$, with $b(x_0, \xi_0) \neq 0$ and $b^{\mathrm{w}}(x, hD)u(h) = O_{L^2}(h^\infty)$. The existence of such b is clear from Theorem 9.1. Now the pseudodifferential calculus shows that

$$b^{\mathrm{w}}(x, hD)a^{\mathrm{w}}(x, hD, h) = c^{\mathrm{w}}(x, hD, h) + r^{\mathrm{w}}(x, hD, h),$$

where $\operatorname{spt} c \subset \operatorname{spt} b$ and $r \in S^{-\infty}(1)$. Theorem 9.1 implies that

$$\|b^{\mathrm{w}}(x, hD)a^{\mathrm{w}}(x, hD, h)u(h)\|_{L^2} = O(h^\infty).$$

This shows that $(x_0, \xi_0) \notin \mathrm{WF}_h(a^{\mathrm{w}} u)$.

3. Now assume that $(x_0, \xi_0) \notin$ ess-spt(a) and use the same b as above. If the support of b is sufficiently close to (x_0, ξ_0), (9.3) implies that $b^{\mathrm{w}}(x, hD)a^{\mathrm{w}}(x, hD, h) = c^{\mathrm{w}}(x, hD, h)$, where $c = O_{L^2}(h^\infty)$. Consequently

$$\|b^{\mathrm{w}}(x, hD)a^{\mathrm{w}}(x, hD, h)u(h)\|_{L^2} = O(h^\infty).$$

□

REMARK. In view of (9.5) we have an alternative definition of ess-spt(a), which makes sense on manifolds:

$$\begin{cases} (x, \xi) \notin \text{ess-spt}(a) \text{ if and only if } (x, \xi) \notin \mathrm{WF}_h(a^{\mathrm{w}} u) \\ \text{ for each family } u = \{u(h)\}_{0<h\leq h_0} \text{ bounded in } L^2. \end{cases} \tag{9.6}$$

Theorem 9.2 also motivates the following

DEFINITION. Let A be an h-dependent family of operators. We define the *wavefront set* of A to be

$$\mathrm{WF}_h(A) := \bigcup \mathrm{WF}_h(u) \cap \mathrm{WF}_h(Au), \tag{9.7}$$

the union taken over all families $\{u(h)\}_{0<h\leq h_0}$ bounded in L^2.

In view of (9.6), if $A = a^{\mathrm{w}}$, then

$$\mathrm{WF}_h(A) = \text{ess-spt}(a)$$

and hence is a closed set.

9.2 APPLICATION: L^∞ BOUNDS

Here we will show how a natural frequency localization condition on approximate solutions to pseudodifferential equations implies h-dependent L^∞ bounds. As an application we will provide bounds on eigenfuction clusters for compact Riemannian manifolds.

We start with the following semiclassical version of the Sobolev inequality (Lemma 3.5):

THEOREM 9.3 (Basic L^∞ bounds). *Suppose that* $\{u(h)\}_{0<h\leq h_0}$ *is bounded in* $L^2(\mathbb{R}^n)$ *and there exists* $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ *such that*

$$\|(1-\psi(hD))u(h)\|_{L^2(\mathbb{R}^n)} = O(h^\infty)\|u(h)\|_{H^k(\mathbb{R}^n)}\,, \textit{for all } k. \tag{9.8}$$

Then

$$\|u(h)\|_{L^\infty(\mathbb{R}^n)} \leq Ch^{-n/2}\|u(h)\|_{L^2(\mathbb{R}^n)}. \tag{9.9}$$

We regard (9.8) as a *frequency localization* condition.

Proof: 1. We can assume that $\|u(h)\|_{L^2} = 1$.

We can also suppose that $u(h)$ is compactly supported. Indeed, if $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, then

$$(1-\psi(hD))\varphi = \varphi(1-\psi(hD)) + r(x,hD)$$

with $r \in S$ and ess-spt(r) compact. We now choose $\psi_1 \in C_{\mathrm{c}}^\infty$ for which

$$(1-\psi_1)(1-\psi) = (1-\psi_1), \quad (1-\psi_1)|_{\text{ess-spt}(r)} = 0.$$

Then

$$\begin{aligned}(1-\psi_1(hD))\varphi u(h) &= (1-\psi_1(hD))\psi(1-\psi(hD))u(h)\\ &\quad + (1-\psi_1(hD))r(x,hD)u(h)\\ &= O_{L^2}(h^\infty).\end{aligned}$$

2. The condition (9.8) implies that

$$\|\langle hD\rangle^k u(h)\| \leq h^{-N_k}$$

for every k. Hence

$$\begin{aligned}\|\langle hD\rangle^k(1-\psi(hD))u(h)\|_{L^2}^2 &= \frac{1}{(2\pi)^n}\|\langle\xi\rangle^k(1-\psi(\xi))\hat{u}\|_{L^2}^2\\ &\leq \frac{1}{(2\pi)^n}\|\langle\xi\rangle^{2k}\hat{u}\|_{L^2}\|(1-\psi(\xi))^2\hat{u}\|_{L^2}\\ &= \|\langle hD\rangle^{2k}u(h)\|_{L^2}\|(1-\psi(hD))^2u(h)\|_{L^2}\\ &= O(h^\infty),\end{aligned}$$

Lemma 3.5 then implies

$$\|(1-\psi(hD))u(h)\|_{L^\infty} = O(h^\infty).$$

3. It remains to estimate $\|\psi(hD)u\|_{L^\infty}$. For this we use the semiclassical inverse Fourier transform (3.22):

$$\|\psi(hD)u\|_{L^\infty} \leq \frac{1}{(2\pi h)^n}\|\psi\|_{L^2}\|\mathcal{F}_h u\|_{L^2} = \frac{1}{(2\pi h)^{n/2}}\|\psi\|_{L^2}\|u\|_{L^2}.$$

□

We will later need the following

LEMMA 9.4 (A simple L^2 estimate). *Suppose that* $a \in S(\mathbb{R}^{n+1}\times\mathbb{R}^n)$ *is real valued, and that*

$$\begin{cases}(hD_t + a^{\mathrm{w}}(x,t,hD_x))u = f\\ u(\cdot,0) = u_0.\end{cases}$$

Then

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \frac{\sqrt{|t|}}{h}\|f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} + \|u_0\|_{L^2(\mathbb{R}^n)}. \tag{9.10}$$

Proof: Since $a^{\mathrm{w}}(t,x,hD)$ is family of bounded operators on $L^2(\mathbb{R}^n)$, the existence of solutions follows from existence theory for (linear) ordinary differential equations in the variable t.

Suppose first that $f \equiv 0$. Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)}^2 &= \mathrm{Re}\langle\partial_t u(\cdot,t), u(\cdot,t)\rangle_{L^2(\mathbb{R}^n)}\\ &= \frac{1}{h}\mathrm{Re}\langle ia^{\mathrm{w}}(x,t,hD)u(\cdot,t), u(\cdot,t)\rangle = 0.\end{aligned}$$

Thus, if we set $E(t)u_0 := u(t)$, we have

$$\|E(t)u_0\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}.$$

If $f \neq 0$, Duhamel's formula gives

$$u(\cdot,t) = E(t)u_0 + \frac{i}{h}\int_0^t E(t-s)f(\cdot,s)\,ds.$$

Hence

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \|u_0\|_{L^2(\mathbb{R}^n)} + \frac{1}{h}\int_0^t \|f(\cdot,s)\|_{L^2(\mathbb{R}^n)}\,ds.$$

The estimate (9.10) is an immediate consequence. □

THEOREM 9.5 (L^∞ bounds for approximate solutions). *Let $m = m(x,\xi)$ an order function. Suppose that $p \in S(m)$ is real valued, and for some compact set $K \subset \mathbb{R}^n \times \mathbb{R}^n$,*

$$\partial_\xi p(x,\xi) \neq 0 \quad \text{if} \quad (x,\xi) \in K, p(x,\xi) = 0. \tag{9.11}$$

If $u = \{u(h)\}_{0<h\leq h_0}$ is bounded in $L^2(\mathbb{R}^n)$ and satisfies the frequency localization condition (9.8)*, if*

$$WF_h(u) \subset K,$$

and if

$$\|p^{\mathrm{w}}(x,hD)u(h)\|_{L^2(\mathbb{R}^n)} = O(h)\|u(h)\|_{L^2(\mathbb{R}^n)}, \tag{9.12}$$

then

$$\|u(h)\|_{L^\infty(\mathbb{R}^n)} \leq Ch^{-(n-1)/2}\|u(h)\|_{L^2(\mathbb{R}^n)}. \tag{9.13}$$

The point is that if $u(h)$ is an approximate solution in the sense of satisfying the estimate (9.12), we can improve the earlier L^∞ estimate (9.9) by a factor of $h^{\frac{1}{2}}$.

Proof: 1. First we observe that, as in Theorem 9.3, we can assume that the functions $u(h)$ are uniformly compactly supported. We also note that the hypothesis on $u(h)$ is local in phase space: if $\chi \in C_c^\infty(T^*\mathbb{R}^n)$ then, normalizing to $\|u(h)\|_{L^2} = 1$, we have

$$\begin{aligned} P(h)\chi^{\mathrm{w}}(x,hD)u(h) &= \chi^{\mathrm{w}}(x,hD)P(h)u(h) + [P(h),\chi^{\mathrm{w}}(x,hD)]u(h) \\ &= O_{L^2}(h). \end{aligned}$$

According to Theorem 9.2, $\mathrm{WF}_h(\chi^{\mathrm{w}}u) \subset K$.

2. Hence it is enough to prove the theorem for $u(h)$ replaced by $\chi^{\mathrm{w}}u(h)$, where χ is supported near a given point in K as a partition of unity argument will then give the bound on $u(h)$.

Suppose that $p \neq 0$ on the support of χ. Then we can use Theorem 4.19 as in part 1 of the proof of Lemma 9.1 to see that $P(h)\chi^{\mathrm{w}}u(h) = O_{L^2}(h)$ implies that $\chi^{\mathrm{w}}u(h) = O_{L^2}(h)$. Theorem 9.3 then shows that

$$\|\chi^{\mathrm{w}}u(h)\|_{L^\infty} \leq Chh^{-n/2} \leq Ch^{-(n-1)/2}.$$

3. Now suppose that p vanishes in the support of χ. By applying a linear change of variables we can assume that $p_{\xi_1} \neq 0$ there. The Implicit Function Theorem shows that in a neighborhood of $\operatorname{spt}\chi$, we have

$$p(x,\xi) = e(x,\xi)(\xi_1 - a(x,\xi')), \tag{9.14}$$

where $\xi = (\xi_1, \xi')$ and $e(x,\xi) > 0$.

We extend e arbitrarily to $e \in S$ so that $e \geq \gamma > 0$ and extend $a(x,\xi')$ to a real valued $a(x,\xi') \in S$. The pseudodifferential calculus shows that

$$\begin{aligned} e^{\mathrm{w}}(x,hD)(hD_{x_1} - a(x,hD_{x'}))(\chi^{\mathrm{w}}u(h)) &= P(h)(\chi^{\mathrm{w}}u(h)) + O_{L^2}(h) \\ &= O_{L^2}(h). \end{aligned}$$

Since e^{w} is elliptic,

$$(hD_{x_1} - a(x,hD_{x'}))(\chi^{\mathrm{w}}u(h)) = O_{L^2}(h). \tag{9.15}$$

4. The proof will be completed once we show

$$\|(\chi^{\mathrm{w}}u)(x_1,\cdot)\|_{L^2(\mathbb{R}^{n-1})} = O(1), \tag{9.16}$$

and for that we use (9.15) and Lemma 9.4. We now apply Theorem 9.3 in x' variables only, that is with $n-1$ replacing n and t replacing x_1. That is allowed since we clearly have

$$\|(1-\psi(hD'))\chi^{\mathrm{w}}u(h)(x_1,\cdot)\|_{L^2(\mathbb{R}^{n-1})} = O(h^\infty),$$

uniformly in x_1. □

REMARK. The bound given in Theorem 9.5 is already optimal in the simplest case in which the assumptions are satisfied: $p(x,\xi) = \xi_1$. Indeed, write $x = (x_1, x')$ and let $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R})$, and $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R}^{n-1})$. Then

$$u(h) := h^{-(n-1)/2}\phi(x_1)\chi(x'/h)$$

satisfies $\|u(h)\|_{L^2} = O(1)$,

$$P(h)u(h) = hD_{x_1}u(h) = O_{L^2}(h);$$

and for any non-trivial choices of ϕ and χ, $\|u(h)\|_{L^\infty} \simeq h^{-(n-1)/2}$.

The condition (9.11) is in general necessary as shown by another simple example. Let $p(x,\xi) = x_1$, and

$$u(h) = h^{-n/2}\phi(x_1/h)\chi(x'/h).$$

Then $\|u(h)\|_{L^2} = O(1)$,

$$P(h)u(h) = hh^{-n/2}(t\phi(t))|_{t=x_1/h}\chi(x'/h) = O_{L^2}(h),$$

and $\|u(h)\|_{L^\infty} \simeq h^{-n/2}$. This is the general bound of Lemma 9.3. □

As an application we present an L^∞ bound on "spectral clusters", that is, linear combinations of eigenfunctions for the Laplacian on a compact manifold. The statement requires the material presented in Appendix E.3.

THEOREM 9.6 (Bounds on eigenfuction clusters). *Suppose that M is an n-dimensional compact Riemannian manifold and let Δ_g be its Laplace-Beltrami operator. Assume that*

$$0 = \lambda_0 < \lambda_1 \leq \cdots \lambda_j \to \infty$$

are the eigenvalues of $-\Delta_g$ and that

$$-\Delta_g \varphi_j = \lambda_j \varphi_j \qquad (j = 1, \dots)$$

are a corresponding orthonormal basis of eigenfunctions.

There exists a constant C such that for any choices of constants $c_j \in \mathbb{C}$, we have the inequality

$$\left\| \sum\nolimits_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j \varphi_j \right\|_{L^\infty} \leq C \mu^{(n-1)/2} \left\| \sum\nolimits_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j \varphi_j \right\|_{L^2}. \tag{9.17}$$

In particular,

$$\|\varphi_j\|_{L^\infty} \leq C \lambda_j^{(n-1)/4} \|\varphi_j\|_{L^2}. \tag{9.18}$$

Proof: Put $h = 1/\mu$, $P(h) := -h^2 \Delta_g - 1$, and

$$u(h) := \sum\nolimits_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j \varphi_j.$$

Then the assumption (9.11) holds everywhere. Also

$$\begin{aligned} \|P(h)u(h)\|_{L^2} &= \left\| \sum\nolimits_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j (h^2 \lambda_j - 1) \varphi_j \right\|_{L^2} \\ &= \left(\sum\nolimits_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} |c_j|^2 (h^2 \lambda_j - 1)^2 \|\varphi_j\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq 2h \|u(h)\|_{L^2}. \end{aligned}$$

Thus the assumption (9.12) holds. On a compact manifold the frequency localization condition (9.8) follows from

$$\|(1 - \varphi(-h^2 \Delta_g)) u(h)\| = O(h^\infty) \|u(h)\|_L^2$$

for $\varphi \in C_c^\infty(\mathbb{R})$ satisfying $\varphi(t) \equiv 1$ for $|t| \leq 2$. But this is a consequence of the Spectral Theorem. □

The estimate (9.17) is essentially optimal. On the other hand the optimality of (9.18) is very rare: see [S-Z] for a recent discussion.

9.3 BEALS'S THEOREM

We next present a semiclassical version of Beals's Theorem, a characterization of pseudodifferential operators in terms of h-dependent bounds on commutators. Beals's Theorem answers a fundamental question: when can a given linear operator be represented using our symbol calculus?

We start with

THEOREM 9.7 (Estimating a symbol by operator norms). *Take $h = 1$. There exist constants $C, M > 0$ such that*

$$\|b\|_{L^\infty} \leq C \sum_{|\alpha| \leq M} \|(\partial^\alpha b)^{\mathrm{w}}(x, D)\|_{L^2 \to L^2}, \tag{9.19}$$

for all $b \in \mathcal{S}'(\mathbb{R}^{2n})$.

Proof. 1. We will first consider the classical quantization

$$b(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} b(x, \xi) e^{i\langle x, \xi\rangle} \hat{u}(\xi)\, d\xi,$$

where by the integration we mean the Fourier transform in $\mathcal{S}'$.

Then if $\varphi = \varphi(x), \psi = \psi(\xi)$ are functions in the Schwartz space $\mathcal{S}$, we can regard $\mathcal{F}(b\bar{\varphi}\hat{\psi}e^{i\langle x,\xi\rangle})$ as a function of the dual variables $(x^*, \xi^*) \in \mathbb{R}^{2n}$. We have

$$\begin{aligned} &\frac{1}{(2\pi)^n} |\mathcal{F}(b\bar{\varphi}\hat{\psi}e^{i\langle x,\xi\rangle})(0,0)| \\ &\qquad = \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} b(x,\xi)\bar{\varphi}(x)\hat{\psi}(\xi) e^{i\langle x,\xi\rangle} dx d\xi \right| \\ &\qquad = |\langle b(x, D)\psi, \varphi\rangle| \leq \|b\|_{L^2 \to L^2} \|\varphi\|_{L^2} \|\psi\|_{L^2}. \end{aligned}$$

Fix $(x^*, \xi^*) \in \mathbb{R}^{2n}$ and rewrite this inequality with $\varphi(x)e^{i\langle x^*, x\rangle}$ replacing $\varphi(x)$ and $\psi(\xi)e^{-i\langle \xi^*, \xi\rangle}$ replacing $\psi(\xi)$, a procedure which does not change the L^2 norms. It follows that

$$\frac{1}{(2\pi)^n} |\mathcal{F}(b\bar{\varphi}\hat{\psi}e^{i\langle x,\xi\rangle})(x^*, \xi^*)| \leq \|b\|_{L^2 \to L^2} \|\varphi\|_{L^2} \|\psi\|_{L^2}. \tag{9.20}$$

2. Now take $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^{2n})$. Select $\varphi, \psi \in \mathcal{S}$ so that

$$\begin{cases} \varphi(x) = 1 & \text{if } (x, \xi) \in \operatorname{spt}\chi \\ \hat{\psi}(\xi) = 1 & \text{if } (x, \xi) \in \operatorname{spt}\chi. \end{cases}$$

Write

$$\varphi = \chi e^{-i\langle x,\xi\rangle}. \tag{9.21}$$

Then

$$\chi(x,\xi) = \varphi(x,\xi)\varphi(x)\hat{\psi}(\xi)e^{i\langle x,\xi\rangle}.$$

According to (3.18),

$$\|\mathcal{F}\chi\|_{L^1} \leq C \sum_{|\alpha|\leq 2n+1} \|\partial^\alpha \chi\|_{L^1};$$

and so (9.21) implies

$$\|\mathcal{F}\varphi\|_{L^1} \leq C \sum_{|\alpha|\leq 2n+1} \|\partial^\alpha \chi\|_{L^1}. \tag{9.22}$$

Thus (9.20) shows that for any $(x^*,\xi^*) \in \mathbb{R}^{2n}$ we have

$$\begin{aligned} |\mathcal{F}(\chi b)(x^*,\xi^*)| &\leq \|\mathcal{F}(\varphi b \bar{\varphi}\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty} \\ &= \frac{1}{(2\pi)^n}\|\mathcal{F}(\varphi) * \mathcal{F}(b\bar{\varphi}\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty} \\ &\leq \frac{1}{(2\pi)^n}\|\mathcal{F}(b\bar{\varphi}\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty}\|\mathcal{F}(\varphi)\|_{L^1} \\ &\leq C\|b\|_{L^2\to L^2}, \end{aligned}$$

the constant C depending on φ, ψ and χ, but not (x^*,ξ^*). Hence

$$\|\mathcal{F}(\chi b)\|_{L^\infty} \leq C\|b\|_{L^2\to L^2} \tag{9.23}$$

with the same constant for any translate of χ.

3. Next, we assert that

$$|\mathcal{F}(\chi b)(x^*,\xi^*)| \leq C\langle (x^*,\xi^*)\rangle^{-2n-1} \sum_{|\alpha|\leq 2n+1} \|(\partial^\alpha b)(x,hD)\|_{L^2\to L^2}. \tag{9.24}$$

To see this, compute

$$\begin{aligned} (x^*)^\alpha(\xi^*)^\beta\mathcal{F}(\chi b)(x^*,\xi^*) &= \int_{\mathbb{R}^{2n}} (x^*)^\alpha(\xi^*)^\beta e^{-i(\langle x^*,x\rangle+\langle \xi^*,\xi\rangle)}\chi b(x,\xi)\,dxd\xi \\ &= \int_{\mathbb{R}^{2n}} e^{-i(\langle x^*,x\rangle+\langle \xi^*,\xi\rangle)} D_x^\alpha D_\xi^\beta(\chi b)\,dxd\xi. \end{aligned}$$

Summing absolute values of the left hand side over all (α,β) with $|\alpha|+|\beta|\leq 2n+1$ and using the estimate (9.23), we obtain the bound

$$\begin{aligned} \|\langle (x^*,\xi^*)\rangle^{2n+1}\mathcal{F}(\chi b)\|_{L^\infty} &\leq C_1 \sum_{|\alpha|+|\beta|\leq 2n+1} \|\mathcal{F}(D_x^\alpha D_\xi^\beta(\chi b))\|_{L^\infty} \\ &\leq C_2 \sum_{|\alpha|\leq 2n+1} \|(\partial^\alpha b)(x,hD)\|_{L^2\to L^2}. \end{aligned}$$

This gives (9.24).

Consequently,

$$\|\chi b\|_{L^\infty} \leq C\|\mathcal{F}(\chi b)\|_{L^1} \leq C \sum_{|\alpha|\leq 2n+1} \|(\partial^\alpha b)(x, hD)\|_{L^2\to L^2}.$$

4. This implies the desired inequality (9.19), except that we used the classical ($t = 1$), and not the Weyl ($t = 1/2$) quantization. To remedy this, recall from Theorem **??** that if

$$b = e^{\frac{i}{2}\langle D_x, D_\xi\rangle}\tilde{b},$$

then

$$\begin{cases} b^{\mathrm{w}}(x, D) = \tilde{b}(x, D) \\ (\partial^\alpha b)^{\mathrm{w}}(x, D) = (\partial^\alpha \tilde{b})(x, D). \end{cases}$$

The continuity statement in Theorem **??** shows that

$$\|b\|_{L^\infty} \leq C \sum_{|\alpha|\leq K} \|\partial^\alpha \tilde{b}\|_{L^\infty},$$

and reduces the argument to the classical quantization.

□

NOTATION. We henceforth write

$$\mathrm{ad}_B A := [B, A];$$

"ad" is called the *adjoint action*.

Recall also that we identify a pair $(x^*, \xi^*) \in \mathbb{R}^{2n}$ with the linear operator $l(x, \xi) = \langle x^*, x\rangle + \langle \xi^*, \xi\rangle$.

THEOREM 9.8 (Semiclassical form of Beals's Theorem). *Let $A : \mathcal{S} \to \mathcal{S}'$ be a continuous linear operator. Then*

(i) $A = a^{\mathrm{w}}$ *for a symbol* $a \in S$

if and only if

(ii) *for all* $N = 0, 1, 2, \ldots$ *and all linear functions* $l_1, \ldots, l_N$*, we have*

$$\|\mathrm{ad}_{l_1(x,hD)} \circ \cdots \circ \mathrm{ad}_{l_N(x,hD)} A\|_{L^2\to L^2} = O(h^N). \tag{9.25}$$

Proof. 1. That (i) implies (ii) follows from the symbol calculus developed in Chapter 4. Indeed, $\|A\|_{L^2\to L^2} = O(1)$ and each commutator with an operator $l_j(x, hD)$ yields a term of order h.

2. That (ii) implies (i) is harder. First of all, the Schwartz Kernel Theorem (Theorem C.1) asserts that we can write

$$Au(x) = \int_{\mathbb{R}^n} K_A(x,y)u(y)\,dy \tag{9.26}$$

for $K_A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. We call K_A the *kernel* of A.

3. We now claim that if we define $a \in \mathcal{S}'(\mathbb{R}^{2n})$ by

$$a(x,\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle w,\xi\rangle} K_A\left(x + \tfrac{w}{2}, x - \tfrac{w}{2}\right)\,dw, \tag{9.27}$$

then

$$K_A(x,y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle}\,d\xi, \tag{9.28}$$

where the integrals are a shorthand for the Fourier transforms defined on $\mathcal{S}'$.

To confirm this, we calculate that

$$\begin{aligned} &\frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle} d\xi \\ &\qquad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y-w,\xi\rangle} K_A\left(\tfrac{x+y}{2} + \tfrac{w}{2}, \tfrac{x+y}{2} - \tfrac{w}{2}\right) dw d\xi \\ &\qquad = K_A(x,y), \end{aligned}$$

since

$$\frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y-w,\xi\rangle} d\xi = \delta_{x-y}(w) \quad \text{in } \mathcal{S}'.$$

In view of (9.26) and (9.28), we see that $A = a^{\mathrm{w}}$, for a defined by (9.27).

4. Now we must show that a belongs to the symbol class S; that is,

$$\sup_{\mathbb{R}^{2n}} |\partial^\alpha a| \le C_\alpha \tag{9.29}$$

for each multiindex α.

To do so we will make use of our hypothesis (9.25) with $l = x_j, \xi_j$, that is, with $l(x, hD) = x_j, hD_j$. We compute

$$
\begin{aligned}
&\mathrm{Op}(hD_{\xi_j} a)u(x) \\
&\quad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} hD_{\xi_j} \left(a\left(\tfrac{x+y}{2}, \xi\right)\right) e^{i\frac{\langle x-y,\xi\rangle}{h}} u(y)\, d\xi dy \\
&\quad = -\frac{1}{(2\pi h)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2}, \xi\right) hD_{\xi_j} \left(e^{\frac{i\langle x-y,\xi\rangle}{h}}\right) u(y)\, d\xi dy \\
&\quad = -\frac{1}{(2\pi h)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2}, \xi\right) e^{\frac{i\langle x-y,\xi\rangle}{h}} (x_j - y_j) u(y)\, d\xi dy \\
&\quad = -[x_j, A]u = -\mathrm{ad}_{x_j} A u(x).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\mathrm{Op}(hD_{x_j} a)u(x) \\
&\quad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a_{x_j}\left(\tfrac{x+y}{2}, \xi\right) e^{\frac{i\langle x-y,\xi\rangle}{h}} u(y)\, d\xi dy \\
&\quad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(D_{x_j} + D_{y_j}) \left(a\left(\tfrac{x+y}{2}, \xi\right)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad e^{\frac{i\langle x-y,\xi\rangle}{h}} u(y)\, d\xi dy \\
&\quad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} hD_{x_j} \left(a\left(\tfrac{x+y}{2}, \xi\right)\right) e^{\frac{i\langle x-y,\xi\rangle}{h}} u(y)\, d\xi dy \\
&\quad + \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2}, \xi\right) e^{\frac{i\langle x-y,\xi\rangle}{h}} (\xi_j - D_{y_j} u(y))\, d\xi dy \\
&\quad = hD_{x_j}(Au) - A(hD_{x_j} u) \\
&\quad = [hD_{x_j}, A]u = \mathrm{ad}_{hD_{x_j}} A u(x).
\end{aligned}
$$

In summary, for $j = 1, \dots, n$,

$$
\begin{cases} \mathrm{ad}_{x_j} A = -\mathrm{Op}(hD_{\xi_j} a) \\ \mathrm{ad}_{hD_{x_j}} A = \mathrm{Op}(hD_{x_j} a). \end{cases} \tag{9.30}
$$

5. Next we convert to the case $h = 1$ by rescaling. For this, define

$$
U_h u(x) := h^{n/4} u(h^{1/2} x)
$$

and check that $U_h : L^2 \to L^2$ is unitary. Then

$$
U_h a^{\mathrm{w}}(x, hD) U_h^{-1} = a^{\mathrm{w}}(h^{1/2} x, h^{1/2} D) = \mathrm{Op}(a_h)
$$

for

$$
a_h(x, \xi) := a(h^{1/2} x, h^{1/2} \xi). \tag{9.31}
$$

Our hypothesis (9.25) is invariant under conjugation by U_h, and is consequently equivalent to

$$\text{ad}_{l_1(h^{1/2}x,h^{1/2}D)} \circ \cdots \circ a_{l_N(h^{1/2}x,h^{1/2}D)}\text{Op}(a_h) = O(h^N). \tag{9.32}$$

But since l_j is linear, $l_j(h^{1/2}x, h^{1/2}D) = h^{1/2}l(x, D)$. Thus (9.32) is equivalent to

$$\text{ad}_{l_1(x,D)} \circ \cdots \circ a_{l_N(x,D)} \circ \text{Op}(a_h) = O(h^{N/2}). \tag{9.33}$$

Taking $l_k(x, \xi) = x_j$ or ξ_j, it follows from (9.33) that

$$\|\text{Op}(\partial^\beta a_h)\| \leq Ch^{\frac{|\beta|}{2}} \tag{9.34}$$

for all multiindices β.

6. Finally, we claim that

$$|\partial^\alpha a_h| \leq C_\alpha h^{|\alpha|/2} \text{ for each multiindex } \alpha. \tag{9.35}$$

But this follows from Theorem 9.7, owing to estimate (9.34):

$$\|a_h^\alpha\|_{L^\infty} \leq C \sum_{|\beta|\leq n+1} \|\text{Op}(\partial^{\alpha+\beta}a_h)\|_{L^2\to L^2} \leq C_\alpha h^{|\alpha|}.$$

Recalling (9.31), we rescale to derive the desired inequality (9.29). □

EXAMPLE: resolvents. Suppose $a \in S$ is real-valued, so that $A = a^{\text{w}}(x, hD)$ is a self-adjoint operator on $L^2(\mathbb{R}^n)$. The resolvent $(A+i)^{-1}$ is then a bounded operator on $L^2(\mathbb{R}^n)$. Can we represent $(A + i)^{-1}$ as a pseudodifferential operator?

Since

$$\text{ad}_l B = -B(\text{ad}_l A)B,$$

we see that the assumptions of Beals's Theorem are satisfied, and consequently $(A + i)^{-1} = b^{\text{w}}(x, hD)$ for some symbol $b \in S$. □

9.4 APPLICATION: EXPONENTIATION OF OPERATORS

In this section we will consider one parameter families of operators which give exponentials of self-adjoint pseudodifferential operators. As we have seen in Theorem 4.4, quantization of exponentiation commutes with quantization for linear symbols. This is of course not true for nonlinear symbols: see Section 10.2 for the subtleties involved in exponention of skew-adjoint pseudodifferential operators.

THEOREM 9.9 (Exponentials and order functions). *Let $m = m(x, \xi)$ be an order function and suppose that $g = g(x, \xi, h)$ satisfies*

$$g(x, \xi) - \log m(x, \xi) = O(1) \tag{9.36}$$

and

$$\partial^\alpha g \in S_\delta \tag{9.37}$$

for $|\alpha| = 1$ and $0 \leq \delta \leq \frac{1}{2}$.

Then the equation

$$\begin{cases} \frac{d}{dt} B(t) = g^{\mathrm{w}}(x, hD) B(t), \\ B(0) = I, \end{cases} \tag{9.38}$$

has a unique solution $B(\cdot)$. Furthermore, for each $t \geq 0$, we have

$$B(t) = b_t^{\mathrm{w}}(x, hD), \tag{9.39}$$

for a symbol

$$b_t \in S(m^t).$$

Here m^t means m raised to the power t.

Theorem 9.9 identifies $\exp(t g^{\mathrm{w}}(x, hD))$ as a quantization of an element of $S_\delta(m^t)$. Thus we are asserting that *on the level of order functions exponentiation commutes with quantization.* Here is an application. Given an operator P, it is often very useful to consider its conjugations of the form

$$P(t) := e^{-t g^{\mathrm{w}}(x, hD)} P e^{t g^{\mathrm{w}}(x, hD)}.$$

As an application of Theorem 9.9 we observe that if g is as above and P is bounded on L^2, then $P(t)$ is bounded on L^2. To see this, apply Theorems 4.13 and 4.16.

To prove Theorem 9.9 we start with

LEMMA 9.10 (Inverting exponentials). *Consider*

$$U(t) := (\exp tg)^{\mathrm{w}}(x, D)$$

as a mapping from $\mathcal{S}(\mathbb{R}^n)$ to itself. There exists $\epsilon_0 = \epsilon_0(g) > 0$ such that the operator $U(t)$ is invertible for $|t| < \epsilon_0$ and

$$U(t)^{-1} = b_t^{\mathrm{w}}(x, D)$$

for a symbol

$$b_t \in S(m^{-t}).$$

Proof: 1. We apply the composition formula given in Theorem 4.13 to obtain

$$U(-t)U(t) = I + e_t^{\mathrm{w}}(x, D)$$

where $e_t \in S$.

More explicitly, we write

$$\begin{aligned} e_t(x_1, \xi) &= \int_0^s e^{sA(D)} A(D)(e^{-tg(x_1,\xi_1)+tg(x_2,\xi_2)})|_{x_2=x_1=x,\xi_2=\xi_1=\xi}\, ds \\ &= \int_0^s ite^{sA(D)} F e^{-tg(x_1,\xi_1)+tg(x_2,\xi_2)}|_{x_2=x_1=x,\xi_2=\xi_1=\xi}\, ds/2, \end{aligned}$$

where

$$A(D) = i\sigma(D_{x_1}, D_{\xi_1}; D_{x_2}, D_{\xi_2})/2,$$
$$F = \partial_{x_1} g(x_1, \xi_1) \cdot \partial_{\xi_2} g(x_2, \xi_2) - \partial_{\xi_1} g(x_1, \xi_1) \cdot \partial_{x_2} g(x_2, \xi_2).$$

2. It follows that $e_t = t\widetilde{e}_t$, for $\widetilde{e}_t \in S$. Therefore

$$e_t^{\mathrm{w}}(x, D) = O_{L^2 \to L^2}(t);$$

this shows that $I + e_t^{\mathrm{w}}(x, D)$ is invertible for $|t|$ small enough. Then Theorem 9.8 implies

$$(I + e_t^{\mathrm{w}}(x, D))^{-1} = c_t^{\mathrm{w}}(x, D)$$

for a symbol $c_t \in S$. Hence $b_t = c_t \# \exp(-tg(x, \xi)) \in S(m^{-t})$. □

Proof of Therem 9.9: 1. We first note that we only need to prove the result in the case $h = 1$ by using the rescaling given in (4.29).

2. The hypotheses on g in (9.36) are equivalent to the statement that $\exp(tg) \in S(m^t)$ for all $t \in \mathbb{R}$. We now observe that

$$\frac{d}{dt}\left(U(-t)\exp(tg^{\mathrm{w}}(x, D))\right) = V(t)\exp(tg^{\mathrm{w}}(x, D)) \tag{9.40}$$

where$V(t) = a_t^{\mathrm{w}}(x, D)$ and $a_t \in S(m^{-t})$. In fact, we see that

$$\frac{d}{dt} U(-t) = -(g\exp(-tg))^{\mathrm{w}}(x, D) \tag{9.41}$$

and

$$U(-t)g^{\mathrm{w}}(x, D) = (\exp(tg)\#g)^{\mathrm{w}}(x, D). \tag{9.42}$$

As before, the composition formula (**??**) gives

$$\exp(-tg)\#g - g\exp(-tg) =$$
$$\int_0^1 \exp(sA(D))A(D)\exp(-tg(x^1, \xi^1)g(x^2, \xi^2)|_{x^1=x^2=x,\xi^1=\xi^2=\xi},$$
$$A(D) = i\sigma(D_{x^1}, D_{\xi^1}; D_{x^2}, D_{\xi^2})/2.$$

From the hypothesis on g we see that $A(D)\exp(tg(x^1,\xi^1))g(x^2,\xi^2)$ is a sum of terms of the form $a(x^1,\xi^1)b(x^2,\xi^2)$ where $a \in S(m^{-t})$ and $b \in S(1)$. The continuity of $\exp(A(D))$ on the spaces of symbols in Theorem **??** gives (9.40).

3. If we put

$$C(t) := -V(t)U(-t)^{-1},$$

then by Lemma 9.10, $C(t) = c_t^{\mathrm{w}}$ where $c_t \in S(1)$. Symbolic calculus shows that c_t depends smoothly on t and

$$(\partial_t + C(t))(U(-t)\exp(tg^{\mathrm{w}}(x,D))) = 0.$$

4. The proof of Theorem 9.9 is now reduced to showing

LEMMA 9.11 (Solving an operator equation). *Suppose that*

$$C(t) = c_t^{\mathrm{w}}(x,D),$$

where the symbols $c_t \in S$ depends continuously on $t \in (-\epsilon_0,\epsilon_0)$.

Assume $q \in S$. Then the solution of

$$\begin{cases} (\partial_t + C(t))Q(t) = 0, \\ \qquad\qquad Q(0) = q^{\mathrm{w}}(x,D) \end{cases} \tag{9.43}$$

is

$$Q(t) = q_t^{\mathrm{w}}(x,D),$$

where $q_t \in S$ depend continuously on $t \in (-\epsilon_0,\epsilon_0)$.

Proof: 1. The Picard existence theorem for ODE shows that $Q(t)$ is bounded on L^2. Assume now that $l_j(x,\xi)$ are linear functions on $\mathbb{R}^n \times \mathbb{R}^n$. Then

$$\begin{aligned} \frac{d}{dt}\mathrm{ad}_{l_1(x,D)} \circ \cdots \circ \mathrm{ad}_{l_N(x,D)} Q(t) + \\ \mathrm{ad}_{l_1(x,D)} \circ \cdots \circ \mathrm{ad}_{l_N(x,D)}(C(t)Q(t)) = 0, \\ \mathrm{ad}_{l_1(x,D)} \circ \cdots \circ \mathrm{ad}_{l_N(x,D)} Q(0) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \end{aligned}$$

If we show that for any choice of $l_j's$ and any N

$$\mathrm{ad}_{l_1(x,D)} \circ \cdots \circ \mathrm{ad}_{l_N(x,D)} Q(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \tag{9.44}$$

then Beals's Theorem concludes the proof.

2. We proceed by induction on N:

$$\begin{aligned} \mathrm{ad}_{l_1(x,D)} \circ \cdots \circ \mathrm{ad}_{l_N(x,D)}(C(t)Q(t)) = \\ C(t)\mathrm{ad}_{l_1(x,D)} \circ \cdots \circ \mathrm{ad}_{l_N(x,D)} Q(t) + R(t), \end{aligned}$$

where $R(t)$ is the sum of terms of the form

$$A_k(t)\mathrm{ad}_{l_1(x,D)} \circ \mathrm{ad}_{l_k(x,D)} Q(t)$$

for $k < N$, $A_k(t) = a_k(t)^{\mathrm{w}}$, and $a_k(t) \in S$ depend continuously on t. This also follows by an inductive based on the derivation property of ad_l:

$$\mathrm{ad}_l(CD) = (\mathrm{ad}_l C)D + C(\mathrm{ad}_l D).$$

Hence by the induction hypothesis $R(t)$ is bounded on L^2, and depends continuously on t. Thus

$$(\partial_t + C(t))\,\mathrm{ad}_{l_1(x,D)} \circ \cdots \circ \mathrm{ad}_{l_N(x,D)} Q(t) = R(t)$$

is bounded on L^2. Since (9.44) is valid at $t = 0$, we obtain it for all $t \in (-\epsilon_0, \epsilon_0)$. □

9.5 INVARIANCE, HALF-DENSITIES

Invariance. We begin with a general discussion concerning the invariance of various quantities under the change of variables

$$\tilde{x} = \gamma(x), \tag{9.45}$$

where $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

Functions. We note first that functions transform under (9.45) by pull-back. This means that we transform u into a function of the new variables $\tilde{x}$ by the rule

$$\tilde{u}(\tilde{x}) = \tilde{u}(\gamma(x)) := u(x), \tag{9.46}$$

for $x \in \mathbb{R}^n$. Observe however that in general the integral of u over a Borel set E is not then invariant:

$$\int_{\gamma(E)} \tilde{u}(\tilde{x})\,d\tilde{x} \neq \int_E u(x)\,dx.$$

Densities. One way to repair this defect is to change our definition (9.46) to include the Jacobian of the transformation γ. We elegantly accomplish this by turning our attention to *densities*, which we denote symbolically as

$$u(x)|dx|.$$

We therefore modify our earlier definition (9.46), now to read

$$\tilde{u}(\tilde{x}) = \tilde{u}(\gamma(x)) := u(x)|\det(\partial\gamma(x))|^{-1}. \tag{9.47}$$

Then we have the invariance assertion

$$\text{“}\,\tilde{u}(\tilde{x})|d\tilde{x}| = u(x)|dx|\text{”},$$

meaning that

$$\int_{\gamma(E)} \tilde{u}(\tilde{x})\,d\tilde{x} = \int_E u(x)\,dx$$

for all Borel sets $E \subseteq \mathbb{R}^n$.

Half-densities. Next recall our general motivation coming from quantum mechanics. The eigenfunctions u we study are then interpreted as *wave functions* and the squares of their moduli are the probability densities in the position representation: the probability of "finding our state in the set E" is given by

$$\int_E |u(x)|^2 \, dx.$$

This probability density should be invariantly defined, and so should not depend on the choice of coordinates x. As above, this means that it is *not* the function $u(x)$ which should be defined invariantly but rather the density $|u(x)|^2 dx$, or up to the phase information, the *half-density*

$$u(x)|dx|^{\frac{1}{2}}.$$

For half-densities we therefore demand that

$$\text{“}\, \tilde{u}(\tilde{x})|d\tilde{x}|^{\frac{1}{2}} = u(x)|dx|^{\frac{1}{2}}\text{”},$$

which means that integrals of the squares should be invariantly defined. To accomplish this, we once again modify our original definition (9.46), this time to become

$$\tilde{u}(\tilde{x}) = \tilde{u}(\gamma(x)) := u(x)|\det(\partial\gamma(x))|^{-\frac{1}{2}}. \tag{9.48}$$

Then

$$\int_{\gamma(E)} |\tilde{u}(\tilde{x})|^2 \, d\tilde{x} = \int_E |u(x)|^2 \, dx.$$

for all Borel subsets $E \subseteq \mathbb{R}^n$.

DISCUSSION. The foregoing formalism is at first rather unintuitive, but turns out later to play a crucial role in the rigorous semiclassical calculus, in particular in the theory of Fourier integral operators, which we will touch upon later. Section 9.2 will demonstrate how the half-density viewpoint fits naturally within the Weyl calculus, and Section 10.2 will explain how half-densities simplify some related calculations for a propagator.

Our Appendix E provides a more careful foundation of these concepts in terms of the *s-density line bundles* over $\mathbb{R}^n$, denoted $\Omega^s(\mathbb{R}^n)$. In this notation, a density is a smooth section of $\Omega^1(\mathbb{R}^n)$ and a half-density is a smooth section of $\Omega^{\frac{1}{2}}(\mathbb{R}^n)$.

We therefore write

$$u|dx| \in C^\infty(\mathbb{R}^n, \Omega^1(\mathbb{R}^n))$$

for densities, and

$$u|dx|^{\frac{1}{2}} \in C^\infty(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n))$$

for half-densities. □

REMARK: Half density operator kernels. Half-densities elegantly appear when we use introduce operator kernels. Suppose that

$$K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \Omega^{\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n)).$$

Then K, acting as an integral kernel, defines a map

$$K \ : \ C_{\mathrm{c}}^\infty(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n)) \ \to C^\infty(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n)),$$

in an invariant way, independently of the choice of densities:

$$\begin{aligned} Ku(x)|dx|^{\frac{1}{2}} &= \int_{\mathbb{R}^n} K(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}u(y)|dy|^{\frac{1}{2}} \\ &:= \left(\int_{\mathbb{R}^n} K(x,y)u(y)dy\right)|dx|^{\frac{1}{2}}. \end{aligned} \tag{9.49}$$

□

9.6 CHANGING VARIABLES

In this section we illustrate the usefulness of half-densities in charterizing invariance properties of quantization under changes of variables.

When we fix the symplectic form $\sigma = d\xi \wedge dx$ on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, the half-density sections over $\mathbb{R}^{n2}$ are identified with functions using the canononical density

$$d\xi_1 \wedge \cdots \wedge d\xi_n \wedge dx_1 \wedge \cdots \wedge dx_n = \frac{1}{n!}\sigma^n.$$

In other words, half-densities transform as functions under symplectic changes of variables, and in particular for symplectic transformations arising as in Example 1 of Section 2.3:

$$(x,\xi) \mapsto (\gamma(x), (\partial\gamma(x)^T)^{-1}\xi). \tag{9.50}$$

We will consider the Weyl quantization of a symbol $a \in \mathcal{S}(\mathbb{R}^{2n})$ *as an operator acting on half-densities.* That is done as in (9.49) by defining

$$\begin{aligned} &K_a(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} \\ &\qquad := \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2},\xi\right) e^{i\frac{\langle x-y,\xi\rangle}{h}} d\xi |dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} \end{aligned} \tag{9.51}$$

and

$$\mathrm{Op}(a)(u|dy|^{\frac{1}{2}})|dx|^{\frac{1}{2}} := \int_{\mathbb{R}^n} K_a(x,y)u(y)\,dy|dx|^{\frac{1}{2}}. \tag{9.52}$$

The arguments of Chapter 4 show that for $a \in \mathcal{S}$ we obtain a bounded operator that quantizes a:

$$\mathrm{Op}(a) : L^2(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n)) \;\to\; L^2(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n)).$$

Next, let $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth diffeomorphism which for simplicity we assume to be the identity outside a compact set. Take $a \in \mathcal{S}$. We write $A = \mathrm{Op}(a)$ for the operator acting on *half-densities.* As above, if we write $\tilde{x} = \gamma(x)$, we define $\tilde{u}$ by

$$\tilde{u}(\tilde{x})|d\tilde{x}|^{\frac{1}{2}} = u(x)|dx|^{\frac{1}{2}}. \tag{9.53}$$

Then $\tilde{A} = (\gamma^{-1})^* A \gamma^*$, acting on half-densities, is given by the rule

$$\tilde{A}\tilde{u}(\tilde{x}) = Au(x), \tag{9.54}$$

when acting on functions.

THEOREM 9.12 (Operators and half-densities). *Let $a \in \mathcal{S}(\mathbb{R}^{2n})$ and let A be its quantization acting on half-densities.*

(i) *Then*

$$(\gamma^{-1})^* A \gamma^* = \mathrm{Op}(\tilde{a}) \tag{9.55}$$

for the symbol

$$\tilde{a}(x,\xi) := a(\gamma^{-1}(x), \partial\gamma(x)^T \xi) + O_{\mathcal{S}}(h^2). \tag{9.56}$$

That is,

$$a(x,\xi) = \tilde{a}(\gamma(x), (\partial\gamma(x)^T)^{-1}\xi) + O_{\mathcal{S}}(h^2). \tag{9.57}$$

(ii) *Consider A acting on functions and define*

$$A_1 = (\gamma^{-1})^* A \gamma^*,$$

then

$$A_1 = \mathrm{Op}(a_1),$$

for the symbol

$$a_1(x,\xi) := a(\gamma^{-1}(x), \partial\gamma(x)^T \xi) + O_{\mathcal{S}}(h). \tag{9.58}$$

That is,

$$a(x,\xi) = a_1(\gamma(x), (\partial\gamma(x)^T)^{-1}\xi) + O_{\mathcal{S}}(h). \tag{9.59}$$

INTERPRETATION. A further motivation for half-densities is that assertion (i) for half-densities (with error term of order $O_{\mathcal{S}}(h^2)$) is more precise than the assertion (ii) for functions (with error term $O_{\mathcal{S}}(h)$). The notation $b = O_{\mathcal{S}}(h)$ means that

$$|x^\alpha \xi^\beta \partial^\gamma b| \le C_{\alpha\beta,\gamma} h$$

for all multiindices α, β, γ: see §A.5. □

Proof. 1. Since $a \in \mathcal{S}$, we have $K_a \in \mathcal{S}$. Take $\tilde{a} \in \mathcal{S}(\mathbb{R}^{2n})$ for which $(\kappa^{-1})^* A \kappa^* = \mathrm{Op}(\tilde{a})$.

2. Remember that

$$Au(x)|dx|^{\frac{1}{2}} = \int_{\mathbb{R}^n} K_a(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} u(y)|dy|^{\frac{1}{2}}$$

for

$$K_a(x,y) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} a\left(\tfrac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle}\, d\xi.$$

Likewise

$$\tilde{A}\tilde{u}(\tilde{x}) = \int_{\mathbb{R}^n} K_{\tilde{a}}(\tilde{x},\tilde{y})|d\tilde{x}|^{\frac{1}{2}}|d\tilde{y}|^{\frac{1}{2}} \tilde{u}(\tilde{y})\,|d\tilde{y}|^{\frac{1}{2}}$$

for

$$K_{\tilde{a}}(\tilde{x},\tilde{y}) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \tilde{a}\left(\tfrac{\tilde{x}+\tilde{y}}{2}, \tilde{\xi}\right) e^{\frac{i}{h}\langle \tilde{x}-\tilde{y},\tilde{\xi}\rangle}\, d\tilde{\xi}.$$

Since $\tilde{u}(\tilde{y})|d\tilde{y}|^{\frac{1}{2}} = u(y)|dy|^{\frac{1}{2}}$ and $d\tilde{y} = |\det \partial\gamma(y)|dy$, it follows that

$$\tilde{A}\tilde{u}(x) = \int_{\mathbb{R}^n} K_{\tilde{a}}(\tilde{x},\tilde{y})|\det \partial\gamma(y)|^{\frac{1}{2}}|\det \partial\gamma(x)|^{\frac{1}{2}} u(y)\, dy.$$

Hence

$$K_a(x,y) = K_{\tilde{a}}(\tilde{x},\tilde{y})|\det \partial\gamma(y)|^{\frac{1}{2}}|\det \partial\gamma(x)|^{\frac{1}{2}}. \tag{9.60}$$

3. Now

$$\begin{aligned} K_{\tilde{a}}(\tilde{x},\tilde{y}) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \tilde{a}\left(\tfrac{\tilde{x}+\tilde{y}}{2}, \tilde{\xi}\right) e^{\frac{i}{h}\langle \tilde{x}-\tilde{y},\tilde{\xi}\rangle}\, d\tilde{\xi} \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \tilde{a}\left(\tfrac{\gamma(x)+\gamma(y)}{2} \tilde{\xi}\right) e^{\frac{i}{h}\langle \gamma(x)-\gamma(y),\tilde{\xi}\rangle}\, d\tilde{\xi}. \end{aligned}$$

We have

$$\gamma(x) - \gamma(y) = \langle g(x,y), x-y\rangle, \tag{9.61}$$

where

$$g(x,y) = \partial\gamma\left(\frac{x+y}{2}\right) + O(|x-y|^2). \tag{9.62}$$

Also

$$\gamma(x)+\gamma(y)=2\gamma\left(\frac{x+y}{2}\right)+O(|x-y|^2). \tag{9.63}$$

Let us also write

$$\tilde{\xi}=(g(x,y)^T)^{-1}\xi. \tag{9.64}$$

Substituting above, we deduce that

$$\begin{aligned} &K_{\tilde{a}}(\tilde{x},\tilde{y}) \\ &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\left[\tilde{a}\left(\gamma(\tfrac{x+y}{2}),(g(x,y)^T)^{-1}\xi\right)+O(|x-y|^2)\right] \\ &\qquad\qquad e^{\frac{i}{h}\langle\kappa(x)-\kappa(y),(g(x,y)^T)^{-1}\xi\rangle}\,d\tilde{\xi}. \end{aligned}$$

We now use the so-called "Kuranishi trick" to rewrite this expression as a pseudodifferential operator. First,

$$\langle\gamma(x)-\gamma(y),(g(x,y)^T)^{-1}\xi\rangle=\langle g(x,y)^{-1}(\gamma(x)-\gamma(y)),\xi\rangle=\langle x-y,\xi\rangle,$$

according to (9.61). Remembering also (9.62), we compute

$$\begin{aligned} &K_{\tilde{a}}(\tilde{x},\tilde{y})= \\ &\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\left[\tilde{a}\left(\gamma(\tfrac{x+y}{2}),(\partial\gamma(\tfrac{x+y}{2})^T)^{-1}\xi\right)+O(|x-y|^2)\right]e^{\frac{i}{h}\langle x-y,\xi\rangle}d\tilde{\xi} \\ &\quad=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\left[a(\tfrac{x+y}{2},\xi)+O(|x-y|^2)\right]e^{\frac{i}{h}\langle x-y,\xi\rangle}d\tilde{\xi}. \end{aligned}$$

Furthermore $d\tilde{\xi}=|\det\, g(x,y)|^{-1}d\xi$ and

$$\det\, g(x,y)=\det\,\partial\gamma(\tfrac{x+y}{2})+O(|x-y|^2).$$

Also, we claim that

$$\left|\det\,\partial\gamma(\tfrac{x+y}{2})\right|^2=|\det\,\partial\gamma(x)||\det\,\partial\gamma(y)|+O(|x-y|^2).$$

This identity is clear if we add a term $\langle A(\frac{x+y}{2}),x-y\rangle$ on the right hand side. But the symmetry under switching x and y shows that $A\equiv 0$.

4. Finally we observe that

$$(x-y)^\alpha e^{\frac{i}{h}\langle x-y,\xi\rangle}=(hD_\xi)^\alpha e^{\frac{i}{h}\langle x-y,\xi\rangle}. \tag{9.65}$$

Hence integrating by parts in the terms with $O(|x-y|^2)$ gives us terms of order $O(h^2)$. So

$$\begin{aligned} &K_{\tilde{a}}(\tilde{x},\tilde{y})= \\ &\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\left(a(\tfrac{x+y}{2},\xi)+O(h^2)\right)e^{\frac{i}{h}\langle x-y,\xi\rangle}d\xi \\ &\qquad\qquad|\det\,\partial\gamma(x)|^{-1/2}|\det\,\partial\gamma(y)|^{-1/2}. \end{aligned}$$

This proves (9.60) with $\tilde{a}$ satisfying (9.56).

5. When A acts on functions, then K_a has to transform as a density. In other words, we need to show

$$K_a(x,y) = K_{a_1}(\tilde{x},\tilde{y})|\det\,\partial\gamma(y)| + O(h), \tag{9.66}$$

instead of (9.60). Since

$$|\det\,\partial\gamma(y)| = |\det\,\partial\gamma(y)|^{1/2}|\det\,\partial\gamma(x)|^{1/2} + O(|x-y|),$$

we see from (9.65) that (9.66) follows from (9.60) with $a_1 = \tilde{a} + O(h)$. □

9.7 NEW SYMBOL CLASSES

Suppose that

$$\gamma : \mathbb{R}^n \to \mathbb{R}^n$$

is a diffeomorphism equal to the identity outside of a compact set. In §9.6 we discussed the operator $(\gamma^{-1})^*\mathrm{Op}(a)\gamma^*$ for $a \in \mathcal{S}(\mathbb{R}^n)$. We want now to introduce more general classes of symbols a than those provided by Chapter 4, and to discuss their invariance under the mapping κ.

DISCUSSION. To motivate the need for these new classes, let m be an order function and recall the class $S(m)$ introduced in Chapter 4:

$$S(m) := \{a \in C^\infty(\mathbb{R}^{2n}) \mid |\partial^\alpha a(x,\xi)| \le C_\alpha m(x,\xi)\}.$$

In view of Theorem 9.12, for a symbol a in $S(m)$ to be invariant under γ, we would need

$$|\partial^\alpha \left(a(\gamma^{-1}(x), \partial\gamma(x)^T\xi)\right)| \le C_\alpha m(x,\xi). \tag{9.67}$$

Since differentiation in x falling on the second set of variables produces factors of the form

$$\partial_{x_j}(\partial\gamma(x)^T\xi),$$

the bound (9.67) is in general false unless

$$|m(x,\xi)| \le C_N\langle\xi\rangle^{-N}$$

for all N. This requirement is clearly too restrictive, as it would exclude all differential operators.

An estimate of the type (9.67) would however hold if differentiation in ξ improves the decay in ξ. This observation leads us to the following definition, in which we restrict to the simplest order functions $\langle\xi\rangle^m$:

DEFINITIONS. (i) The *classical symbols* are

$$S^{m,k} = \{a \in C^\infty(\mathbb{R}^{2n}) \mid |\partial_x^\alpha\partial_\xi^\beta a| \le C_{\alpha\beta}h^{-k}\langle\xi\rangle^{m-|\beta|}\}. \tag{9.68}$$

(ii) We also write

$$\Psi^{m,k} = \{a^{\mathrm{w}}(x, hD) \mid a \in S^{m,k}\}$$

to denote the corresponding class of operators obtained by the quantization procedure described in Theorem 4.10.

Since $S^{m,k} \subset S^k(\langle\xi\rangle^m)$ all the results of Chapter 4 are applicable but due to the improvement under differentiation in ξ there are many new features, important in the study of partial differential equations. We will only present the improved composition formula and the change of variables formula.

THEOREM 9.13 (Composition for $\Psi^{m,k}$). *Suppose that $a \in S^{m_1,k_1}$ and $b \in S^{m_2,k_2}$. Then*

$$a^{\mathrm{w}}(x, hD) \circ b^{\mathrm{w}}(x, hD) = c^{\mathrm{w}}(x, hD)$$

where $c \in S^{m_1+m_2,k_1+k_2}$ is given by (**??**).

Moreover,

$$c(x,\xi) = \sum_{k=0}^{N} \frac{1}{k!}\left(\frac{ih}{2}\sigma(D_x, D_\xi, D_y, D_\eta)\right)^k a(x,\xi)b(y,\eta)|_{x=\xi, y=\eta} + O_{S^{m_1+m_2-N-1,k_1+k_2}}(h^{N+1}) \tag{9.69}$$

where

$$\sigma(D_x, D_\xi, D_y, D_\eta) := \langle D_\xi, D_y\rangle - \langle D_x, D_\eta\rangle.$$

REMARK. Similar statements hold for the usual quantization:

$$a(x, hD) \circ b(x, hD) = c_1(x, hD) = e^{ih\langle D_\xi, D_y\rangle} a(x,\xi)b(y,\eta)|_{y=x,\eta=\xi}$$

and

$$c_1(x,\xi) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)(hD_x)^\alpha b(x,\xi) + O_{S^{m_1+m_2-N-1,k_1+k_2}}(h^{N+1}).$$

Proof: 1. To simplify the notation let us make the harmless assumption that $k_1 = k_2 = 0$. Since $S^{m_j,0} \subset S(\langle\xi\rangle^{m_j})$ the validity of (**??**) follows from Theorem 4.6. Similarly, (9.69) is valid but with an error $O_{S(\langle\xi\rangle^{m_1+m_2})}(h^{N+1})$.

2. Now we observe that

$$\left(\frac{ih}{2}\sigma(D_x, D_\xi, D_y, D_\eta)\right)^k a(x,\xi)b(y,\eta)|_{x=\xi,y=\eta} \in S^{m_1+m_2-k,-k},$$

and we will show that the remainder satisfies

$$c(x,\xi) - \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{ih}{2} \sigma(D_x, D_\xi, D_y, D_\eta) \right)^k a(x,\xi) b(y,\eta)|_{x=\xi, y=\eta} \in S^{N+1}(\langle \xi \rangle^{m_1+m_2-N-1}). \tag{9.70}$$

Since N is arbitrary it follows that

$$|\partial_x^\alpha \partial_\xi^\beta c(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m_1+m_2-|\beta|},$$

uniformly for $0 < h \leq 1$.

3. To check (9.70) we recall that (**??**) (and hence (9.69)) was proved by consider the action of $\exp A(D)$, where A is a quadratic form, on symbol classes – see Theorem **??**. Using Taylor's formula we write $\exp(ihA(D)) =$

$$\sum_{k=0}^{N} \frac{(ihA(D))^k}{k!} + \frac{1}{N!} \int_0^1 (1-t)^N \exp(ithA(D))(ihA(D))^{N+1} dt.$$

In our case we have $A(D) = h\sigma(D_x, D_\xi; D_y, D_\eta)/2$, so that

$$c(x,\xi) = \exp(ihA(D)) a(x,\xi) b(y,\eta)|_{x=y, \eta=\xi},$$

and the remainder in (9.70) is given by

$$\frac{1}{N!} \int_0^1 (1-t)^N \exp(ithA(D))(ihA(D))^{N+1} (a(x,\xi) b(y,\eta))|_{x=y, \xi=\eta} dt.$$

Since $a \in S^{m_1,0}$ and $b \in S^{m_2,0}$, we see that

$$(hA(D))^{N+1}(a(x,\xi)b(y,\eta)) \in \sum_{k=0}^{N+1} S^{N+1}(\langle \xi \rangle^{m_1-k} \langle \eta \rangle^{m_2-N-1+k}).$$

Theorem **??** shows that $\exp(ithA(D))$:

$$S^{N+1}(\langle \xi \rangle^{m_1-k} \langle \eta \rangle^{m_2-N-1+k}) \longrightarrow S^{N+1}(\langle \xi \rangle^{m_1-k} \langle \eta \rangle^{m_2-N-1+k}),$$

with bounds uniform for $0 \leq t \leq 1$, $0 < h \leq 1$. Also,

$$S^{N+1}(\mathbb{R}^{2n}_{(x,\xi)} \times \mathbb{R}^{2n}_{(y,\eta)}, \langle \xi \rangle^{m_1-k} \langle \eta \rangle^{m_2-N-1+k})|_{x=y, \xi=\eta} = S^{N+1}(\mathbb{R}^{2n}_{(x,\xi)}, \langle \xi \rangle^{m_1+m_2-N-1}),$$

from which (9.70) follows. $\square$

We also record the following useful lemma:

LEMMA 9.14 (Schwartz kernels of operators in $\Psi^{m,k}$).

(i) *Suppose that $a \in S^{m,k}$ and that $K_a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is the Schwartz kernel of* $\mathrm{Op}(a)$*. Then*

$$K_a(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta) \tag{9.71}$$

for the diagonal $\Delta := \{(x,x) : x \in \mathbb{R}^n\}$. Furthermore, we have the estimates

$$|(hD_x)^\alpha (hD_y)^\beta K_a(x,y)| \leq C_N h^{-k} \left(\frac{h}{|x-y|} \right)^N \tag{9.72}$$

for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ and $N > |\alpha| + |\beta| + m + n$.

(ii) *If K_a satisfies*

$$|\partial_x^\alpha \partial_y^\beta K_a(x,y)| \leq C_N \left(\frac{h}{\langle x-y \rangle} \right)^N \tag{9.73}$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, then K_a is the Schwartz kernel of $\mathrm{Op}(a)$ *for $a \in S^{-\infty,-\infty}$.*

The lemma shows one of the many advantages of symbol classes $S^{m,k}$ since we now have smoothness and rapid decay away from the diagonal.

Proof. 1. We can consider either the Weyl quantization, Op, or the standard quantization, Op_1. That follows from Theorem **??** since as in Part 3 of the proof of Theorem 9.13 we see that

$$\exp\left(i(t-s)h\langle D_x, D_\xi \rangle\right) \ : \ S^{m,k} \longrightarrow S^{m,k}. \tag{9.74}$$

So for simplicity of notation we opt for $\mathrm{Op}_1(a)$.

2. Suppose first that $a \in \mathcal{S}$ so that

$$K_a(x,y) = \frac{1}{(2\pi h)^n} \int a(x,\xi) e^{i\langle x-y,\xi\rangle/h} d\xi,$$

and the integral is taken in the usual sense. We note that

$$(x-y)^\gamma K_a(x,y) = \frac{1}{(2\pi h)^n} \int (-hD_\xi)^\gamma a(x,\xi) e^{i\langle x-y,\xi\rangle/h} d\xi,$$

and hence

$$|(x-y)^\gamma K_a(x,y)| \leq C_\gamma h^{|\gamma|} \|\langle \xi \rangle^{n+1} \partial_\xi^\gamma a\|_\infty \int \langle \xi \rangle^{-n-1} d\xi.$$

Observe next that

$$\sup_{|\gamma|=N} |(x-y)^\gamma| \geq n^{-\frac{N}{2}} |x-y|^N.$$

(It is enough to prove this inequality for $y = 0$, $|x| = 1$. It then says that we can choose γ, $\gamma_1 + \cdots \gamma_n = N$, so that $|x_1|^{\gamma_1} \cdots |x_n|^{\gamma_n} \geq n^{-N/2}$

for any $x_1^2 + \cdots x_n^2 = 1$. The last condition implies that there exists m, $1 \leq m \leq n$ such that $|x_m| \geq 1/\sqrt{n}$. We then take $\gamma_j = \delta_{jm} N$.)

Therefore we obtain

$$|K_a(x,y)| \leq C_N \left(\frac{h}{|x-y|} \right)^N \sup_{|\gamma|=N} \| \langle \xi \rangle^{n+1} \partial_\xi^\gamma a \|_\infty . \tag{9.75}$$

Similarly,

$$|(hD_x)^\alpha (hD_y)^\beta ((x-y)^\gamma K_a(x,y))| \leq$$
$$C_{\gamma,\alpha,\beta} \sup_{|\rho| \leq |\alpha|} h^{|\gamma|} \| \langle \xi \rangle^{n+1+|\alpha|+|\beta|} \partial_\xi^\gamma \partial_x^\rho a \|_\infty ,$$

and consequently,

$$|(hD_x)^\alpha (hD_y)^\beta K_a(x,y)| \leq$$
$$C_{N,\alpha,\beta} \left(\frac{h}{|x-y|} \right)^N \sup_{|\rho|\leq|\alpha|, |\gamma|=N} \| \langle \xi \rangle^{n+1+|\alpha|+|\beta|} \partial_x^\rho \partial_\xi^\gamma a \|_\infty . \tag{9.76}$$

3. If $a \in S^{m,k}$ we observe that seminorms appearing on the right hand side of (9.76) are finite and bounded by h^{-k} if $N > |\alpha| + |\beta| + m + n$. Approximation by symbols in $\mathcal{S}$ concludes the proof of (**??**).

4. The second conclusion follows from applying the inverse Fourier transform:

$$a(x,\xi) = \int K_a(x, x-z) e^{i\langle z,\xi \rangle/h} dz.$$

Since

$$|\partial_x^\alpha \partial_z^\beta K_a(x,x-z)| \leq C_N \frac{h^N}{\langle z \rangle^N},$$

we see that $a \in S^{-\infty,-\infty}$. □

Before restating Theorem 9.12 in this more general setting we will discuss usual quantization acting on functions. Suppose first that $a \in \mathcal{S}$. Then

$$(\gamma^{-1})^* \mathrm{Op}_1(a) \gamma^* = \mathrm{Op}_1(a_\gamma), \tag{9.77}$$

where

$$a_\gamma(\gamma(x), \eta) = e^{-\frac{i}{h}\langle \gamma(x), \eta \rangle} a(x, hD) e^{\frac{i}{h}\langle \gamma(\cdot),\eta \rangle}. \tag{9.78}$$

In fact, Theorem **??** shows that

$$\begin{aligned} a_\gamma(y,\eta) &= e^{-\frac{i}{h}\langle y,\eta\rangle} \mathrm{Op}_1(a_\gamma) e^{\frac{i}{h}\langle y,\eta \rangle} \\ &= e^{-\frac{i}{h}\langle y,\eta\rangle} (\gamma^{-1})^* a(x,hD) \gamma^* (e^{-\frac{i}{h}\langle \cdot,\eta\rangle}), \end{aligned}$$

which is the same as (9.78).

THEOREM 9.15 (Changing variables I). *Suppose that* $a \in S^{m,k}$ *where* $S^{m,k}$ *is defined by* (9.68). *Then* (9.77) *defines* $a_\gamma \in S^{m,k}$ *for which* (9.78) *holds. Moreover,*

$$a_\gamma(\gamma(x), \eta) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \partial\gamma(x)^T \eta)(hD_y)^\alpha e^{\frac{i}{h}\langle \rho_x(y), \eta \rangle}|_{y=x} + O_{S^{m-N-1,k}}(h^{N+1}), \tag{9.79}$$

where

$$\rho_x(y) = \gamma(y) - \gamma(x) - \partial\gamma(x)(y - x).$$

In particular,

$$a_\gamma(\gamma(x), \eta) = a(x, \partial\gamma(x)^T \eta) + O_{S^{m-1,k}}(h). \tag{9.80}$$

Proof. 1. We need to show is that a_γ defined by (9.78) is in $S^{m,k}$. Since that will imply

$$(\gamma^{-1})^* \mathrm{Op}_1(a) \gamma^* u = \mathrm{Op}_1(a_\gamma) u, \quad u(x) = \exp(i\langle x, \eta \rangle / h),$$

the operator identity (9.77) will follow since $\exp(i\langle x, \eta\rangle/h)$, $\eta \in \mathbb{R}^n$, are dense in $\mathcal{S}'(\mathbb{R}^n_x)$.

2. We now claim that $a_\gamma(\gamma(x), \eta) =$

$$\frac{1}{(2\pi h)^n} \iint a(x, \xi) \chi_x(y) \chi_\eta(\xi) e^{i(\langle x-y, \xi\rangle + \langle \gamma(y) - \gamma(x), \eta\rangle)/h} dy d\xi + O_S(\langle \eta \rangle^{-\infty} h^\infty), \tag{9.81}$$

where

$$\chi_x(y) := \chi(x - y), \quad \chi_\eta(\xi) := \chi((\xi - \partial\gamma(x)^T \eta)/\langle \eta \rangle),$$
$$\chi \in C_c^\infty(\mathbb{R}, [0, 1]), \quad \chi|_{[-1,1]} = 1, \quad \chi|_{\complement[-2,2]} = 0.$$

In fact, on the support of $1 - \chi_x(y)\chi_\eta(\xi)$ the phase is not stationary:

$$d_{y,\xi}(\langle x - y, \xi \rangle + \langle \kappa(y) - \kappa(x), \eta \rangle) = 0 \Longleftrightarrow \begin{cases} x = y \\ \text{and} \\ \xi = \partial\kappa(x)^T \eta. \end{cases}$$

Consequently the now standard integration-by-parts argument, which we leave to the reader, gives (9.81).

3. We rewrite the main part in (9.81) as follows:

$$\frac{1}{(2\pi \tilde{h})^n} \iint a_\eta(x, \tilde{\xi}) \chi_x(y) \tilde{\chi}_\eta(\tilde{\xi}) e^{i(\langle x-y, \tilde{\xi}\rangle + \langle \gamma(y) - \gamma(x), \eta/\langle \eta \rangle \rangle)/\tilde{h}} dy d\tilde{\xi}$$

$$a_\eta(x, \tilde{\xi}) := a(x, \langle \eta \rangle \tilde{\xi}), \quad \tilde{\chi}_\eta(\tilde{\xi}) := \chi_\eta(\tilde{\xi}\langle \eta \rangle), \quad \tilde{h} := h/\langle \eta \rangle.$$

The support of the integrand is contained in a fixed compact set, and

$$h^k \langle \eta \rangle^{-m} a_\eta \chi_\eta \in S(1),$$

uniformly in h and η. Hence we can apply the method of stationary phase (Theorem 3.15) to obtain an expansion in powers of $\tilde{h} = h/\langle \eta \rangle$. By computing the leading term we easily check that (9.80) holds.

4. To obtain a formula for terms in the full expansion (9.79) we use the method which already appeared in the second proof of Theorem 3.10.

Since we will not use (9.79) the argument is only sketched with full details given in [H2, Theorem 18.1.17]: the integrand in (9.81) can be rewritten as follows

$$\frac{1}{(2\pi h)^n} \left[a(x,\xi) e^{i\langle \rho_x(y),\eta\rangle/h} \chi_x(y)\chi_\eta(\xi) \right] e^{i\langle x-y, \xi - \partial\gamma(x)^T\eta\rangle/h},$$

where $\rho_x(y)$ is given by (9.79). If we consider the term in square brackets as the amplitude, and change variables to

$$z := x - y, \quad w := \xi - \partial\gamma(x)^T \eta,$$

this becomes the integral in the statement of Theorem **??** (with z and w playing the roles of x and y there). Since $\rho_x(y)$ vanishes to second order,

$$\rho_x(x) = 0, \quad d_y\rho_x(x) = 0,$$

at the critical point $x = y$, the differentiation of the oscillatory term in the amplitude combined with the decay of $\partial_\xi^\alpha a$ shows that the terms in (9.79) are in $S^{m-|\alpha|/2, k-|\alpha|/2}$. Hence the formal expansion makes sense but to control the error terms we need arguments similar to those in second proof of Theorem 3.10. □

We can now give the generalization of Theorem 9.12. The proof given in §9.6 can be adapted to the present setting using the integration by parts arguments from the proof of Lemmas 9.14.

THEOREM 9.16 (Changing variables II). *Let $a \in S^{m,k}(\mathbb{R}^{2n})$ and let A be its quantization acting on half-densities.*

(i) *Consider A acting on half-densities. Then*

$$(\gamma^{-1})^* A \gamma^* = \mathrm{Op}(\tilde{a}) \tag{9.82}$$

for

$$\tilde{a}(x,\xi) := a(\gamma^{-1}(x), \partial\gamma(x)^T\xi) + O_{S^{m-2,k}}(h^2). \tag{9.83}$$

That is,

$$a(x,\xi) = \tilde{a}(\gamma(x), (\partial\gamma(x)^T)^{-1}\xi) + O_{S^{m-2,k}}(h^2). \tag{9.84}$$

(ii) *When we consider A acting on functions and define*

$$A_1 = (\gamma^{-1})^* A \gamma^*,$$

then

$$A_1 = \mathrm{Op}(a_1),$$

for

$$a_1(x,\xi) := a(\gamma^{-1}(x), \partial\gamma(x)^T\xi) + O_{S^{m-1,k}}(h). \tag{9.85}$$

That is,

$$a(x,\xi) = a_1(\gamma(x), (\partial\gamma(x)^T)^{-1}\xi) + O_{S^{m-1,k}}(h). \tag{9.86}$$

□

10. Quantizing symplectic transformations

This final chapter presents some more advanced topics, mostly concerning how (and why) to quantize symplectic transformations.

10.1 DEFORMATION AND QUANTIZATION

Throughout this chapter, we identify $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. In this section $\gamma : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denotes a symplectomorphism:

$$\gamma^*\sigma = \sigma \quad \text{ for } \sigma = \sum_{j=1}^n d\xi_j \wedge dx_j,$$

normalized so that $\gamma(0,0) = (0,0)$. Our goal is to *quantize* γ *locally*, meaning to find a unitary operator $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

$$F^{-1}AF = B \qquad \text{near } (0,0)$$

for $A = \mathrm{Op}(a)$, where $a \in S$ and $B = \mathrm{Op}(b)$ for

$$b = \gamma^* a + O(h).$$

This can be useful in practice, since sometimes we can design κ so that $\kappa^* a$ is more tractable than a.

The basic strategy will be (i) finding a family $\{\kappa_t\}_{0\le t\le 1}$ of symplectomorphisms so that $\gamma_0 = I$ and $\gamma_1 = \gamma$; (ii) quantizing the functions q_t generating this flow of mappings; and then (iii) solving an associated operator ODE (10.7).

10.1.1 Deformations. We begin by deforming γ to the identity mapping. So assume U_0 and U_1 are simply connected neighborhoods of $(0,0)$ and $\gamma : U_0 \to U_1$ is a symplectomorphism such that $\kappa(0,0) = (0,0)$.

THEOREM 10.1 (Deforming symplectomorphisms). *There exists a continuous, piecewise smooth family*

$$\{\gamma_t\}_{0\le t\le 1}$$

of local symplectomorphisms $\gamma_t : U_0 \to U_t =: \gamma_t(U_0)$ *such that*

(i) $\gamma_t(0,0) = 0 \qquad (0 \le t \le 1)$
(ii) $\gamma_1 = \gamma$, $\gamma_0 = I$.
(iii) *Also,*

$$\frac{d}{dt}\gamma_t = (\gamma_t)_* H_{q_t} \qquad (0 \le t \le 1) \tag{10.1}$$

for a smooth family of functions $\{q_t\}_{0 \le t \le 1}$.

REMARK. The statement (10.1) means that for each function $a \in C^\infty(U_1)$, we have

$$\frac{d}{dt}\gamma_t^* a = H_{q_t}\gamma_t^* a. \tag{10.2}$$

In fact,

$$\frac{d}{dt}\gamma_t^* a = \langle da, d\gamma_t/dt\rangle = \langle da, (\gamma_t)_* H_{q_t}\rangle = H_{q_t}\gamma_t^* a,$$

where $\langle\cdot,\cdot\rangle$ is the pairing of differential 1-forms and vectorfields on U_t. □

Proof. 1. We first consider the case that γ is given by a linear symplectomorphism $K : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$K^* J K = J \tag{10.3}$$

for

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Since K is an invertible matrix, we have the unique polar decomposition

$$K = QP,$$

where Q is orthogonal and P is positive definite. From (10.3) we deduce that

$$Q^{*-1}P^{*-1} = K^{*-1} = JQJ^{-1}JPJ^{-1};$$

whence the uniqueness of Q and P implies

$$Q^{*-1} = JQJ^{-1}, \quad P^{*-1} = JPJ^{-1}.$$

That is, both Q and P are symplectic. Furthermore, we can write

$$P = \exp A,$$

where $A = A^*$ and $JA + AJ = 0$.

2. We identify $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$, under the relation $(x, y) \leftrightarrow x + iy$. Since

$$\langle x + iy, x' + iy'\rangle_{\mathbb{C}^n} = \langle (x, y), (x', y')\rangle_{\mathbb{R}^n} + i\sigma((x, y), (x', y')),$$

the fact that Q is orthogonal and symplectic implies it is unitary:

$$Q = Q^{*-1} = -JQJ.$$

(Similarly, any unitary transformation on $\mathbb{C}^n$ gives an orthogonal symplectic transformation in $\mathbb{R}^n \times \mathbb{R}^n$.)

We can now write

$$Q = \exp iB,$$

where $B^* = B$ is Hermitian on $\mathbb{C}^n$. A smooth deformation to the identity is now clear:

$$K_t := \exp(itB)\exp(tA) \qquad (0 \le t \le 1).$$

3. For the general case that γ is nonlinear, set $K := \partial\gamma(0,0)$. Then for $1/2 \le t \le 1$,

$$\gamma_t := K_{2-2t}^{-1} \circ \gamma$$

is a piecewise smooth family of symplectomorphisms satisfying

$$\gamma_1 = \gamma, \quad \partial\gamma_{1/2}(0,0) = I.$$

For $0 \le t \le 1/2$, we set

$$\gamma_t(m) := \frac{1}{2t}\gamma_{1/2+t}(2tm).$$

4. Define $V_t := \frac{d}{dt}\gamma_t$; we must show

$$V_t = (\kappa_t)_* H_{q_t}$$

for some function q_t. According to Cartan's formula (Theorem B.3):

$$\mathcal{L}_{V_t}\sigma = d\sigma \lrcorner V_t + d(\sigma \lrcorner V_t).$$

But $\mathcal{L}_{V_t}\sigma = \frac{d}{dt}\gamma_t^*\sigma = \frac{d}{dt}\sigma = 0$, since $\gamma_t^*\sigma = \sigma$. Furthermore, $d\sigma = 0$, and consequently $d(\sigma \lrcorner V_t) = 0$. Owing to Poincaré's Lemma (Theorem B.4), we have

$$\gamma_t^*(\sigma \lrcorner V_t) = dq_t$$

for a function q_t; and this means that $V_t = (\kappa_t)_* H_{q_t}$. □

To define our symbol classes, we hereafter consider the order function

$$m := (1 + |x|^2 + |\xi|^2)^{\frac{k}{2}}$$

for some positive integer k.

THEOREM 10.2 (Quantizing one parameter families of symplectomorphisms). *Let $\{\gamma_t\}_{0\leq t\leq 1}$ be a smooth family of symplectomorphisms of $\mathbb{R}^{2n}$, such that*

$$\gamma_0 = I, \quad \frac{d}{dt}\gamma_t = (\gamma_t)_* H_{q_t},$$

where $q_t \in S(m)$ is a smooth family of real valued symbols.

Then there exists a family of unitary operators

$$F(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

such that

$$F(0) = I,$$

and for all $A = \mathrm{Op}(a)$ with $a \in \mathcal{S}$, we have

$$F(t)^{-1} \circ A \circ F(t) = B(t) \qquad (0 \leq t \leq 1) \tag{10.4}$$

for

$$B(t) = \mathrm{Op}(b_t), \tag{10.5}$$

where

$$b_t = \gamma_t^* a + hc_t \tag{10.6}$$

for $c_t \in \mathcal{S} \cap S$.

Proof. 1. We define

$$Q(t) := \mathrm{Op}(q_t) : \mathcal{S} \to \mathcal{S} \qquad (0 \leq t \leq 1),$$

and recall that

$$Q(t)^* = Q(t).$$

Since $Q(t)$ depends smoothly on t as an operator on $\mathcal{S}$, we can solve the operator ODE

$$\begin{cases} hD_t F(t) + F(t)Q(t) = 0 & (0 \leq t \leq 1) \\ F(0) = I \end{cases} \tag{10.7}$$

for $F(t) : \mathcal{S} \to \mathcal{S}$. Then

$$\begin{cases} hD_t F(t)^* - Q(t)F(t)^* = 0 & (0 \leq t \leq 1) \\ F(0)^* = I. \end{cases} \tag{10.8}$$

2. We claim that

$$F(t) \text{ is unitary on } L^2(\mathbb{R}^n).$$

To confirm this, let us calculate using (10.7) and (10.8):

$$\begin{aligned} hD_t(F(t)F(t)^*) &= hD_tF(t)F(t)^* + F(t)hD_tF(t)^* \\ &= -F(t)Q(t)F(t)^* + F(t)Q(t)F(t)^* = 0. \end{aligned}$$

Hence $F(t)F(t)^* \equiv I$. On the other hand,

$$\begin{aligned} hD_t(F(t)^*F(t) - I) &= Q(t)F(t)^*F(t) - F(t)^*F(t)Q(t) \\ &= [Q(t), F(t)^*F(t) - I]. \end{aligned}$$

with $F(0)^*F(0) - I = 0$. Since this equation for $F(t)^*F(t) - I$ is homogeneous, it follows that $F(t)^*F(t) \equiv I$.

3. Now define

$$B(t) := F(t)^{-1}AF(t), \tag{10.9}$$

and first show that

$$B(t) \ : \ \mathcal{S}' \longrightarrow \mathcal{S}. \tag{10.10}$$

This from a stronger statement showing that for any N,

$$B_N(t) := \langle x\rangle^N \langle hD_x\rangle^N B(t) \langle x\rangle^N \langle hD_x\rangle^N = O(1) \ : \ L^2 \longrightarrow L^2, \tag{10.11}$$

that is the bound does not depend on h. To see this we note that

$$B_N(t) = F_N(t)^{-1} B_N(0) F_N(t),$$

where $F_N(t)$ is defined in the same way as $F(t)$ but with $Q(t)$ replaced by

$$Q_N(t) := \langle hD_x\rangle^{-N} \langle x\rangle^{-N} Q(t) \langle x\rangle^N \langle hD_x\rangle.$$

Theorem 4.13 shows that $Q_N(t) = \mathrm{Op}(q_N(t))$, $q_N(t) \in S(1)$, and hence it is bounded on L^2. The inverse, $F_N(t)^{-1} = G_N(t) : \mathcal{S} \to \mathcal{S}$, is obtained by solving

$$hD_tG_N(t) - Q_N(t)G_N(t) = 0 \quad (0 \leq t \leq 1), \qquad G_N(0) = I.$$

Since $Q_N(t) = Q_N(t)^* + O_{L^2\to L^2}(h)$, we see that

$$\frac{d}{dt}\|G_N(t)u\|^2 = \frac{2}{h}\mathrm{Im}\langle Q_N(t)G_N(t)u, G_N(t)\rangle \leq C\|G_N(t)u\|^2,$$

and hence, by Gronwall's inequality, $G_N(t)$, and hence $F_N(t)$, are bounded on L^2, uniformly with respect to h.

This concludes the proof of (10.11) since

$$B_N(0) = \langle x\rangle^N \langle hD_x\rangle^N A \langle x\rangle^N \langle hD_x\rangle^N = O(1) \ : \ L^2 \longrightarrow L^2$$

by the assumption that $A = \mathrm{Op}(a)$, $a \in \mathcal{S}$.

4. We assert that

$$B(t) = \mathrm{Op}(b_t) \tag{10.12}$$

for

$$b_t = \gamma_t^* a + O(h), \quad b_t \in \mathcal{S} \cap S(1). \tag{10.13}$$

To prove this, define the family of pseudodifferential operators

$$\tilde{B}(t) := \mathrm{Op}(\gamma_t^* a).$$

We calculate

$$\begin{aligned} hD_t\tilde{B}(t) &= \frac{h}{i}\mathrm{Op}\left(\frac{d}{dt}\gamma_t^* a\right) = \frac{h}{i}\mathrm{Op}(H_{q_t}\gamma_t^* a) \\ &= \frac{h}{i}\mathrm{Op}(\{q_t, \gamma_t^* a\}) = [Q(t), \tilde{B}(t)] + E(t), \end{aligned}$$

and the pseudodifferential calculus implies that

$$\|E(t)\|_{L^2 \to L^2} = O(h^2)$$

where $E(t) = \mathrm{Op}(e(t))$ for a symbol $e(t) \in S^{-2}$.

Therefore

$$\begin{aligned} hD_t(F(t)\tilde{B}(t)F(t)^{-1}) &= (hD_tF(t))\tilde{B}(t)F(t)^{-1} + F(t)(hD_t\tilde{B}(t))F(t)^{-1} \\ &\quad + F(t)\tilde{B}(t)hD_t(F(t)^{-1}) \\ &= -F(t)Q(t)\tilde{B}(t)F(t)^{-1} + F(t)([Q(t), \tilde{B}(t)] \\ &\quad + E(t))F(t)^{-1} + F(t)\tilde{B}(t)Q(t)F(t)^{-1} \\ &= F(t)E(t)F(t)^{-1} = O(h^2). \end{aligned}$$

Integrating and dividing by h gives

$$F(t)\tilde{B}(t)F(t)^{-1} = A + \frac{i}{h}\int_0^t F(s)E(s)F(s)^{-1}ds = A + O(h), \tag{10.14}$$

so that $\tilde{B}(t) - B(t) = O(h)$.

5. We will now construct families of pseudodifferential operators $B_k(t)$ so that for each m

$$B(t) = \tilde{B}(t) + B_1(t) + \cdots + B_m(t) + O_{L^2 \to L^2}(h^{m+1}), \quad B_j \in \Psi^{-j}. \tag{10.15}$$

For that let

$$\tilde{e}(t) = (\kappa_t)^* \int_0^t (\kappa_s^{-1})^* e(s)ds,$$

and set $\tilde{E}(t) = \mathrm{Op}(\tilde{e}(t))$. We observe that

$$hD_t\tilde{E}(t) = [Q(t), \tilde{E}] + \frac{h}{i}\left(E(t) + E_1(t)\right),$$

where $E_1(t) = \mathrm{Op}(e_1(t))$, $e_1(t) \in S^{-3}$ by the pseudodifferential calculus. Then, as in Step 4 above,

$$\begin{aligned} hD_t\left(F(t)\tilde{E}(t)F(t)^{-1}\right) &= -F(t)[Q(t),\tilde{E}(t)]F(t)^{-1} \\ &\qquad + F(t)hD_t\left(\tilde{E}(t)\right)F(t)^{-1} \\ &= \frac{h}{i}\left(F(t)E(t)F(t)^{-1} + F(t)E_1(t)F(t)^{-1}\right). \end{aligned}$$

Integrating in t gives

$$F(t)\tilde{E}(t)F(t)^{-1} = \int_0^t F(s)E(s)F(s)^{-1}ds + \int_0^t F(s)E_1(s)F(s)^{-1}ds.$$

This we now substitute in (10.14) obtaining

$$\begin{aligned} \tilde{B}(t) - B(t) &= \frac{i}{h}\tilde{E}(t) - F(t)^{-1}\left(\frac{i}{h}\int_0^t F(s)E_1(s)F(s)^{-1}ds\right)F(t) \\ &= \frac{i}{h}\tilde{E}(t) + O_{L^2\to L^2}(h^2). \end{aligned}$$

Setting $B_1(t) = i\tilde{E}(t)/h \in \Psi^{-1}$, and continuing inductively gives $B_k(t)$ satisfying (10.15).

6. It remains to show that $B(t)$ is a pseudodifferential operator. To do so, we invoke Beals's Theorem 9.8 by showing that for any linear $l_1, \cdots, l_M$, we have the estimate

$$\mathrm{ad}_{l_M}\cdots\mathrm{ad}_{l_1}B(t) = O_{L^2\to L^2}(h^M). \tag{10.16}$$

But this statement is clear from Steps 3 and 5: for any P we can find a pseudodifferential operator $\mathrm{Op}(b_t^P)$, with $b_t^N \in S^{-1}$, such that

$$\begin{aligned} B(t) &= \mathrm{Op}(b_t^P) + R_P(t), \\ \langle x\rangle^N\langle hD_x\rangle^N R_P(t)\langle x\rangle^N\langle hD_x\rangle^N &= O(h^P)\ :\ L^2 \to L^2. \end{aligned}$$

Since

$$\mathrm{ad}_{l_M}\cdots\mathrm{ad}_{l_1}\mathrm{Op}(b_t^N) = O(h^M),$$

and, by a trivial estimate,

$$\mathrm{ad}_{l_M}\cdots\mathrm{ad}_{l_1}R_P(t) = O(h^P)$$

(10.16) follows by choosing $M \geq P$. $\square$

REMARK. The argument used in Step 2 of the proof shows that if in Theorem 10.2 we have

$$a(x,\xi;h) \sim a_0(x,\xi) + ha_1(x,\xi) + \cdots + h^N a_N(x,\xi) + \cdots,$$

for $a_j \in S$, then

$$b_t(x,\xi;h) \sim \gamma_t^* a_0(x,\xi) + h b_t^1(x,\xi) + \cdots h^N b_t^N(x,\xi) + \cdots .$$

However, the higher order terms are difficult to compute. □

10.1.2 Locally defined symplectomorphisms. The requirement that the family of symplectomorphism be global on $\mathbb{R}^{2n}$ is very strong and often invalid in interesting situations. So we now discuss quantization of *locally defined symplectomorphisms*, for which the quantization formula (10.4) holds only locally.

THEOREM 10.3 (Local quantization). *Let $\gamma : U_0 \to U_1$ be a symplectomorphism fixing $(0,0)$ and defined in a neighbourhood of U_0.*

Then there exists a unitary operator

$$F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

such that for all $A = \mathrm{Op}(a)$ with $a \in S$, we have

$$F^{-1}AF = B, \tag{10.17}$$

where $B = \mathrm{Op}(b)$ for a symbol $b \in S$ satisfying

$$b|_{U_0} := \gamma^*(a|_{U_1})|_{U_0} + O(h). \tag{10.18}$$

Proof. 1. According to Theorem 10.1, there exists a piecewise smooth family of symplectomorphisms $\gamma_t : U_0 \to U_t$, $(0 \le t \le 1)$ such that $\gamma = \gamma_1$, $\gamma_0 = I$, and

$$\frac{d}{dt}\gamma_t = (\gamma_t)_* H_{q_t} \qquad (0 \le t \le 1)$$

within U, for a smooth family $\{q_t\}_{0 \le t \le 1}$.

We extend q_t smoothly to be equal to 0 in $\mathbb{R}^{2n} - U_0$ and then define a family of global symplectomorphisms $\tilde{\gamma}_t$ using the now globally defined functions q_t. Observe that

$$\tilde{\gamma}_t|_{U_0} = \gamma_t : U_0 \to U_t;$$

and hence

$$\tilde{\gamma}_t^*(a)|_{U_0} = \gamma_t^*(a|_{U_t})|_{U_0}. \tag{10.19}$$

2. We now apply Theorem 10.2, to obtain the family of operators $\{F(t)\}_{0 \le t \le 1}$. We observe that since the supports of the functions q_t lie in a fixed compact set, the proof of Theorem 10.2 shows that (10.4) holds for $a \in S$. That is,

$$F(t)^{-1}AF(t) = \mathrm{Op}(b(t)) = B(t)$$

for

$$b(t) = \tilde{\gamma}_t^* a + O(h).$$

We now put

$$F := F(1), \quad B := B(1).$$

Then (10.19) shows that formula (10.17) is valid. □

10.1.3 Microlocality. It will prove useful to formulate the theorems above without reference to the global properties of the operator F.

DEFINITIONS. (i) Let U, V be open, bounded subsets of $\mathbb{R}^{2n}$, and assume

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is linear.

We say that T is *tempered* if for each seminorm $\|\cdot\|_1$ on $\mathcal{S}(\mathbb{R}^n)$, there exists another seminorm $\|\cdot\|_2$ and a constant $N \in \mathbb{R}$ such that

$$\|Tu\|_1 = O(h^{-N})\|u\|_2 \tag{10.20}$$

for all $u \in \mathcal{S}$.

(ii) Given two tempered operators T and S, we say that

$$T \equiv S \quad \textit{microlocally} \quad \text{on} \quad U \times V \tag{10.21}$$

if there exist open sets $\tilde{U} \supseteq U$ and $\tilde{V} \supseteq V$ such that

$$A(T - S)B = O(h^\infty)$$

as a mapping $\mathcal{S} \to \mathcal{S}$, for all A, B such that

$$\mathrm{WF}_h(A) \subset \tilde{V}, \ \mathrm{WF}_h(B) \subset \tilde{U}.$$

(iii) In particular, we say

$$T \equiv I \quad \textit{microlocally} \ \text{ near } U \times U$$

if there exists an open set $\tilde{U} \supseteq U$ such that

$$A - TA = A - AT = O(h^\infty)$$

as mappings $\mathcal{S} \to \mathcal{S}$, for all A with $\mathrm{WF}_h(A) \subset \tilde{U}$.

(iv) We will say that T is *microlocally invertible* near $U \times U$ if there exists an operator S such that $TS \equiv I$ and $ST \equiv I$ microlocally near $U \times U$.

When no confusion is likely, we write

$$S = T^{-1}$$

and call S a *microlocal inverse* of T.

LEMMA 10.4 (Wavefront sets and composition). *If*

$$\mathrm{WF}_h(A) \cap U = \emptyset$$

and $B = \mathrm{Op}(b)$ *for* $b \in S$, *then*

$$\mathrm{WF}_h(BA) \cap U = \emptyset. \tag{10.22}$$

Proof. The symbol of BA is $b \# a = O(h^\infty)$ in U. □

LEMMA 10.5 (Tempered unitary transformations). *The unitary transformations* $F(t)$ *given by Theorem 10.2 are tempered.*

Proof. Up to powers of h, each seminorm on $\mathcal{S}$ is bounded from above and below by these specific seminorms:

$$u \mapsto \|A_N u\| \quad \text{for} \quad A_N := (1 + |x|^2 + |hD|^2)^N.$$

We observe that the operators A_N are invertible and selfadjoint and that, in the notation of the proof of Theorem 10.2

$$A_N Q(t) A_N^{-1} = Q_N(t) = \mathrm{Op}(q_t^N),$$

for $q_t^N \in S(m)$ such that $q_t^N - q_t \in S^{-1}(m)$.

We then have

$$hD_t A_N F(t) A_N^{-1} = A_N F(t) A_N^{-1} Q_N(t);$$

and hence the same arguments as in Step 3 of the proof of Theorem 10.4,

$$\|A_N F(t) u\|^2 \leq C \|A_N u\|^2.$$

Consequently for any seminorm $\| \cdot \|_1$ on $\mathcal{S}$, there exists a seminorm $\| \cdot \|_2$ and N such that

$$\|F(t) u\|_1 \leq O(h^{-N}) \|u\|_2.$$

□

The previous two lemmas and Theorem 10.3 give

THEOREM 10.6 (More on local quantization). *Let* $\gamma : U_0 \to U_1$ *be a symplectomorphism fixing* $(0, 0)$ *and defined in a neighbourhood of* U_0. *Suppose* U *is open,* $\overline{U} \subset\subset U_0 \cap U_1$.

Then there exists a tempered operator

$$F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

such that F *is microlocally invertible near* $U \times U$ *and for all* $A = \mathrm{Op}(a)$, *with* $a \in S$,

$$F^{-1} A F = B \qquad \textit{microlocally near } U \times U, \tag{10.23}$$

where $B = \mathrm{Op}(b)$ *for a symbol* $b \in S$ *satisfying*

$$b := \gamma^* a + O(h). \tag{10.24}$$

In (10.24) we do not specify the neighbourhoods, as we did in (10.19), since the statement needs to make sense only locally near $U \times U$.

The last theorem has the following converse which we include for completeness:

THEOREM 10.7 (Converse). *Suppose that* $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is a tempered operator such that for every* $A = \mathrm{Op}(a)$ *with* $a \in S$, *we have*

$$AF \equiv FB$$

microlocally near $(0,0)$, *for*

$$B = \mathrm{Op}(b), \quad b = \gamma^* a + O(h),$$

where $\kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *is a symplectomorphism, defined locally near* U, *with* $\kappa(0,0) = (0,0)$.

Then there exists a pseudodifferential operator F_0, *elliptic near* U, *and a family of self-adjoint pseudodifferential operators* $Q(t)$, *such that*

$$F = F(1) \quad \text{microlocally near } U \times U,$$

where

$$\begin{cases} hD_t F(t) + F(t)Q(t) = 0 & (0 \le t \le 1) \\ F(0) = F_0. \end{cases}$$

Proof. 1. From Theorem 10.1 we know that there exists a family of local symplectomorphisms, γ_t, satisfying $\gamma_t(0,0) = (0,0)$, $\gamma_1 = \gamma$ and $\gamma_0 = I$. Since we are working locally, there exists a function q_t so that γ_t is generated by its Hamiltonian vectorfield H_{q_t}.

As in the proof of Theorem 10.3 we extend this function to be zero outside a compact set. Let us now consider the dynamics

$$\begin{cases} hD_t F(t) = Q(t)F(t) & (0 \le t \le 1) \\ F(1) = CFC, \end{cases}$$

where C is a pseudodifferential operator with $\mathrm{WF}_h(I - C) \cap U = \emptyset$.

2. We claim that $F(0)$ satisfies

$$\mathrm{Op}(a)F(0) = F(0)\mathrm{Op}(a + h\tilde{a}) \tag{10.25}$$

for $a, \tilde{a} \in \mathcal{S} \cap S^0(1)$. To establish this, let us introduce $V(t)$ satisfying

$$\begin{cases} hD_t V(t) + V(t)Q(t) = 0 & (0 \leq t \leq 1) \\ V(0) = I. \end{cases}$$

Then using Theorem 10.3 and the assumption that $b = \gamma^* a + 0(h)$, we deduce that

$$\begin{aligned} \mathrm{Op}(a)F(t)V(t) &= F(t)\mathrm{Op}(b)V(t) \\ &= F(t)V(t)(V(t)^{-1}\mathrm{Op}(b)V(t)) \\ &= F(t)V(t)\mathrm{Op}(a + h\tilde{a}). \end{aligned}$$

Putting $t = 0$ gives (10.25).

3. We now use Beals's Theorem to conclude that $F(0) \in \Psi^0$. We verify the hypothesis by induction: suppose we know that

$$\mathrm{ad}_{\mathrm{Op}(b_1)} \cdots ad_{\mathrm{Op}(b_N)} F(0) = O(h^N),$$

for any $b_j \in S^0(1)$. Then by (10.25)

$$\mathrm{Op}(b_{N+1})F(0) - F(0)\mathrm{Op}(b_{N+1}) = h\mathrm{Op}(\tilde{b}_{N+1})F(0);$$

and hence

$$\begin{aligned} &\|\mathrm{ad}_{\mathrm{Op}(b_1)} \cdots ad_{\mathrm{Op}(b_N)} \mathrm{ad}_{\mathrm{Op}(b_{N+1})} F(0)\|_{L^2 \to L^2} = \\ &\qquad \|h\mathrm{ad}_{\mathrm{Op}(b_1)} \cdots \mathrm{ad}_{\mathrm{Op}(b_N)}(\mathrm{Op}(\tilde{b}_{N+1})F(0))\|_{L^2 \to L^2} = O(h^{N+1}), \end{aligned}$$

according to the induction hypothesis and the derivation property

$$\mathrm{ad}_A(BC) = B(\mathrm{ad}_A C) + (\mathrm{ad}_A B)C.$$

Hence Beals's Theorem applies and shows that $F(0)$ is a pseudodifferential operator. By construction, $F(1) = CFC \equiv F$ near $(0, 0)$. □

10.1.4. Quantization of linear symplectic maps.

CHECK DEFINITION OF J Consider first the simple linear symplectic transformation $\gamma = J$; that is,

$$\gamma(x, \xi) = (-\xi, x) \tag{10.26}$$

on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

Then we can take for $0 \leq t \leq 1$,

$$\gamma_t(x, \xi) = \left(\cos\left(\frac{t\pi}{2}\right) x - \sin\left(\frac{t\pi}{2}\right) \xi, \sin\left(\frac{t\pi}{2}\right) x + \cos\left(\frac{t\pi}{2}\right) \xi\right);$$

so that

$$\frac{d\gamma_t}{dt} = (\gamma_t)_* H_q,$$

for

$$q := \frac{\pi}{4}(|x|^2 + |\xi|^2).$$

THEOREM 10.8 (J quantized). *The operator F associated with the transformation* (10.26) *as in Theorem 10.3 is*

$$Fu(x) := \frac{e^{-\frac{\pi}{4}i}}{(2\pi h)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{i\langle x,y\rangle}{h}} u(y)\, dy = \frac{e^{-\frac{\pi}{4}i}}{(2\pi h)^{\frac{n}{2}}} \mathcal{F}_h u. \tag{10.27}$$

Proof. 1. To verify this, we first show that for $a \in \mathcal{S}'$ we have

$$a^{\mathrm{w}}(x, hD) \circ F = F \circ a^{\mathrm{w}}(-hD, x); \tag{10.28}$$

that is, the conclusion of Theorem 10.2 holds without any error terms. As in the proof of that theorem, we see that

$$hD_t A_t = \frac{\pi}{4}[-h^2\Delta + |x|^2, A_t]$$

for

$$A_t := F(t)^{-1} a^{\mathrm{w}}(x, hD) F(t).$$

Let $l(x, \xi)$ be a linear function on $\mathbb{R}^{2n}$ and consider the exponential symbol

$$a_t(x, \xi) := \exp(\gamma_t^* l(x, \xi)/h)$$

and its Weyl quantization

$$a_t^{\mathrm{w}}(x, hD) = \exp(\gamma_t^* l(x, hD)/h).$$

An explicit computation reveals that

$$hD_t a_t(x, hD) = \frac{\pi}{4}[-h^2\Delta + |x|^2, a_t(x, hD)].$$

Since any Weyl operator is a superposition of exponentials of l's (recall (**??**)), assertion (10.28) follows.

2. Suppose now that $\widetilde{F}$ is another unitary operator for which (10.28) holds. Then $\tilde{F} = cF$ for $c \in \mathbb{C}$, $|c| = 1$, as follows from applying Lemma 3.3 to $L = F^* \widetilde{F}$. Since the Fourier transform satisfies (10.28) and $(2\pi h)^{-n/2} \mathcal{F}_h$ is unitary, we deduce that

$$F = \frac{c}{(2\pi h)^{\frac{n}{2}}} \mathcal{F}_h.$$

3. Thus it remains to compute the constant c. For this, let us put $u_0 = \exp(-|x|^2/2)$ and consider the ODE

$$\begin{cases} hD_t u(t) = \frac{\pi}{4}(-h^2\Delta + |x|^2) u(t), \\ \qquad u(0) = u_0. \end{cases}$$

Recalling (10.7), we see that $u(t) = F(t)^* u_0$. Since u_0 is the ground state of the harmonic oscillator with eigenvalue h, we learn that $u(t) = a(t)u_0$, where $a(t)$ solves the ODE

$$\begin{cases} \frac{d}{dt}a(t) = \frac{\pi i}{4}a(t) \\ a(0) = 1; \end{cases}$$

that is, $a(t) = \exp(\pi i t/4)$. Finally, we note that

$$e^{\pi i/4}u_0 = F(1)^* u_0 = \bar{c}(2\pi h)^{-n/2}\mathcal{F}_h u_0 = \bar{c}u_0;$$

whence

$$c = \exp(-\pi i/4).$$

□

REMARK. The family of canonical transformations γ_t $(0 \leq t \leq 1)$, used here can be extended to a periodic family of canonical transformations: $\gamma_{t+4} = \gamma_t$ $(t \in \mathbb{R})$. Extending $F(t)$ using (10.7), we see that the argument above gives

$$F(4k) = (-1)^k I, \quad \gamma_{4k} = I.$$

Consequently on the quantum level the deformation produces an additional shift in the phase. This shift has an important geometric and physical interpretation and is related to the *Maslov index*. For a brief discussion and references see [**?**, Sect.7]. □

REMARK: Quantizing linear symplectic mappings. Using Step 1 in the proof of Theorem 10.1, we can in fact quantize any linear symplectic transformation. So given

$$K : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \qquad K = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$C^* A = A^* C, \quad D^* B = B^* D, \quad D^* A - B^* C = I,$$

we can construct $F_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ satisfying

$$F_K^* F_K = F_K F_K^* = I, \quad a^{\mathrm{w}}(x, hD) \circ F_K = F_K \circ (K^* a)^{\mathrm{w}}(x, hD).$$

The operator F_K is unique up to a multiplicative factor; and hence

$$F_{K_1} \circ F_{K_2} = c F_{K_1 \circ K_2}, \quad |c| = 1.$$

The association $K \mapsto F_K$ can in fact be chosen so that $c = \pm 1$; therefore it is *almost* a representation of the group of symplectic transformations. To make it a representation, one has to move to the double cover of the symplectic group, the so-called the metaplectic group. Unitary operators quantizing linear symplectic transformations are consequently called *metaplectic operators*: see Dimassi–Sjöstrand [D-S, Appendix to Chapter 7] for a self-contained presentation in the semiclassical spirit, and Folland [**?**, Chapter 4] for more and for references. □

EXAMPLE: A invertible. For reasons already apparent in the discussion of the Fourier transform, there cannot be a general formula for the kernel F_K in terms of the entries A, B, C, D of K.

But if $\det A \neq 0$, we have for $u \in \mathcal{S}$ the formula

$$F_K u(x) = \frac{(\det A)^{-\frac{1}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\varphi(x,\eta) - \langle y,\eta\rangle)} u(y)\, dy d\xi, \tag{10.29}$$

where

$$\varphi(x,\eta) := -\frac{1}{2}\langle CA^{-1}x, x\rangle + \langle A^{-1}x, \eta\rangle + \frac{1}{2}\langle A^{-1}B\eta, \eta\rangle. \tag{10.30}$$

We will refer to this formula in our next example. □

10.2 SEMICLASSICAL ANALYSIS OF PROPAGATORS

In this section we consider the flow of symplectic transformations

$$\gamma_t = \exp(tH_p), \tag{10.31}$$

generated by the real-valued symbol $p \in S(m)$.

Let $P = \mathrm{Op}(p)$. Then in the notation of Theorem 10.2, $F(t) = e^{-itP/h}$ solves

$$\begin{cases} (hD_t + P)F(t)u = 0 \\ \qquad\qquad F(0)u = u \end{cases}$$

for $u \in \mathcal{S}$. In this case, Theorem 10.2 reproduces Egorov's Theorem 8.2: if $a \in \mathcal{S}$, then

$$e^{itP/h}\mathrm{Op}(a)e^{-itP/h} = \mathrm{Op}(b_t),$$

for

$$b_t = (\exp tH_p)^* a + O(h).$$

A Fourier integral representation formula. Our goal now is to find for small times $t_0 > 0$ a microlocal representation of $F(t)$ as an oscillatory integral. In other words, we would like to find an operator $U(t)$ so that for each h dependent family, $u \in \mathcal{S}$ with $\mathrm{WF}_h(u) \subset\subset \mathbb{R}^{2n}$, we have

$$\begin{cases} hD_t U(t)u + PU(t)u = O(h^\infty) \qquad (-t_0 \leq t \leq t_0) \\ \qquad\qquad\qquad U(0)u = u. \end{cases} \tag{10.32}$$

Using Duhamel's formula, we can then deduce that

$$F(t) - U(t) = O(h^\infty).$$

THEOREM 10.9 (Oscillatory integral representation). *We have the representation*
(10.33)
$$U(t)u(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\varphi(t,x,\eta)-\langle y,\eta\rangle)} b(t,x,\eta;h)u(y)\,dyd\eta,$$
for the phase φ and amplitude b as defined below.

The proof will appear after the following constructions of the phase and amplitude.

Construction of the phase function. We start by finding the phase function φ as a local generating function associated with the symplectomorphisms (10.31). (Recall the discussion in §2.3 of generating functions.)

Let U denote a bounded open set containing $(0,0)$.

LEMMA 10.10 (Hamilton–Jacobi equation). *If $t_0 > 0$ is small enough, there exists a smooth function*
$$\varphi = \varphi(t,x,\eta)$$
defined in $(-t_0,t_0)\times U \times U$, such that
$$\gamma_t(y,\eta) = (x,\xi)$$
locally if and only if
$$\xi = \partial_x\varphi(t,x,\eta), \quad y = \partial_\eta\varphi(t,x,\eta). \tag{10.34}$$
Furthermore, φ solves the Hamilton–Jacobi equation
$$\begin{cases} \partial_t\varphi(t,x,\eta) + p(x,\partial_x\varphi(t,x,\eta)) = 0 \\ \varphi(0,x,\eta) = \langle x,\eta\rangle. \end{cases} \tag{10.35}$$

Proof. 1. We know that for points (y,η) lying in a compact subset of $\mathbb{R}^{2n}$, the flow
$$(y,\eta)\mapsto \gamma_t(y,\eta) \tag{10.36}$$
is surjective near $(0,0)$ for times $0 \le t \le t_0$, provided t_0 is small enough. This is so since $\gamma_0(y,\eta) = (y,\eta)$.

2. To show the existence of φ, consider
$$\Lambda := \{(t,p(y,\eta);\gamma_t(y,\eta);y,\eta) : t\in\mathbb{R}, (y,\eta)\in\mathbb{R}^{2n}\},$$

This is a surface in $\mathbb{R}^2 \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, a typical point of which we will write as $(t, \tau, x, \xi, y, \eta)$. Introduce the one-form

$$V := -\tau dt + \sum_{j=1}^{n} \xi_j dx_j + \sum_{j=1}^{n} y_j d\eta_j.$$

That γ_t is a symplectic implies $dV|_\Lambda = 0$. By Poincaré's Lemma (Theorem B.4), there exists a smooth function φ such that

$$d\varphi = V.$$

In view of (10.36) we can use (t, x, η) as coordinates on $\Lambda \cap ((-t_0, t_0) \times U \times U)$; and hence

$$-\tau dt + \sum_{j=1}^{n} \xi_j dx_j + \sum_{j=1}^{n} y_j d\eta_j = \partial_t \varphi dt + \sum_{j=1}^{n} \partial_{x_j} \varphi dx_j + \sum_{j=1}^{n} \partial_{\eta_j} \varphi d\eta_j.$$

Comparing the terms on the two sides gives (10.34) and (10.35). □

Construction of the amplitude. The amplitude b in (10.33) must satisfy

$$(hD_t + p^{\mathrm{w}}(x, hD))(e^{i\varphi(t,x,\eta)/h} b(t, x, \eta; h)) = O(h^\infty);$$

and so

$$(\partial_t \varphi + hD_t + e^{-i\varphi/h} p^{\mathrm{w}}(x, hD) e^{i\varphi/h}) b(t, x, \eta; h) = O(h^\infty), \tag{10.37}$$

for (x, η) in a neighbourhood of $U \times U$, $0 \leq t \leq t_0$.

We will build b as an expansion in powers of h:

$$b(t, x, \eta; h) \sim b_0(t, x, \eta) + h b_1(t, x, \eta) + h^2 b_2(t, x, \eta) + \cdots . \tag{10.38}$$

Once all the terms b_j are computed, Borel's Theorem 4.11 produces the amplitude b.

LEMMA 10.11 (Calculation of b_0). *We have*

$$b_0(t, x, \eta) = (\det \partial^2_{\eta x} \varphi(t, x, \eta))^{\frac{1}{2}}. \tag{10.39}$$

Note that $\det \partial^2_{\eta x} \varphi > 0$ for $0 \leq t \leq t_0$, if t_0 is sufficiently small.

Proof. 1. We first observe that

$$e^{-i\varphi/h} p^{\mathrm{w}}(x, hD) e^{i\varphi/h} = q_t(x, hD; h),$$

where

$$q_t(x, \xi; h) = p(x, \partial_x \varphi + \xi) + O(h^2). \tag{10.40}$$

In fact, writing $\varphi(x) - \varphi(y) = F(x,y)(x-y)$, we easily check that

$$e^{-i\varphi/h}p^{\mathrm{w}}(x,hD)e^{i\varphi/h}u =$$
$$\iint a\left(\frac{x+y}{2}, \xi + F(x,y)\right) e^{i\langle x-y,\xi\rangle/h}u(y)dyd\xi,$$

where

$$F(x,y) = \partial_x\varphi\left(\frac{x+y}{2}\right) + O((x-y)^2).$$

Hence,

$$e^{-i\varphi/h}p^{\mathrm{w}}(x,hD)e^{i\varphi/h}u = \iint (a((x+y)/2, \xi + \partial_x\varphi((x+y)/2))$$
$$+\langle e(x,y,\xi)(x-y),(x-y)\rangle)e^{i\langle x-y,\xi\rangle/h}u(y)dyd\xi,$$

where the entries of the matrix valued function e are in S. Integration by parts based on (9.65) gives (10.40).

2. Recalling from Lemma 10.10 that $\partial_t\varphi = -p(x,\partial_x\varphi)$, we then deduce from (10.37) that

$$(hD_t + f_t^{\mathrm{w}}(x,hD,\eta))b(t,x,\eta) = O(h^2), \tag{10.41}$$

where

$$f_t(x,\xi) := p(x, \partial_x\varphi(t,x,\eta) + \xi) - p(x,\partial_x\varphi(t,x,\eta)),$$

and where η considered as a parameter. So

$$f_t(x,\xi,\eta) = \sum_{j=1}^n \xi_j\partial_{\xi_j}p(x,\partial_x\varphi(t,x,\eta)) + O(|\xi|^2).$$

Hence for $g = g(t,x,\eta) \in S$,

$$f_t^{\mathrm{w}}(x,hD,\eta)g = \frac{1}{2}\sum_{j=1}^n \left((\partial_{\xi_j}p)hD_{x_j}g + hD_{x_j}\left(\partial_{\xi_j}p\, g\right)\right) + O(h^2),$$

in which expression the derivatives of p are evaluated at $(x,\partial_x\varphi(t,x,\eta))$. Consequently b_0 satisfies:

$$hD_tb_0 + \frac{1}{2}\sum_{j=1}^n(\partial_{\xi_j}p)hD_{x_j}b_0 + hD_{x_j}(\partial_{\xi_j}p\; b_0) = 0.$$

This we rewrite as

$$(\partial_t + V_t + \frac{1}{2}\operatorname{div} V_t)b_0 = 0 \tag{10.42}$$

with

$$V_t := \sum(\partial_{\xi_j}p)\partial_{x_j}.$$

3. To understand this equation geometrically, we consider $b_0(t,\cdot,\eta)$ as a function on

$$\Lambda_{t,\eta} := \{(x, \partial_x\varphi(t,x,\eta))\}.$$

Then

$$\gamma_{s,t} \ : \Lambda_{t-s,\eta} \ \to \Lambda_{t,\eta},$$

$$\frac{d}{ds}\gamma_{s,t}^* u|_{s=0} = H_p|_{\Lambda_{t,\eta}} u = V_t u,$$

for $u \in C^\infty$. But equation (10.42) can be further rewritten as

$$\frac{d}{dt}\gamma_t^* b_0(t,\cdot,\eta) = -\frac{1}{2}\gamma_t^*(\text{div } V_t \; b_0(t,\cdot,\eta)). \tag{10.43}$$

We claim next that

$$\gamma_t^* b_0(t,x,\eta) = |\partial\gamma_t|^{-\frac{1}{2}}\,, \tag{10.44}$$

is the solution of (10.43) satisfying $b_0(0,x,\eta) = 1$. Here γ_t is considered as a function $\Lambda_{0,\eta} \to \Lambda_{t,\eta}$. In fact,

$$\begin{aligned}
\frac{d}{dt}|\partial\gamma_t|^{-\frac{1}{2}} &= \frac{d}{ds}|\partial\gamma_t \circ \kappa_{s,t}|_{s=0}^{-\frac{1}{2}} \\
&= \frac{d}{ds}|\partial\gamma_t|^{-\frac{1}{2}}\gamma_t^* \, |\partial\gamma_{s,t}|_{s=0}^{-\frac{1}{2}} \\
&= -\frac{1}{2}\gamma_t^* \text{div } V_t \, |\partial\gamma_t|^{-\frac{1}{2}}\,.
\end{aligned}$$

4. To obtain an explicit formula for b_0, we recall that

$$\gamma_t^{-1} \ : \ (x, \partial_x\varphi(t,x,\eta)) \to (\partial_\eta\varphi(t,x,\eta), \eta).$$

Hence

$$\partial(\gamma_t^{-1}|_{\Lambda_{t,\eta}}) = \partial^2_{\eta x}\varphi(t,x,\eta),$$

and consequently, from (10.44), we see that (10.39) holds. □

Proof of Theorem 10.9 Using the same argument for the higher order terms in b, we can find its full expansion with all the equations valid in $(-t_0,t_0)\times U$. That shows that $U(t)$ given by (10.33) satisfies (10.32), and thereby completes the proof of Theorem 10.9. □

EXAMPLE. Revisiting example (10.29), we see that for the phase (10.30) the corresponding amplitude is

$$b_0 = (\det \partial^2_{x\eta}\varphi(x,\eta))^{1/2} = (\det A)^{-1/2}.$$

□

REMARK: Amplitudes as half-densities. The somewhat cumbersome derivation of the formula for b_0, the leading term of the amplitude

b in (10.33), becomes much more natural when we use half-densities, introduced earlier in Section 9.1.

We first make a general observation. If $a := u|dx|^{\frac{1}{2}}$ is a half-density, and γ_t is a family of diffeomorphisms generated by a family of vector-fields:

$$\frac{d}{dt}\gamma_t = (\gamma_t)_* V_t,$$

then

$$\mathcal{L}_{V_t} a := \frac{d}{dt}\gamma_t^* a = (V_t u + (\operatorname{div} V_t/2)u)|dx|^{\frac{1}{2}}. \tag{10.45}$$

Indeed,

$$\gamma_t^* a = \gamma_t^* u |\partial\gamma_t|^{\frac{1}{2}} |dx|^{\frac{1}{2}};$$

and if we define

$$\gamma_{s,t}(x) := \gamma_{t+s}(\gamma_t^{-1}(x)), \quad \frac{d}{ds}\gamma_{s,t}(x)|_{s=0} = V_t(x),$$

then

$$\frac{d}{dt}|\partial\gamma_t|^{\frac{1}{2}} = \frac{d}{ds}|\partial\gamma_t \circ \kappa_{s,t}|^{\frac{1}{2}} = \frac{1}{2}|\partial\gamma_t|^{\frac{1}{2}}\kappa_t^* \operatorname{div} V_t.$$

This means that if we consider $b_0(t,x,\eta)|dx|^{\frac{1}{2}}$ as a half-density on $\Lambda_{t,\eta}$ then (10.42) becomes

$$(d/dt)\gamma_t^*(b_0|dx|^{\frac{1}{2}}) = (\partial_t + \mathcal{L}_{V_t})(b_0(t,x,\eta)|dx|^{\frac{1}{2}}) = 0.$$

This is the same as

$$\gamma_t^*(b_0(t,x,\eta)|dx|^{\frac{1}{2}}|_{\Lambda_{\varphi_{t,\eta}}}) = |dx|^{\frac{1}{2}}|_{\Lambda_{\varphi_{0,\eta}}}.$$

It follows that $\gamma_t^* b_0 = |\partial\gamma_t|^{-1/2}$, the same conclusion as before.

It is appealing that the amplitude, interpreted as a half-density, is invariant under the flow. When coordinates change, and in particular when we move to larger times at which (10.34) and (10.35) are no longer valid, the statement about the amplitude as a half-density remains simple. □

REMARK: A more general version of oscillatory integral representation.

If we examine the proof of Theorem 10.9 we notice that we did not use the fact that $P = p^{\mathrm{w}}(x,hD)$ is t independent. That means that we can consider the solution of a more general problem,

$$\begin{cases} (hD_t + P(t))F(t)u = 0 \\ \qquad\qquad F(0)u = u \end{cases} \tag{10.46}$$

where

$$P(t) = p^{\mathrm{w}}(t,x,hD), \quad p(t,x,\xi) \in C^\infty(\mathbb{R}_t, S(\mathbb{R}^{2n}_{x,\xi}, m)).$$

For the approximate solution of this problem we still have the same oscillatory integral representation as the one give in Theorem 10.9. In particular that means that we have an oscillatory integral representation of the family of operators defined in Theorem 10.3 for small values of t there.

For the yet more general problem of p depending on h we refer to [**?**, Section 7] and references given there. Here we note that the proof works for $P(t) = p^{\mathrm{w}}(t,x,hD) + h^2 p_2^{\mathrm{w}}(t,x,hD)$ and that form of operators acting on half-densities is invariant (see Theorem 9.12).

10.3 SEMICLASSICAL STRICHARTZ ESTIMATES, L^p BOUNDS

In this section we will use Theorem 10.9 to obtain L^p bounds on approximate solutions to

Let $a = a(t,x,\xi) \in C^\infty(\mathbb{R}, S(T^*\mathbb{R}^k, m))$. We introduce the following nondegeneracy condition at (t,x,ξ):

$$\partial_\xi^2 a(t,x,\xi) \ \text{ is non-degenerate .} \tag{10.47}$$

REMARK. The Hessian, $\partial_\xi^2 f(\xi_0)$, of a smooth function $f(\xi)$ is not invariantly defined unless $\partial_\xi f(\xi_0) = 0$. However the statement (10.47) is invariant if only *linear* transformations in ξ are allowed. That is the case for symbol transformation induced by changes of variables in x, see Theorem 9.12.

We consider the problem which essentially the same as (10.46):

$$\begin{cases} (hD_t + A(t))F(t,r)u = 0 \\ \qquad\qquad F(r,r)u = u \end{cases} \tag{10.48}$$

where $r \in \mathbb{R}$. As discussed in the remark at the end of Section 10.2, Theorem 10.9 gives a description of $F(t,r)$ for small values of t.

THEOREM 10.12 (Semiclassical Strichartz estimates). *Suppose that $a(t) \in C^\infty(\mathbb{R}_t, S(T^*\mathbb{R}^k, m))$, is real valued, $\chi \in C_c^\infty(T^*\mathbb{R}^k)$, and that (10.47) holds in $\operatorname{spt}(\chi)$, $t \in \mathbb{R}$. With $A(t) := a^{\mathrm{w}}(t,x,hD)$, let $F(t,r)$ be the solution of (10.48). Then for $\psi \in C_c^\infty(\mathbb{R})$ with support sufficiently close to 0, any $I \subset\subset \mathbb{R}$, and*

$$U(t,r) := \psi(t)F(t,r)\chi^{\mathrm{w}}(x,hD) \quad \textit{or} \quad U(t,r) := \psi(t)\chi^{\mathrm{w}}(x,hD)F(t,r)$$

we have

$$\text{(10.49)}\qquad \begin{gathered}\sup_{r\in I}\left(\int_{\mathbb{R}}\|U(t,r)f\|^p_{L^q(\mathbb{R}^k)}dt\right)^{\frac{1}{p}}\leq Bh^{-\frac{1}{p}}\|f\|_{L^2(\mathbb{R}^k)},\\ \frac{2}{p}+\frac{k}{q}=\frac{k}{2},\quad 2\leq p\leq\infty,\quad 1\leq q\leq\infty,\ (p,q)\neq(2,\infty).\end{gathered}$$

Proof: 1. In view of Theorem C.10 we need to show that

$$\text{(10.50)}\qquad \|U(t,r)U(s,r)^*f\|_{L^\infty(X,\mu)}\leq Ah^{-k/2}|t-s|^{-k/2},\quad t,s\in\mathbb{R},$$

with constants independent of $r\in I$. We can put $r=0$ in the argument and drop the dependence on r in U and F.

2. We use Theorem 10.9. The construction there and the assumption that $\chi\in C^\infty_{\mathrm{c}}$ show that

$$U(t)=\widetilde{U}(t)+E(t),$$

where

$$E(t)=O(h^\infty):\mathcal{S}'\to\mathcal{S},$$

and the Schwartz kernel of $\widetilde{U}(t)$ is

$$\text{(10.51)}\qquad \begin{gathered}\widetilde{U}(t,x,y)=\frac{1}{(2\pi h)^k}\int_{\mathbb{R}^k}e^{\frac{i}{h}(\varphi(t,x,\eta)-\langle y,\eta\rangle)}\tilde{b}(t,y,x,\eta;h)d\eta,\\ \tilde{b}\in S\cap C^\infty_{\mathrm{c}}(\mathbb{R}^{1+3k}),\quad \varphi(0,x,\eta)=\langle x,\eta\rangle,\\ \partial_t\varphi(t,x,\eta)+a(t,x,\partial_x\varphi(t,x,\eta))=0.\end{gathered}$$

3. Hence we only need to prove (10.50) with U replaced by $\widetilde{U}$ and that means that we need an L^∞ bound on the Schwartz kernel of $W(t,s):=\widetilde{U}(t)\widetilde{U}(s)^*$:

$$W(t,s,x,y)=\frac{1}{(2\pi h)^{2k}}\int_{\mathbb{R}^{3k}}e^{\frac{i}{h}(\varphi(t,x,\eta)-\varphi(s,y,\zeta)-\langle z,\eta-\zeta\rangle)}\,B\,dzd\zeta d\eta,$$

where

$$B=B(t,s,x,y,z,\eta,\zeta;h)\in S\cap C^\infty_{\mathrm{c}}(\mathbb{R}^{2+6k}).$$

4. The phase is nondegenerate in (z,ζ) variables and stationary for $\zeta=\eta$, $z=\partial_\zeta\varphi(s,y,\zeta)$. Hence we can apply Theorem 3.15 to obtain

$$W(t,s,x,y)=\frac{1}{(2\pi h)^k}\int_{\mathbb{R}^k}e^{\frac{i}{h}(\varphi(t,x,\eta)-\varphi(s,y,\eta))}\,B_1(t,s,x,y,\eta;h)\,d\eta,$$

where $B_1\in S\cap C^\infty_{\mathrm{c}}(\mathbb{R}^{2+3k})$. We now rewrite the phase as follows:

$$\begin{aligned}\widetilde{\varphi}:=\varphi(t,x,\eta)-\varphi(s,y,\eta)&=(t-s)\left(a\left(0,x,\eta\right)+O(|t|+|s|)\right)\\ &\quad+\langle x-y,\eta+sF(s,x,y,\eta)\rangle,\quad F\in C^\infty(\mathbb{R}^{1+3k}),\end{aligned}$$

where using (10.51) we wrote

$$\varphi(s,x,\eta)-\varphi(s,y,\eta)=\langle x-y,\eta\rangle+\langle x-y,sF(s,x,y,\eta)\rangle.$$

5. The phase is stationary when

$$\partial_\eta\widetilde{\varphi}=(I+s\partial_\eta F)(x-y)+(t-s)(\partial_\eta a+O(|t|+|s|))=0,$$

and in particular, for s small, having a stationary point implies

$$x-y=O(t-s),$$

as then $(I+s\partial_\eta F)$ is invertible. The Hessian is given by

$$\begin{aligned}\partial_\eta^2\widetilde{\varphi}&=s\left(\partial_\eta^2F\right)(x-y)+(t-s)\left(\partial_\eta^2a+O(|t|+|s|)\right)\\&=(t-s)\left(\partial_\eta^2a+O(|t|+|s|)\right),\end{aligned}$$

where $\partial_\eta^2a=\partial_\eta^2a(0,x,\eta)$.

6. Hence, for t and s sufficiently small, that is for a suitable choice of the support of ψ in the definition of $U(\cdot)$, the nondegeneracy assumption (10.47) implies that at the critical point

$$\partial_\eta^2\widetilde{\varphi}=(t-s)\psi(x,y).$$

If $|t-s|>Mh$ where M is a large constant we can use the stationary phase estimate in Theorem 3.15 to see that

$$|W(t,s,x,y)|\leq Ch^{-k/2}|t-s|^{-k/2}.$$

When $|t-s|<Mh$ we see that the crude upper bound on the integral gives

$$\begin{aligned}|W(t,s,x,y)|&\leq\frac{1}{(2\pi h)^{2k}}\int_{\mathbb{R}^{3k}}|B(t,s,x,y,z,\eta,\zeta;h)|dzd\zeta d\eta\\&\leq Ch^{-k}\leq C'h^{-k/2}|t-s|^{-k/2},\end{aligned}$$

which is what we need to apply Theorem C.10. □

We formulate the following nondegeneracy assumptions at $(x_0,\xi_0)\in T^*\mathbb{R}^n$:

$$p(x_0,\xi_0)=0\implies\partial_\xi p(x_0,\xi_0)\neq0. \tag{10.52}$$

Consequently, the set

$$\{\xi\ :\ p(x_0,\xi)=0\}$$

is a smooth hypersurface in $\mathbb{R}^n$ near ξ_0. We then assume that the second fundamental form of this hypersuface is nondegenerate at ξ_0.

We can reformulate this as follows. By a linear change of variables assume that $\partial_\xi p(x_0,\xi_0)=(\rho,0,\cdots,0)$, $\rho\neq0$. Then near (x_0,ξ_0),

$$p(x,\xi)=e(x,\xi)(\xi_1-a(x,\xi')),$$

and our assumption on the curvature becomes

$$\partial^2_{\xi'} a(x_0, \xi'_0) \ \text{ is nondegenerate.} \tag{10.53}$$

THEOREM 10.13 (L^p bounds on approximate solutions). *Suppose that $u(h)$, $\|u(h)\|_{L^2} = 1$, satisfies the frequency localization condition* (9.8)*. Suppose also that* (10.53) *is satisfied in $WF_h(u)$, and that*

$$p^{\mathrm{w}}(x, hD)u(h) = O_{L^2}(h). \tag{10.54}$$

Then for $p = 2(n+1)/(n-1)$, and any $K \subset\subset \mathbb{R}^n$,

$$\|u(h)\|_{L^p(K)} = O(h^{-1/p}). \tag{10.55}$$

REMARK. The first example in the remark after Theorem 9.5 shows that the curvature condition (10.53) is in general necessary. In fact, if $P(h) = hD_{x_1}$ and $u(h) = h^{-(n-1)/2}\chi(x_1)\chi(x'/h)$ then for $p = 2(n+1)/(n-1)$,

$$\|u\|_{L^p} \simeq h^{(n-1)(1/p-1/2)} = h^{-(n-1)/(n+1)} \neq O(h^{-1/p}).$$

However for the simplest case in which (10.53) holds,

$$p(x, \xi) = \xi_1 - \xi_2^2 - \cdots - \xi_n^2,$$

the estimate (10.55) is optimal. To see that put

$$u(h) := h^{-(n-1)/4}\chi_0(x_1)\exp(-|x'|^2/2h),$$

where $x = (x_1, x')$, $\chi_0 \in C^\infty_{\mathrm{c}}(\mathbb{R})$. Then

$$(-h^2\Delta_{x'} + |x'|^2)u(h) = (n-1)h\, u(h),$$

$\|u(h)\|_{L^2} \simeq 1$, $|x'|^{2k}u(h) = O_{L^2}(h^k)$. Hence,

$$p^{\mathrm{w}}(x, hD)u(h) = O_{L^2}(h),$$

and

$$\|u(h)\|_{L^p(\mathbb{R}^n)} \simeq h^{(n-1)(2/p-1)/4} = h^{-1/p}, \quad p = 2(n+1)/(n-1).$$

Before proving Theorem 10.13 we prove a lemma which is a consequence of Theorem 10.12

LEMMA 10.14. *In the notation of Theorem 10.13 we have*

$$\| \int_0^t U(t,s)\mathbf{1}_I(s)f(s,x)ds\|_{L^p(\mathbb{R}_t \times \mathbb{R}^k_x)} \leq Ch^{-1/p}\int_{\mathbb{R}} \|f(s,x)\|_{L^2(\mathbb{R}^k_x)}ds. \tag{10.56}$$

Proof: 1. We apply the integral version of Minkowski's inequality:

$$\begin{aligned}&\| \int_0^t U(t,s)\mathbf{1}_I(s)f(s,x)ds\|_{L^p(\mathbb{R}_t\times\mathbb{R}^k_x)}\\ &\qquad \leq C\int_{I\cap\mathbb{R}_+}\|\mathbf{1}_{[s,\infty)}(t)U(t,s)f(s,x)\|_{L^p(\mathbb{R}_t\times\mathbb{R}^k_x)}ds\\ &\qquad \leq C\int_{I\cap\mathbb{R}_+}\|U(t,s)f(s,x)\|_{L^p(\mathbb{R}_t\times\mathbb{R}^k_x)}ds.\end{aligned}$$

2. Now we use the estimate (10.49) with $p = q = 2(n+1)/(n-1)$:

$$\|U(t,s)f(s,x)\|_{L^p(\mathbb{R}_t\times\mathbb{R}^k_x)} \leq Ch^{-1/p}\|f(s,x)\|_{L^2(\mathbb{R}^k_x)},$$

from which (I is compact) (10.56) follows. □

Proof of Theorem 10.13: 1. We follow the same procedure as in the proof of Theorem 9.5. As in that case the condition (10.54) is local in phase space, that is, it implies that for any $\chi \in C^\infty_{\rm c}(T^*\mathbb{R}^n)$,

$$p^{\rm w}(x,hD)\chi^{\rm w}(x,hD)u(h) = O(h),$$

in L^2.

2. We factorize $p(x,\xi)$ as in (9.14) and we conclude that for χ with sufficiently small support,

$$(hD_{x_1} - a(x,hD_{x'}))\chi^{\rm w}(x,hD)u(h) = O_{L^2}(h).$$

Let

$$f(x_1,x',h) = (hD_{x_1} - a(x,hD_{x'}))(\chi^{\rm w}u(h)).$$

Since $\|f\|_{L^2} = O(h)$, we see

$$\int_{\mathbb{R}}\|f(x_1,\cdot)\|_{L^2(\mathbb{R}^{n-1})}dx_1 \leq C\|f\|_{L^2(\mathbb{R}^n)} = O(h). \tag{10.57}$$

3. We now apply Theorem 10.12 with $t = x_1$ and x replaced by $x' \in \mathbb{R}^{n-1}$, that is $k = n-1$. The assumption (10.53) shows that $\partial^2_{\xi'}a$ is nondegenerate in the support of χ. We can choose ψ and χ in the definition of $U(t)$ in the statement of Theorem 10.12 so that

$$\chi^{\rm w}(x,hD)u(x_1,x',h) = \frac{i}{h}\int_0^{x_1}U(t,s)f(s,x')ds + O_{\mathcal{S}}(h^\infty).$$

Let us choose $p = q$, $k = n-1$ in (10.49), that is,

$$p = q = \frac{2(n+1)}{n-1}.$$

Then, using (10.49), (10.57), and (10.56),

$$\begin{aligned}\|\chi^{\mathrm{w}}(x,hD)u\|_{L^p} &\leq \frac{1}{h}\,h^{-1/p}\int_{\mathbb{R}}\|f(s,\cdot,h)\|_{L^2(\mathbb{R}^{n-1})}ds + O(h^\infty)\\ &= O(h^{-1/p}).\end{aligned}$$

A partition of unity argument used in the proof of Theorem 9.5 concludes the proof. □

As a corollary we obtain Sogge's bounds on spectral clusters on Riemannian manifolds:

THEOREM 10.15 (L^p bounds on eigenfuctions). *Suppose that M is an n-dimensional compact Riemannian manifold and let Δ_g be its Laplace-Beltrami operator. If*

$$0 = \lambda_0 < \lambda_1 \leq \cdots \lambda_j \to \infty$$

is the complete set of eigenvalues of $-\Delta_g$, and

$$-\Delta_g \varphi_j = \lambda_j \varphi_j$$

are the corresponding eigenfunctions, then for any $c_j \in \mathbb{C}$, $j = 0, 1, \cdots$,

$$\|\sum_{\mu\leq\sqrt{\lambda_j}\leq\mu+1} c_j\varphi_j\|_{L^p} \leq C\mu^{\sigma(p)}\|\sum_{\mu\leq\sqrt{\lambda_j}\leq\mu+1} c_j\varphi_j\|_{L^2},$$

$$\sigma(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right) & \text{for } \ 2 \leq p \leq \frac{2(n+1)}{n-1}, \\ \frac{n-1}{2}-\frac{n}{p} & \text{for } \ \frac{2(n+1)}{n-1} \leq p \leq \infty. \end{cases} \tag{10.58}$$

In particular

$$\|\varphi_j\|_{L^p} \leq C\lambda_j^{\sigma(p)/2}\|\varphi_j\|_{L^2}. \tag{10.59}$$

Proof: 1. We argue as in the proof of Theorem 9.6 but now we need to check is the curvature assumption (10.53): at any point (x_0, ξ_0) and for suitable coordinates

$$p(x_0,\xi) = |\xi|^2 - 1, \quad \xi_0 = (1, 0, \cdots, 0).$$

The hypersurface $p(x_0,\xi) = 0$ is the unit sphere in $\mathbb{R}^n_\xi$ and it has a nondegenerate second fundamental form.

2. Complex interpolation [H1, Theorem 7.1.12] between the estimate in Theorem 9.6, the trivial L^2 estimate, and the estimate in Theorem 10.13 gives the full result. □

10.4 MORE SYMPLECTIC GEOMETRY

To further apply the local theory of quantized symplectic transformations to the study of semiclassical operators we will need two results from symplectic geometry. The first is a stronger form of Darboux's Theorem 2.11, which we state without proof.

THEOREM 10.16 (Variant of Darboux's Theorem). *Let A and B be two subsets of $\{1, \cdots, n\}$, and suppose that*

$$p_j(x,\xi) \quad (j \in A), \quad q_k(x,\xi) \quad (k \in B)$$

are smooth, real-valued functions defined in a neighbourhood of $(0,0) \in \mathbb{R}^{2n}$, with linearly independent gradients at $(0,0)$.

If

$$\begin{gathered}\{q_i, q_j\} = 0 \quad (i,j \in A), \quad \{p_k, p_l\} = 0 \quad (k,l \in B), \\ \{p_k, q_j\} = \delta_{kj} \quad (j \in A, k \in B),\end{gathered} \tag{10.60}$$

then there exists a symplectomorphism κ, locally defined near $(0,0)$, such that $\kappa(0,0) = (0,0)$ and

$$\kappa^* q_j = x_j \quad (j \in A), \quad \kappa^* p_k = \xi_j \quad (k \in B). \tag{10.61}$$

See Hörmander [H2, Theorem 21.1.6] for an elegant exposition.

The next result is less standard and comes from the work of Duistermaat and Sjöstrand: consult Hörmander [H2, Lemma 21.3.4] for the proof.

THEOREM 10.17 (Symplectic integrating factor). *Let p and q be smooth, real-valued functions defined near $(0,0) \in \mathbb{R}^{2n}$, satisfying*

$$p(0,0) = q(0,0) = 0, \quad \{p,q\}(0,0) > 0. \tag{10.62}$$

Then there exists a smooth, positive function u for which

$$\{up, uq\} \equiv 1 \tag{10.63}$$

in a neighborhood of $(0,0)$.

10.5 NORMAL FORMS FOR OPERATORS WITH REAL SYMBOLS

Operators of real principal type. Recall that we are taking our order function to be

$$m := (1 + |x|^2 + |\xi|^2)^{\frac{k}{2}}.$$

Set $P = p^{\mathrm{w}}(x, hD; h)$, where

$$p(x, \xi; h) \sim p_0(x, \xi) + hp_1(x, \xi) + \cdots + h^N p_N(x, \xi) + \cdots ,$$

for $p_j \in S(m)$. We assume that the *real-valued* principal symbol p_0 satisfies

$$p_0(0,0) = 0, \quad \partial p_0(0,0) \neq 0; \tag{10.64}$$

and then say that P is an operator of *real principal type* at $(0, 0)$.

THEOREM 10.18 (Normal form for real principal type operators). *Suppose that $P = p^{\mathrm{w}}(x, hD; h)$ is a semiclassical real principal type operator at $(0, 0)$.*

Then there exist
(i) *a local canonical transformation κ defined near $(0, 0)$, such that $\kappa(0, 0) = (0, 0)$ and*

$$\kappa^* \xi_1 = p_0; \tag{10.65}$$

and
(ii) *an operator T, quantizing κ in the sense of Theorem 10.6, such that*

$$T^{-1} \ \text{ exists microlocally near } ((0,0),(0,0)) \tag{10.66}$$

and

$$TPT^{-1} = hD_{x_1} \quad \text{microlocally near } ((0,0),(0,0)). \tag{10.67}$$

INTERPRETATION. The point is that using this theorem, we can transplant various mathematical objects related to P to others related to hD_{x_1}, which are much easier to study. A simple example is given by the following estimate:

$$\|u\| \leq \frac{C}{h} \|Pu\|,$$

when $u = u(h) \in \mathcal{S}$ has $\mathrm{WF}_h(u)$ in a small neighbourhood of $(0, 0)$. □

Proof. 1. Theorem 10.16 applied with $A = \emptyset$ and $B = \{1\}$, provides κ satisfying (10.65) near $(0, 0)$. Then Theorem 10.1 gives us a family of symplectic transfomations γ_t for $0 \leq t \leq 1$.

Let $F(t)$ be defined using the family γ_t in Theorem 10.6, and put $T_0 = F(1)$. Then

$$T_0 P - hD_{x_1} = E \ \text{ microlocally near } (0, 0). \ ,$$

for $E = \mathrm{Op}(e)$, $e \in S^{-1}$.

2. We now look for a symbol $a \in S$ so that a is elliptic at $(0,0)$ and

$$hD_{x_1} + E = AhD_{x_1}A^{-1} \text{ microlocally near } (0,0)$$

for $A := \mathrm{Op}(a)$. This is the same as solving

$$[hD_{x_1}, A] + EA = 0.$$

Since $P = p_0^{\mathrm{w}} + hp_1^{\mathrm{w}} + h^2 p_2^{\mathrm{w}} + \cdots$, the Remark after the proof of Theorem 10.2 shows that

$$e(x, \xi; h) = he_0(x, \xi) + h^2 e_1(x, \xi) + \cdots .$$

Hence we can find $a_0 \in S$ such that $a_0(0,0) \neq 0$ and

$$\frac{1}{i}\{\xi_1, a_0\} + e_0 a_0 = 0$$

near $(0,0)$.

Define $A_0 := \mathrm{Op}(a_0)$; then

$$[hD_{x_1}, A_0] + EA_0 = \mathrm{Op}(r_0)$$

for a symbol $r_0 \in S^{-2}$.

3. We now inductively find $A_j = \mathrm{Op}(a_j)$, for $a_j \in S^{-j}$, satisfying

$$[hD_{x_1}, A_0 + A_1 + \cdots + A_N] + E(A_0 + A_1 + \cdots A_N) = \mathrm{Op}(r_N),$$

for $r_N \in S^{-N-2}(1)$. We then put

$$A \sim A_1 + A_2 + \cdots + A_N + \cdots ,$$

which is elliptic near $(0,0)$. Finally, define

$$T := A^{-1} T_0.$$

This operator quantizes κ in the sense of Theorem 10.6. $\square$

10.6 NORMAL FORMS FOR OPERATORS WITH COMPLEX SYMBOLS

Operators of complex principal type. Assume as before that $P = p^{\mathrm{w}}(x, hD; h)$ has the symbol

$$p(x, \xi; h) \sim p_0(x, \xi) + hp_1(x, \xi) + \cdots + h^N p_N(x, \xi) + \cdots$$

with $p_j \in S(m)$. We now allow $p(x, \xi)$ to be *complex-valued*, and still say that P is *principal type* at $(0,0)$ if

$$p_0(0,0) = 0, \quad \partial p_0(0,0) \neq 0.$$

Discussion. If $\partial(\operatorname{Re} p_0)$ and $\partial(\operatorname{Im} p_0)$ are linearly independent, then the submanifold of $\mathbb{R}^{2n}$ where P is *not* elliptic has codimension two –

as opposed to codimension one in the real-valued case. The symplectic form restricted to that submanifold is non-degenerate if

$$\{\operatorname{Re} p_0, \operatorname{Im} p_0\} \neq 0.$$

Under this assumption a combination of Theorems 10.16 and 10.17 shows that there exists a canonical transformation κ, defined near $(0,0)$, and a smooth positive function u such that

$$\kappa^*(\xi_1 \pm i x_1) = u p_0.$$

That is, after a multiplication by a function we obtain the symbol of the creation or annihilation operator for the harmonic oscillator in the (x_1, ξ_1) variables. (Recall the discussion of the harmonic oscillator in Section 6.1.)

THEOREM 10.19 (Normal form for the complex symplectic case). *Suppose that $P = p^{\mathrm{w}}(x, hD; h)$ is a semiclassical principal type operator at $(0,0)$, with principal symbol p_0 satisfying*

$$p_0(0,0) = 0, \quad \pm\{\operatorname{Re} p_0, \operatorname{Im} p_0\}(0,0) > 0. \tag{10.68}$$

Then there exist
(i) *a local canonical transformation κ defined near $(0,0)$ and a smooth function u such that $\kappa(0,0) = (0,0)$, $u(0,0) > 0$, and*

$$\kappa^*(\xi_1 \pm i x_1) = u p_0;$$

and (ii) *an operator T, quantizing κ in the sense of Theorem 10.6, and a pseudodifferential operator A, elliptic at $(0,0)$, such that*

$$T^{-1} \quad \textit{exists microlocally near } ((0,0),(0,0)) \tag{10.69}$$

and

$$TPT^{-1} = A(hD_{x_1} \pm i x_1) \quad \textit{microlocally near } ((0,0),(0,0)). \tag{10.70}$$

INTERPRETATION. We can transplant mathematical objects related to P to others related to $A(hD_{x_1} \pm i x_1)$, which are clearly much easier to study. □

Proof. 1. We start as in the proof of Theorem 10.18. To simplify the notation, let us assume

$$\{\operatorname{Re} p_0, \operatorname{Im} p_0\} > 0.$$

As noted above, using Theorems 10.16 and 10.17 we can find a smooth function u, with $u(0,0) > 0$, and a local canonical transformation κ such that $\kappa(0,0) = (0,0)$ and $\kappa^*(\xi_1 + i x_1) = u p_0$.

Quantizing as before, we obtain an operator T_0 satisfying

$$T_0 P = Q(hD_{x_1} + ix_1 + E)T_0, \tag{10.71}$$

where $Q = \mathrm{Op}(q)$ for a function q satisfying

$$\gamma^* q = 1/u$$

and $E = \mathrm{Op}(e)$ for some $e \in S^{-1}$.

2. We now need to find pseudodifferential operators B and C, elliptic at $(0,0)$, and such that

$$(hD_{x_1} + ix_1 + E)B \equiv C(hD_{x_1} + ix_1), \tag{10.72}$$

microlocally near $(0,0)$. As in the proof of Theorem 10.18, we have

$$E = \mathrm{Op}(e), \quad e = he_0(x,\xi) + h^2 e_1(x,\xi) + \cdots .$$

We will find the symbols of B and C by computing successive terms in their expansions:

$$b \sim b_0 + hb_1 + \cdots + h^N b_N + \cdots ,$$
$$c \sim c_0 + hc_1 + \cdots + h^N c_N + \cdots .$$

3. Let us rewrite (10.72) as

$$(hD_{x_1} + ix_1 + E)B - C(hD_{x_1} + ix_1) = \mathrm{Op}(r),$$

for

$$r(x,\xi) = r_0(x,\xi) + hr_1(x,\xi) + \cdots + h^N r_N(x,\xi) + \cdots ,$$

with

$$r_0 = (\xi_1 + ix_1)(b_0 - c_0),$$
$$r_1 = (\xi_1 + ix_1)(b_1 - c_1) + e_0 b_0 + \{\xi_1 + ix_1, b_0\}/2i - \{c_0, \xi_1 + ix_1\}/2i.$$

Here we used composition formula in Weyl calculus (see Theorem 4.6).

We want to choose b and c so that $r_j \equiv 0$ for all j. For $r_0 = 0$ we simply need $b_0 = c_0$. Then to obtain $r_1 = 0$ we have to solve

$$-i(\partial_{x_1} - i\partial_{\xi_1})b_0 + e_0 b_0 + (\xi_1 + ix_1)(b_1 - c_1) = 0.$$

4. We first find b_0 such that

$$\begin{cases} -i(\partial_{x_1} - i\partial_{\xi_1})b_0 + e_0 b_0 = O(x_1^\infty) \\ \qquad\qquad b_0|_{x_1=0} = 1; \end{cases}$$

that is, the left hand side vanishes to infinite order at $x_1 = 0$, and $b_0 = 1$ there. The derivatives $\partial_{x_1}^k e_0|_{x_1=0}$ determine $\partial_{x_1}^k b_0|_{x_1=0}$. Then Borel's Theorem 4.11 produces a smooth function b_0 with these prescribed derivatives.

5. With $b_0 = c_0$ chosen that way we see that

$$t_1 := (-i(\partial_{x_1} - i\partial_{\xi_1})b_0 + e_0 b_0)/(\xi_1 + ix_1)$$

is a smooth function: the numerator vanishes to infinite order on the zero set of the denominator. If we put

$$c_1 = b_1 + t_1 \tag{10.73}$$

then $r_1 = 0$.

6. Now, using (10.73) the same calculation as before, we see that

$$r_3 = (\xi_1 + ix_1)(b_2 - c_2) + e_0 b_1 - i\{\xi_1 + ix_1, b_1\} + \tilde{r}_3,$$

where $\tilde{r}_3$ depends only on $b_0 = c_0$, t_1, and e. Hence $\tilde{r}_3$ is already determined. We proceed as in Step 4 and first solve

$$\begin{cases} -i(\partial_{x_1} - i\partial_{\xi_1})b_1 + e_0 b_1 + \tilde{r}_3 = O(x_1^\infty) \\ \qquad\qquad\qquad\qquad b_1|_{x=1} = 0. \end{cases}$$

This determines b_1 (and hence c_1). We continue in the same way to determine b_2 (and hence c_2). An iteration of the argument completes the construction of b and c, for which (10.72) holds microlocally near $(0, 0)$.

7. Finally, we put $T = B^{-1}T_0$, where B^{-1} is the microlocal inverse of B near $(0, 0)$, and $A = B^{-1}QC$, to obtain the statement of the theorem. □

10.7 SEMICLASSICAL PSEUDOSPECTRA

We present in this last section an application to the so-called *semiclassical pseudospectrum.* Recall from Chapter 6 that if $P = P(h) = -h^2\Delta + V(x)$ and V is real-valued, satisfying

$$V \in S(\langle x\rangle^m), \quad |\xi^2 + V(x)| \geq (1 + |\xi|^2 + |x|^m)/C \text{ for } |x| \geq C, \tag{10.74}$$

then the spectrum of P is discrete. (We deduced this from the meromorphy of the resolvent of P, $R(z) = (P - z)^{-1}$.)

Quasimodes. Because of the Spectral Theorem, which is applicable as V is real, we also know that approximate location of eigenvalues is implied by the existence of approximate eigenfunctions, called *quasimodes*. Indeed suppose that

$$\|(P - z(h))u(h)\| = O(h^\infty), \ \|u(h)\| = 1. \tag{10.75}$$

Then there exist $E(h)$ and $v(h)$ such that

$$(P - E(h))v(h) = 0, \ \|v(h)\| = 1, \ |E(h) - z(h)| = O(h^\infty). \tag{10.76}$$

In other words, if we can solve (10.75), then the approximate eigenvalue $z(h)$ is in fact close to a true eigenvalue $E(h)$ (although $u(h)$ need not be close to a true eigenfunction $v(h)$.)

Nonnormal operators. But it is well known that this is not the case for nonnormal operators P, for which the commutator $[P^*, P]$ does not vanish. Now if If $p = |\xi|^2 + V(x)$, then the symbol of this commutator is

$$\frac{1}{i}\{\bar{p}, p\} = 2\{\operatorname{Re} p, \operatorname{Im} p\}; \tag{10.77}$$

and when this is nonzero we are in the situation discussed in Theorem 10.19. This discussion leads us to

THEOREM 10.20 (Quasimodes). *Suppose that $P = -h^2\Delta + V(x)$ and that*

$$z_0 = \xi_0^2 + V(x_0), \quad \operatorname{Im}\langle \xi_0, \partial V(x_0)\rangle \neq 0. \tag{10.78}$$

Then there exists a family of functions $u(h) \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|(P - z_0)u(h)\|_{L^2} = O(h^\infty), \quad \|u(h)\|_{L^2} = 1. \tag{10.79}$$

Moreover, we can choose $u(h)$ so that

$$\operatorname{WF}_h(u(h)) = \{(x_0, \xi_0)\}, \quad \operatorname{Im}\langle \xi_0, \partial V(x_0)\rangle < 0. \tag{10.80}$$

Proof. We first replace V by a compactly supported potential agreeing with V near x_0. Our function $u(h)$ will be constructed with support near x_0.

By changing the sign of ξ_0 if necessary, but without changing z_0, we can assume that

$$\{\operatorname{Re} p, \operatorname{Im} p\}(x_0, \xi_0) = 2\operatorname{Im}\langle \xi_0, \partial V(x_0)\rangle < 0.$$

According Theorem 10.19, $P - z_0$ is microlocally conjugate to

$$A(hD_{x_1} - ix_1) \text{ near } ((x_0, \xi_0), (0, 0)).$$

Let

$$u_0(x, h) := \exp(-|x|^2/2h);$$

so that

$$(hD_{x_1} - ix_1)u_0(h) = 0, \quad \operatorname{WF}_h(u_0(h)) = \{(0, 0)\}.$$

Following the notation of Theorem 10.19, we define $u(h) := T^{-1}u_0(h)$. Then $\operatorname{WF}_h(u(h)) = \{(x_0, \xi_0)\}$ and

$$(P - z_0)u(h) \equiv T^{-1}A(hD_{x_1} - ix_1)T(T^{-1}u_0) \equiv 0.$$

$\square$

REMARK. If $p(x,\xi) = |\xi|^2 + V(x)$, the potential V satisfies (10.74), and

$$\{p(x,\xi) : (x,\xi) \in \mathbb{R}^{2n}\} \neq \mathbb{C},$$

then the operator P still has a discrete spectrum. This follows from the proof of Theorem 6.7, once we have a point at which $P - z$ is elliptic. Such a point is produced if there exists z not in the set of values of $p(x,\xi)$. However, the hypothesis of Theorem 10.20 holds in a dense open subset of the interior of the closure of the range of p. □

EXAMPLE. It is also clear that more general operators can be considered. As a simple one-dimensional example, take

$$P = (hD_x)^2 + ihD_x + x^2$$

with

$$p(x,\xi) = \xi^2 + i\xi + x^2, \quad \{\operatorname{Re} p, \operatorname{Im} p\} = -2x.$$

Hence there is a quasimode corresponding to any point in the interior of the range of p, namely $\{z : \operatorname{Re} z \geq (\operatorname{Im} z)^2\}$. On the other hand, since

$$e^{x/2h} P e^{-x/2h} = (hD)^2 + x^2 + \frac{1}{4},$$

P has the discrete spectrum $\{1/4 + nh : n \in \mathbb{N}\}$. Since the spectrum lies inside an open set of quasimodes, it is unlikely to have any true physical meaning. □

Appendix A. Notation

A.1 BASIC NOTATION.

$\mathbb{R}_+ = (0, \infty)$

$\mathbb{R}^n = n$-dimensional Euclidean space

x, y denote typical points in $\mathbb{R}^n$: $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

$\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$

$z = (x, \xi), w = (y, \eta)$ denote typical points in $\mathbb{R}^n \times \mathbb{R}^n$:
$z = (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, $w = (y_1, \dots, y_n, \eta_1, \dots, \eta_n)$

$\mathbb{T}^n = n$-dimensional flat torus $= \mathbb{R}^n/\mathbb{Z}^n$

$\mathbb{C} =$ complex plane

$\mathbb{C}^n =$ n-dimensional complex space

$\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i =$ inner product on $\mathbb{C}^n$

$|x| = \langle x, x\rangle^{1/2}$

$\langle x\rangle = (1 + |x|^2)^{1/2}$

$\mathbb{M}^{m\times n} = m \times n$-matrices

$\mathbb{S}^n = n \times n$ real symmetric matrices

$A^T =$ transpose of the matrix A

I denotes both the identity matrix and the identity mapping.

$J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$

$\sigma(z, w) = \langle Jz, w\rangle =$ symplectic inner product

$\#S =$ cardinality of the set S

$|E| =$ Lebesgue measure of the set $E \subseteq \mathbb{R}^n$

A.2 FUNCTIONS, DIFFERENTIATION.

The support of a function is denoted "spt", and a subscript "c" on a space of functions means those with compact support.

- Partial derivatives:

$$\partial_{x_j} := \frac{\partial}{\partial x_j}, \quad D_{x_j} := \frac{1}{i}\frac{\partial}{\partial x_j}$$

• Multiindex notation: A multiindex is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$, the entries of which are nonnegative integers. The size of α is

$$|\alpha| := \alpha_1 + \dots + \alpha_n.$$

We then write for $x \in \mathbb{R}^n$:

$$x^\alpha := {x_1}^{\alpha_1} \dots {x_n}^{\alpha_n},$$

where $x = (x_1, \dots, x_n)$.

Also

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$$

and

$$D^\alpha := \frac{1}{i^{|\alpha|}} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}.$$

(WARNING: Our use of the symbols "D" and "D^α" differs from that in the PDE textbook [E].)

If $\varphi : \mathbb{R}^n \to \mathbb{R}$, then we write

$$\partial\varphi := (\varphi_{x_1}, \dots, \varphi_{x_n}) = \text{gradient},$$

and

$$\partial^2\varphi := \begin{pmatrix} \varphi_{x_1x_1} & \dots & \varphi_{x_1x_n} \\ & \ddots & \\ \varphi_{x_nx_1} & \dots & \varphi_{x_nx_n} \end{pmatrix} = \text{Hessian matrix}$$

Also

$$D\varphi := \frac{1}{i}\partial\varphi.$$

If φ depends on both the variables $x, y \in \mathbb{R}^n$, we put

$$\partial_x^2\varphi := \begin{pmatrix} \varphi_{x_1x_1} & \dots & \varphi_{x_1x_n} \\ & \ddots & \\ \varphi_{x_nx_1} & \dots & \varphi_{x_nx_n} \end{pmatrix}$$

and

$$\partial_{x,y}^2\varphi := \begin{pmatrix} \varphi_{x_1y_1} & \dots & \varphi_{x_1y_n} \\ & \ddots & \\ \varphi_{x_ny_1} & \dots & \varphi_{x_ny_n} \end{pmatrix}.$$

• Jacobians: Let

$$x \mapsto y = y(x)$$

be a diffeomorphism, $y = (y^1, \dots, y^n)$. The Jacobian matrix is

$$\partial y = \partial_x y := \begin{pmatrix} \frac{\partial y^1}{\partial x_1} & \dots & \frac{\partial y^1}{\partial x_n} \\ & \ddots & \\ \frac{\partial y^n}{\partial x_1} & \dots & \frac{\partial y^n}{\partial x_n} \end{pmatrix}_{n\times n}.$$

• Poisson bracket: If $f, g : \mathbb{R}^n \to \mathbb{R}$ are C^1 functions,

$$\{f, g\} := \langle \partial_\xi f, \partial_x g \rangle - \langle \partial_x f, \partial_\xi g \rangle \;= \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}.$$

• The *Schwartz space* is

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty \text{ for all multiindices } \alpha, \beta\}.$$

We say

$$\varphi_j \to \varphi \quad \text{in } \mathcal{S}$$

provided

$$\sup_{\mathbb{R}^n} |x^\alpha D^\beta(\varphi_j - \varphi)| \to 0$$

for all multiindices α, β

We write $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ for the space of *tempered distributions*, which is the dual of $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$. That is, $u \in \mathcal{S}'$ provided $u : \mathcal{S} \to \mathbb{C}$ is linear and $\varphi_j \to \varphi$ in $\mathcal{S}$ implies $u(\varphi_j) \to u(\varphi)$.

We say

$$u_j \to u \quad \text{in } \mathcal{S}'$$

provided

$$u_j(\varphi) \to u(\varphi) \quad \text{for all } \varphi \in \mathcal{S}.$$

A.3 ELEMENTARY OPERATORS.

Multiplication operator: $M_\lambda f(x) = \lambda f(x)$

Translation operator: $T_\xi f(x) = f(x - \xi)$

Reflection operator: $Rf(x) := f(-x)$

A.4 OPERATORS.

A^* = adjoint of the operator A

$[A, B] = AB - BA$ = commutator of A and B

$\sigma(A)$ = symbol of the pseudodifferential operator A

$\operatorname{spec}(A)$ = spectrum of A.

$\operatorname{tr}(A)$ = trace of A.

We say that the operator B is of *trace class* if

$$\operatorname{tr}(B) := \sum \sqrt{\lambda_j} < \infty,$$

where the $\lambda_j \geq 0$ are the eigenvalues of the symmetric matrix B^*B.

• If $A : X \to Y$ is a bounded linear operator, we define the operator norm

$$\|A\| := \sup\{\|Au\|_Y \mid \|u\|_X \leq 1\}.$$

We will often write this norm as

$$\|A\|_{X\to Y}$$

when we want to emphasize the spaces between which A maps.

The space of bounded linear operators from X to Y is denoted by $L(X, Y)$; and the space of bounded linear operators from X to itself is denoted $L(X)$.

A.5 ESTIMATES.

• We write

$$f = O(h^\infty) \quad \text{as } h \to 0$$

if for each positive integer N there exists a constant C_N such that

$$|f| \leq C_N h^N \quad \text{for all } 0 < h \leq 1.$$

• If we want to specify boundedness in the space X, we write

$$f = O_X(h^N)$$

to mean

$$\|f\|_X = O(h^N).$$

• If A is a bounded linear operator between the spaces X, Y, we will often write

$$A = O_{X\to Y}(h^N)$$

to mean

$$\|A\|_{X\to Y} = O(h^N).$$

A.6. SYMBOL CLASSES.

We record from Chapter 4 the various definitions of classes for symbols $a = a(x, \xi, h)$.

• Given an order function m on $\mathbb{R}^{2n}$, we define the corresponding class of symbols:

$$\begin{aligned} S(m) \ := \ &\{a \in C^\infty \mid \text{for each multiindex } \alpha \\ &\qquad \text{there exists a constant } C_\alpha \text{ so that } |\partial^\alpha a| \leq C_\alpha m\}. \end{aligned}$$

• We as well define

$$S^k(m) := \{a \in C^\infty \mid |\partial^\alpha a| \leq C_\alpha h^{-k} m \ \text{ for all multiindices } \alpha\}$$

and

$$S^k_\delta(m) := \{a \in C^\infty \mid |\partial^\alpha a| \le C_\alpha h^{-\delta|\alpha|-k} m \ \text{ for all multiindices } \alpha\}.$$

The index k indicates how singular is the symbol a as $h \to 0$; the index δ allows for increasing singularity of the derivatives of a.

• Write also

$$S^{-\infty}(m) := \{a \in C^\infty \mid \text{for each } \alpha \text{ and } N,\ \ |\partial^\alpha a| \le C_{\alpha,N} h^N m\}.$$

So if a is a symbol in $S^{-\infty}(m)$, then a and all of its derivatives are $O(h^\infty)$ as $h \to 0$.

• If the order function is the constant function $m \equiv 1$, we will usually not write it:

$$S^k := S^k(1),\ S^k_\delta = S^k_\delta(1).$$

• We will also omit zero superscripts:

$$\begin{aligned} S &:= S^0 = S^0(1) \\ &= \{a \in C^\infty(\mathbb{R}^{2n}) \mid |\partial^\alpha a| \le C_\alpha \ \text{ for all multiindices } \alpha\}. \end{aligned}$$

• We will sometimes write

$$a = O_S(h^N),$$

to mean that for all α

$$|\partial^\alpha a| \le C_\alpha h^N.$$

We use similar notation for other spaces with seminorms.

A.7 PSEUDODIFFERENTIAL OPERATORS.

We cross reference the following terminology from Appendix E. Let M denote a manifold.

• A linear operator $A : C^\infty(M) \to C^\infty(M)$ is called a *pseudodifferential operator* if there exist integers m, k such that for each coordinate patch U_γ, and there exists a symbol $a_\gamma \in S^{m,k}$ such that for any $\varphi, \psi \in C^\infty_c(U_\gamma)$

$$\varphi A(\psi u) = \varphi \gamma^* a^w_\gamma(x, hD)(\gamma^{-1})^*(\psi u)$$

for each $u \in C^\infty(M)$.

• We write

$$A \in \Psi^{m,k}(M)$$

and also put

$$\Psi^k(M) := \Psi^{0,k}(M), \quad \Psi(M) := \Psi^{0,0}(M).$$

Appendix B. Differential forms

In this section we provide a minimalist review of differential forms on $\mathbb{R}^N$. For more a detailed and fully rigorou)description of differential forms on manifolds we refer to [W, Chapter 2].

NOTATION.

(i) If $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$, then $dx_j, d\xi_j \in (\mathbb{R}^{2n})^*$ satisfy

$$\begin{aligned} dx_j(u) &= dx_j(x, \xi) = x_j \\ d\xi_j(u) &= d\xi_j(x, \xi) = \xi_j. \end{aligned}$$

(ii) If $\alpha, \beta \in (\mathbb{R}^{2n})^*$, then

$$(\alpha \wedge \beta)(u, v) := \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

for $u, v \in \mathbb{R}^{2n}$. More generally, for $\alpha_j \in (\mathbb{R}^{2n})^*$, $j = 1, \cdots, m \leq 2n$, and $u = (u_1, \cdots, u_m)$, an m-tuple of $u_k \in \mathbb{R}^{2n}$,

$$(\alpha_1 \wedge \cdots \wedge \alpha_m)(u) = \det([\alpha_j(u_k)]_{1 \leq j,k \leq 2n}). \tag{B.1}$$

(iii) If $f : \mathbb{R}^n \to \mathbb{R}$, the *differential* of f, is the 1-form

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

(iv) An m-form on $\mathbb{R}^n$ is given by

$$w = \sum_{i_1 < i_2 < \cdots < i_m} f_{i_1 \cdots i_m}(x) dx_{i_1} \wedge \cdots dx_{i_m}, \quad f_{i_1 \cdots i_m} \in C^\infty(\mathbb{R}^n).$$

Its action at x on m-tuples of vectors is given using (ii).

(v) The differential of m-form is defined by induction using (iii) and $d(fg) = df \wedge g + f dg$, where f is a function and g is an $(m-1)$-form. It satisfies $d^2 = 0$.

THEOREM B.1 (Alternative definition of d). *Suppose w is a differential 2-form, and $u \in C^\infty(\mathbb{R}^n, \mathbb{R}^3)$, $u = (u_1, u_3, u_3)$ is a 3-tuple of vectorfields. Then*

$$\begin{aligned} dw(u) = {} & u_1 \left(w(u_2, u_3)\right) + u_2 \left(w(u_3, u_1)\right) + u_3 \left(w(u_1, u_2)\right) \\ & - w([u_1, u_2], u_3) - w([u_2, u_3], u_1) - w([u_3, u_1], u_2). \end{aligned} \tag{B.2}$$

1. Both sides of (B.2) are linear in w and trilinear in u.
2. When, say, u_1 is multiplied by $f \in C^\infty(\mathbb{R}^n)$, then

$$dw(fu_1, u_2, u_3) = f dw(u),$$

and the right hand side of (B.2) is equal to

$$\begin{aligned} &fu_1\left(w(u_2, u_3)\right) + u_2\left(fw(u_3, u_1)\right) + u_3\left(fw(u_1, u_2)\right) \\ &- w([fu_1, u_2], u_3) - w([u_2, u_3], fu_1) - w([u_3, fu_1], u_2), \end{aligned}$$

and this is equal to the right hand side of (B.2) multiplied by f. In fact,

$$[fu_1, u_2] = f[u_1, u_2] - (u_2 f)u_1, \quad [u_3, fu_1] = f[u_3, u_1] + (u_3 f)u_1,$$

and

$$\begin{aligned} u_2\left(fw(u_3, u_1)\right) &= fu_2\left(w(u_3, u_1)\right) + (u_2 f)w(u_3, u_1), \\ u_3\left(fw(u_1, u_2)\right) &= fu_3\left(w(u_1, u_2)\right) + (u_3 f)w(u_1, u_2). \end{aligned}$$

3. Hence we only need to check this identity for u constant and for $w = w_1 dw_2 \wedge dw_3$, where $w_1 \in C^\infty$, and w_2, w_3 are coordinate functions (that is are among $x_1, \cdots x_n$). Then

$$dw(u) = \det\left([u_j w_i]_{1 \le i,j \le n}\right),$$

and the right hand side of (B.2) is given by (remember that now $u_j w_i$, $i = 2, 3$ are constants) by the expansion of this determinant with respect to the first row, $(u_1 w_1, u_2 w_1, u_3 w_1)$. □

DEFINITION. If η is a differential m-form and V a vector field, then the *contraction of* η *by* V, denoted

$$V \lrcorner \eta,$$

is the $(m-1)$-form defined by

$$(V \lrcorner \eta)(u) = \eta(V, u),$$

where u is an $(m-1)$-tuple of vectorfields. We use the consistent convention that for 0-forms, that is for functions, $V \lrcorner f = 0$.

We note the following property of contraction which can be deduced from (B.1): if v is a k-form and w is an m-form, then

$$\text{(B.3)} \qquad V \lrcorner (v \wedge w) = (V \lrcorner v) \wedge w + (-1)^k v \wedge (V \lrcorner w).$$

DEFINITIONS. Let $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping.

(i) If V is a vector field on $\mathbb{R}^n$, the *push-forward* is

$$\kappa_* V = \partial \kappa(V).$$

(ii) If η is a 1-form on $\mathbb{R}^n$, the *pull-back* is

$$(\kappa^*\eta)(u) = \eta(\kappa_* u).$$

THEOREM B.2 (Differentials and pull-backs). *Let w be a differential m-form. We have*

$$d(\kappa^* w) = \kappa^*(dw). \tag{B.4}$$

Proof. 1. We first prove this for functions: $d(\kappa^* f) = d(\kappa(f)) = \sum_{j=1}^n \frac{\partial y_i}{\partial x_j}\frac{\partial f}{\partial y_i} dx_j$. Furthermore,

$$\kappa^*(df) = \kappa^*\left(\sum_{i=1}^n \frac{\partial f}{\partial y_i} dy_i\right) = \sum_{i=1}^n \frac{\partial f}{\partial y_i}\kappa^*(dy_i).$$

2. The proof now follows by induction on the order of the differential form: any m-form can be written as a linear combination of forms $f dg$ where f is a function, and g is $(m-1)$-form. □

DEFINITION. If V is a vector field generating the flow φ_t, then the *Lie derivative* of w is

$$\mathcal{L}_V w := \frac{d}{dt}((\varphi_t)^* w)|_{t=0}.$$

Here w denotes a function, a vector field or a form. We recall that φ_t is generated by a time independent vectorfield, V, $\varphi_t = \exp(tV)$, means that

$$(d/dt)\varphi_t(m) = V(\varphi_t(m)), \quad \varphi_0(m) = m.$$

EXAMPLE S. (i) If f is a function,

$$\mathcal{L}_V f = V(f).$$

(ii) If W is a vector field

$$\mathcal{L}_V W = [V, W].$$

Since for differential forms, w, $d(\phi_t)^* w = \phi_t^*(dw)$, we see that $\mathcal{L}_V$ commutes with d:

$$d(\mathcal{L}_V w) = \mathcal{L}_V(dw). \tag{B.5}$$

We also note that $\mathcal{L}_V$ is a derivation: for a function $f \in C^\infty$ and a differential form w,

$$\mathcal{L}_V(fw) = (\mathcal{L}_V f)w + f\mathcal{L}_V w. \tag{B.6}$$

THEOREM B.3 (Cartan's formula). *If w is a differential form,*

$$\mathcal{L}_V w = d(V \lrcorner w) + (V \lrcorner dw). \tag{B.7}$$

Proof. 1. We proceed by induction on the order of differential forms. For 0-forms, that is for functions, we have

$$\mathcal{L}_V f = Vf = V \lrcorner df = d(V \lrcorner f) + (V \lrcorner df),$$

since by our convention $V \lrcorner f = 0$.

2. Any m-form is a linear combinations of forms fdg where f is a function and g in an $(m-1)$-form. Then, using (B.5), (B.6), $d^2 = 0$, and the induction hypothesis,

$$\begin{aligned} \mathcal{L}_V(fdg) &= (\mathcal{L}_V f)dg + f\mathcal{L}_V dg \\ &= (Vf)dg + fd(\mathcal{L}_V g) \\ &= (Vf)dg + fd(d(V \lrcorner g) + V \lrcorner dg) \\ &= (Vf)dg + fd(V \lrcorner dg). \end{aligned} \tag{B.8}$$

3. The right hand side of (B.7) for $w = fdg$ is equal to

$$\begin{aligned} &d(V \lrcorner (fdg)) + V \lrcorner (d(fdg)) = \\ &\qquad f(V \lrcorner dg) + df \wedge (V \lrcorner dg) + V \lrcorner (df \wedge dg). \end{aligned} \tag{B.9}$$

Now we can use (B.3) with $w = df$, $v = dg$, $k = 1$, to obtain

$$V \lrcorner (df \wedge dg) = (Vf)dg - df \wedge (V \lrcorner dg).$$

Inserting this in (B.9) and the comparison with (B.8) gives (B.7) for $w = fdg$ and hence for all differential m-forms. □

THEOREM B.4 (Poincaré's Lemma). *If α is a k-form defined in the open ball $U = B^0(0, R)$ and if*

$$d\alpha = 0,$$

then there exists a $(k-1)$ form ω in U such that

$$d\omega = \alpha.$$

Proof. 1. Let $\Omega^k(U)$ denote the space of k-forms on U. We will build a linear mapping

$$H : \Omega^k(U) \to \Omega^{k-1}(U)$$

such that

$$d \circ H + H \circ d = I. \tag{B.10}$$

Then

$$d(H\alpha) + Hd\alpha = \alpha$$

and so $d\omega = \alpha$ for $\omega := H\alpha$.

2. Define $A : \Omega^k(U) \to \Omega^k(U)$ by

$$A(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \left(\int_0^1 t^{k-1} f(tp)\, dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Set

$$X := \langle x, \partial_x \rangle = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}.$$

We claim

$$A\mathcal{L}_X = I \quad \text{on } \Omega^k(U). \tag{B.11}$$

and

$$d \circ A = A \circ d. \tag{B.12}$$

Assuming these assertions, define

$$H := A \circ X \lrcorner .$$

By Cartan's formula, Theorem B.3,

$$\mathcal{L}_X = d \circ (X \lrcorner) + X \lrcorner \circ d.$$

Thus

$$\begin{aligned} I = A\mathcal{L}_X &= A \circ d \circ (X \lrcorner) + A \circ X \lrcorner \circ d \\ &= d(A \circ X \lrcorner) + (A \circ X \lrcorner) \circ d \\ &= d \circ H + H \circ d; \end{aligned}$$

and this proves (B.10).

3. To prove (B.11), we compute

$$\begin{aligned} &A\mathcal{L}_X(f dx_{i_1} \wedge \cdots \wedge dx_{i_2}) \\ &\quad = A \left[\left(kf + \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j} \right) (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \right] \\ &\quad = \int_0^1 kt^{k-1} f(tp) + \sum_{j=1}^n t^{k-1} x_j \frac{\partial f}{\partial x_j}(tp) \\ &\qquad\qquad\qquad dt dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &\quad = \int_0^1 \frac{d}{dt}(t^k f(tp))\, dt dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &\quad = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

4. To verify (B.12), note

$$\begin{aligned}
&A \circ d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\
&\qquad = A\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\
&\qquad = \left(\int_0^1 t^{k-1} \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(tp) dx_j dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&\qquad = d\left(\left(\int_0^1 t^{k-1} f(tp) dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\
&\qquad = d \circ A(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}).
\end{aligned}$$

□

Appendix C. Functional analysis

THEOREM C.1 (Schwartz Kernel Theorem). *Let $A : \mathcal{S} \to \mathcal{S}'$ be a continuous linear operator.*

Then there exists a distribution $K_A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$Au(x) = \int_{\mathbb{R}^n} K_A(x,y)u(y)\,dy \tag{C.1}$$

for all $u \in \mathcal{S}$.

We call K_A the *kernel* of A.

THEOREM C.2 (Inverse Function Theorem). *Let X, Y denote Banach spaces and assume*

$$f : X \to Y$$

is C^1. Select a point $x_0 \in X$ and write $y_0 := f(x_0)$.

(i) *(Right inverse) If there exists $A \in L(Y,X)$ such that*

$$\partial f(x_0)A = I,$$

then there exists $g \in C^1(Y,X)$ such that

$$f \circ g = I \quad \text{near } y_0.$$

(ii) *(Left inverse) If there exists $B \in L(Y,X)$ such that*

$$B\partial f(x_0) = I,$$

then there exists $g \in C^1(Y,X)$ such that

$$g \circ f = I \quad \text{near } x_0.$$

THEOREM C.3 (Approximate inverses). *Let X, Y be Banach spaces and suppose $A : X \to Y$ is a bounded linear operator. Suppose there exist bounded linear operators $B_1, B_2 : Y \to X$ such that*

$$\begin{cases} AB_1 = I + R_1 & \text{on } Y \\ B_2A = I + R_2 & \text{on } X, \end{cases} \tag{C.2}$$

where

$$\|R_1\| < 1, \ \|R_2\| < 1.$$

Then A is invertible.

Proof. The operator $I + R_1$ is invertible, with

$$(I + R_1)^{-1} = \sum_{k=0}^{\infty} (-1)^k R_1^k,$$

this series converging since $\|R_1\| < 1$. Hence

$$AC_1 = I \quad \text{for } C_1 := B_1(I + R_1)^{-1}.$$

Likewise,

$$(I + R_2)^{-1} = \sum_{k=0}^{\infty} (-1)^k R_2^k;$$

and

$$C_2 A = I \quad \text{for } C_2 := (I + R_2)^{-1} B_2.$$

So A has a left and a right inverse, and is consequently invertible, with $A^{-1} = C_1 = C_2$. □

THEOREM C.4 (Norms of powers of operators). *Let $A \in L(H_1, H_2)$, where H_1, H_2 are Hilbert spaces.*
(i) *Then*

$$\|A\| = \sup_{\|u\|,\|v\|=1} |\langle Au, v\rangle|, \ \|A\| = \|A^*\|, \ \|A\|^2 = \|A^*A\|.$$

(ii) *If A is self-adjoint, $\|A\|^m = \|A^m\|$ for all $m \in \mathbb{N}$.*

Proof. 1. We may assume $\|A\| > 0$. Note that

$$|\langle Au, v\rangle| \le \|Au\|\|v\| \le \|A\|\|u\|\|v\| = \|A\|$$

for any two unit vectors u, v. Thus $\sup_{\|u\|,\|v\|=1} |\langle Au, v\rangle| \le \|A\|$.

Now if $u \notin \ker(A)$, we can put $v = Au/\|Au\|$. Consequently,

$$\sup_{\|u\|,\|v\|=1} |\langle Au, v\rangle| \ge \sup_{\substack{u \notin \ker(A) \\ \|u\|=1}} \frac{1}{\|Au\|} |\langle Au, Au\rangle| = \sup_{\substack{u \notin \ker(A) \\ \|u\|=1}} \|Au\| = \|A\|;$$

and therefore

$$\begin{aligned} \|A\|^2 = \sup_{\|u\|=1} \|Au\|^2 &= \sup_{\|u\|=1} |\langle A^*Au, u\rangle| \\ &\le \sup_{\|u\|,\|v\|=1} |\langle A^*Au, v\rangle| = \|A^*A\|. \end{aligned}$$

Also, for any u, v with norm one we have

$$|\langle A^*Au, v\rangle| = |\langle Au, Av\rangle| \le \|Au\|\|Av\| \le \|A\|^2.$$

Taking the supremum over u, v gives us the inequality $\|A^*A\| \le \|A\|^2$.

2. A simple induction now yields $\|A\|^{2^k} = \|A^{2^k}\|$ for all natural numbers k. For a general m, find an n such that $m+n$ is a power of 2. Then $\|A^{n+m}\| = \|A\|^{n+m}$, and so

$$\|A\|^m\|A\|^n = \|A\|^{m+n} = \|A^{m+n}\| = \|A^m A^n\| \leq \|A\|^m\|A\|^n.$$

Therefore the inequality signs above must be equalities, and this implies $\|A^m\| = \|A\|^m$. □

THEOREM C.5 (Cotlar–Stein Theorem). *Let H_1, H_2 be Hilbert spaces and $A_j \in L(H_1, H_2)$ for $j = 1, \ldots$. Assume*

$$\sup_j \sum_{k=1}^{\infty} \|A_j^* A_k\|^{1/2} \leq C, \quad \sup_j \sum_{k=1}^{\infty} \|A_j A_k^*\|^{1/2} \leq C.$$

Then the series $A := \sum_{j=1}^{\infty} A_j$ converges in strong operator topology and

$$\|A\| \leq C.$$

Proof. 1. Let us first assume that $A_j = 0$ for $j > J$ so that A is well defined. Since A^*A is self-adjoint, the previous theorem implies

$$\|A\|^{2m} = \|(A^*A)^m\|.$$

In addition,

$$(A^*A)^m = \sum_{j_1,\ldots,j_{2m}=1}^{\infty} A_{j_1}^* A_{j_2} \ldots A_{j_{2m-1}}^* A_{j_{2m}} =: \sum_{j_1,\ldots,j_{2m}} a_{j_1,\ldots,j_{2m}}.$$

Now

$$\|a_{j_1,\ldots,j_{2m}}\| \leq \|A_{j_1}^* A_{j_2}\|\|A_{j_3}^* A_{j_4}\| \ldots \|A_{j_{2m-1}}^* A_{j_{2m}}\|,$$

and also

$$\|a_{j_1,\ldots,j_{2m}}\| \leq \|A_{j_1}\|\|A_{j_2} A_{j_3}^*\| \ldots \|A_{j_{2m-2}} A_{j_{2m-1}}^*\|\|A_{j_{2m}}\|.$$

Multiply these estimates and take square roots:

$$\|a_{j_1,\ldots,j_{2m}}\| \leq C\|A_{j_1}^* A_{j_2}\|^{1/2}\|A_{j_2} A_{j_m}^*\|^{1/2} \ldots \|A_{j_{2m-1}}^* A_{j_{2m}}\|^{1/2}.$$

Consequently,

$$\begin{aligned}
\|A\|^{2m} &= \|(A^*A)^m\| \leq \sum_{j_1,\ldots,j_{2m}=1}^{\infty} \|a_{j_1,\ldots,j_{2m}}\| \\
&\leq C \sum_{j_1,\ldots,j_{2m}=1}^{\infty} \|A_{j_1} A_{j_2}^*\|^{1/2} \ldots \|A_{j_{2m-1}}^* A_{j_{2m}}\|^{1/2} \\
&\leq JCC^{2m},
\end{aligned}$$

where the J factor came from having $2m$ sums and only $2m-1$ factors in the summands.

Hence

$$\|A\| \le J^{\frac{1}{2m}} C^{\frac{2m+1}{2m}} \to C \quad \text{as } m \to \infty.$$

2. To consider the general case, take $u \in E$, and suppose $u = A_k^* v$ for some k. Then

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} A_j u \right\| &= \left\| \sum_{j=1}^{\infty} A_j A_k^* v \right\| \\ &\le \sum_{j=1}^{\infty} \|A_j A_k^*\|^{1/2} \|A_j A_k^*\|^{1/2} \|v\| \\ &\le C^2 \|v\|. \end{aligned}$$

Thus $\sum_{j=1}^{\infty} A_j u$ converges for $u \in \Sigma := \operatorname{span}\{A_k^*(E) \mid k = 1, \dots, n\}$ and so also for $u \in \bar{\Sigma}$. If u is orthogonal to $\bar{\Sigma}$, then $u \in \ker(A_k)$ for all k; in which case $\sum_{j=1}^{\infty} A_j u = 0$. □

Henceforth H denotes a complex Hilbert space, with inner product $\langle \cdot, \cdot \rangle$.

THEOREM C.6 (Spectrum of self-adjoint operators). *Suppose $A : H \to H$ is a bounded self-adjoint operator.*

(i) *Then $(A - \lambda)^{-1}$ exists and is a bounded linear operator on H for $\lambda \in \mathbb{C} - \operatorname{spec}(A)$, where $\operatorname{spec}(A) \subset \mathbb{R}$ is the spectrum of A.*

(ii) *If $\operatorname{spec}(A) \subseteq [a, \infty)$, then*

$$\langle Au, u \rangle \ge a \|u\|^2 \quad (u \in A). \tag{C.3}$$

THEOREM C.7 (Maximin and minimax principles). *Suppose that $A : H \to H$ is self-adjoint and semibounded, meaning $A \ge -c_0$. Assume also that $(A + 2c_0)^{-1} : H \to H$ is a compact operator.*

Then the spectrum of A is discrete: $\lambda_1 \le \lambda_2 \le \lambda_3 \cdots$; and furthermore

(i)

$$\lambda_j = \max_{\substack{V \subseteq H \\ \operatorname{codim} V < j}} \min_{\substack{v \in V \\ v \ne 0}} \frac{\langle Av, v \rangle}{\|v\|^2}, \tag{C.4}$$

(ii)

$$\lambda_j = \min_{\substack{V \subseteq H \\ \dim V \le j}} \max_{\substack{v \in V \\ v \ne 0}} \frac{\langle Av, v \rangle}{\|v\|^2}. \tag{C.5}$$

In these formulas, V denotes a linear subspace of H.

DEFINITIONS. (i) Let $Q : H \to H$ be a bounded linear operator. We define the *rank* of Q to be the dimension of the range $Q(H)$.

(ii)If A is an operator with real and discrete spectrum, we set

$$N(\lambda) := \#\{\lambda_j \mid \lambda_j \leq \lambda\}$$

to count the number of eigenvalues less than or equal to λ.

THEOREM C.8 (Estimating $N(\lambda)$). *Let A satisfy the assumptions of Theorem C.7.*

(i) *If*

$$\text{(C.6)} \qquad \begin{cases} \text{for each } \delta > 0, \text{ there exists an operator } Q, \\ \text{with rank } Q \leq k, \text{ such that} \\ \langle Au, u\rangle \geq (\lambda - \delta)\|u\|^2 - \langle Qu, u\rangle \textit{ for } u \in H, \end{cases}$$

then

$$N(\lambda) \leq k.$$

(ii) *If*

$$\text{(C.7)} \qquad \begin{cases} \text{for each } \delta > 0, \text{ there exists a subspace } V \\ \text{with } \dim V \geq k, \text{ such that} \\ \langle Au, u\rangle \leq (\lambda + \delta)\|u\|^2 \text{ for } u \in V, \end{cases}$$

then

$$N(\lambda) \geq k.$$

Proof. 1. Set $W := Q(H)^T$. Thus codim $W =$ rank Q $\leq k$. Therefore the maximin formula (C.4) implies

$$\begin{aligned} \lambda_k &= \max_{\substack{V \subseteq H \\ \text{codim} V < k}} \min_{\substack{v \in V \\ v \neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2} \geq \min_{\substack{v \in W \\ v \neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2} \\ &= \min_{\substack{v \in W \\ v \neq 0}} \left(\lambda - \delta - \frac{\langle Qv, v\rangle}{\|v\|^2} \right) = \lambda - \delta, \end{aligned}$$

since $\langle Qv, v\rangle = 0$ if $v \in Q(H)^T$. Hence $\lambda \leq \lambda_k + \delta$. This is valid for all $\delta > 0$, and so

$$N(\lambda) = \max\{j \mid \lambda_j \leq \lambda\} \leq k.$$

This proves assertion (i).

2. The minimax formula (C.5) directly implies that

$$\lambda_k \leq \max_{\substack{v\in V\\ v\neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2} \leq \lambda + \delta.$$

Hence $\lambda_k \leq \lambda + \delta$. This is valid for all $\delta > 0$, and so

$$N(\lambda) = \max\{j \mid \lambda_j \leq \lambda\} \geq k.$$

This is assertion (ii).

□

THEOREM C.9 (Lidskii's Theorem). *Suppose that B is an operator of trace class on $L^2(M, \Omega^{\frac{1}{2}}(M))$, given by the integral kernel*

$$K \in C^\infty(M \times M; \Omega^{\frac{1}{2}}(M \times M)).$$

Then K_Δ, the restriction to the diagonal $\Delta := \{(m, m) \ : m \in M\}$, has a well-defined density; and

$$\text{tr } B = \int_\Delta K_\Delta. \tag{C.8}$$

We will also use the following general result of Keel-Tao [**?**]:

THEOREM C.10 (Abstract Strichartz estimates). *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $U \in L^\infty(\mathbb{R}, L(L^2(X))$ satisfy*

$$\begin{gathered} \|U(t)\|_{\mathcal{B}(L^2(X))} \leq A, \quad t \in \mathbb{R}, \\ \|U(t)U(s)^* f\|_{L^\infty(X,\mu)} \leq Ah^{-\mu}|t-s|^{-\sigma}\|f\|_{L^1(X,\mu)}, \quad t, s \in \mathbb{R}, \end{gathered} \tag{C.9}$$

where $A, \sigma, \mu > 0$ are fixed.

The for every pair p, q satisfying

$$\frac{2}{p} + \frac{2\sigma}{q} = \sigma, \quad 2 \leq p \leq \infty, \quad 1 \leq q \leq \infty, \quad (p, q) \neq (2, \infty),$$

we have

$$\left(\int_{\mathbb{R}} \|U(t)f\|^p_{L^q(X,\mu)} dt\right)^{\frac{1}{p}} \leq Bh^{-\frac{\mu}{p\sigma}}\|f\|_{L^2(X,\mu)}. \tag{C.10}$$

We should stress that in the application to bounds on approximate solution (Section 10.3) we only use the "interior" exponent $p = q$ which does not require the full power of [**?**] – see [S]. For the reader's convenience we present the proof of that case.

Proof of the case $p = q$: 1. A rescaling in time easily reduces the estimate to the case $h = 1$.

2. The estimate we want reads

$$\|U(t)f\|_{L^p(\mathbb{R}_t\times X)} \le B\|f\|_{L^2(X)}, \quad \frac{1}{p} = \frac{\sigma}{2(1+\sigma)}. \tag{C.11}$$

Let p' denote the exponent dual to p: $1/p + 1/p' = 1$. Then, since $L^{p'}$ is dual to L^p, (C.11) is equivalent to

$$\int_{\mathbb{R}\times X} U(t)f(x)\; G(t,x)\; d\mu(x)dt \le \|f\|_{L^2(X)}\|G\|_{L^{p'}(\mathbb{R}\times X)},$$

for all $G \in L^{p'}(\mathbb{R}\times X)$, and that in turn means that

$$\| \int_{\mathbb{R}} U(t)^* G(t) dt\|_{L^2(X)} \le C\|G\|_{L^{p'}(\mathbb{R}\times X)}.$$

In other words,

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \langle U(t)^*G(t), U(s)^*F(s)\rangle\; dtds\right| \le C\|G\|_{L^{p'}(\mathbb{R}\times X)}\|F\|_{L^{p'}(\mathbb{R}\times X)}. \tag{C.12}$$

3. We now apply the Riesz-Thorin interpolation theorem (see for instance [H1, Theorem 7.1.12]) to $U(t)U(s)^*$ with fixed $t, s \in \mathbb{R}$. The two *fixed time* estimates provided by the hypothesis (C.9) give:

$$\|U(t)U(s)^*\|_{\mathcal{B}(L^{p'},L^p)} \le A^{3-2/p'}|t-s|^{-\sigma(2/p'-1)}, \quad 1 \le p' \le 2,$$

and in particular,

$$|\langle U(t)^*G(t), U(s)^*F(s)\rangle| \le A^{3-2/p'}|t-s|^{-\sigma(2/p'-1)}\|G(t)\|_{L^{p'}(X)}\|F(s)\|_{L^{p'}(X)}. \tag{C.13}$$

4. Finally we invoke the Hardy-Littlewood-Sobolev inequality which says that if $K_a(t) = |t|^{-1/a}$ and $1 < a < \infty$ then

$$\|K_a * u\|_{L^r(\mathbb{R})} \le C\|u\|_{L^{p'}(\mathbb{R})}, \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{a}, \quad 1 < p' < r, \tag{C.14}$$

see [H1, Theorem 4.5.3]. To obtain (C.12) from (C.13) we apply (C.14) with

$$\frac{1}{a} = \sigma\left(\frac{2}{p'} - 1\right), \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{a}, \quad p = r,$$

which has a unique solution

$$p = \frac{2(1+\sigma)}{\sigma}.$$

This completes the proof. □

Appendix D. Fredholm theory

In this appendix we will describe the role of the Schur complement formula in spectral theory, in particular in analytic Fredholm theory. Our presentation follows [S-Z2].

D.1 Grushin problems

Linear algebra. The *Schur complement formula* states for two-by-two systems of matrices that if

$$\begin{pmatrix} P & R_- \\ R_+ & R_0 \end{pmatrix}^{-1} = \begin{pmatrix} E & E_+ \\ E_- & E_0 \end{pmatrix},$$

then P is invertible if and only if E_0 is invertible, with

$$P^{-1} = E - E_+ E_0^{-1} E_-, \quad E_0^{-1} = R_0 - R_+ P^{-1} R_-. \tag{D.1}$$

Generalization. We can generalize to problems of the form

$$\begin{pmatrix} P & R_- \\ R_+ & O \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{D.2}$$

where

$$P : X_1 \to X_2, \quad R_+ : X_1 \to X_+, \quad R_- : X_- \to X_2,$$

for appropriate Banach spaces X_1, X_2, X_+, X_-. We call (D.2) a *Grushin problem.* (In practice, we start with an operator P and build a Grushin problem by choosing $R_\pm$, in which case it is normally sufficient to take $R_0 = 0$.)

If the Grushin problem (D.2) is invertible, we call it *well-posed* and we write its inverse as follows:

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_0 \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{D.3}$$

for operators

$$E : X_2 \to X_1, \quad E_0 : X_+ \to X_-, \quad E_+ : X_+ \to X_1, \quad E_- : X_2 \to X_-.$$

LEMMA D.1 (The operators in a Grushin problem). *If* (D.2) *is well-posed, then the operators R_+, E_- are surjective, and the operators E_+, R_- are injective.*

D.2 Fredholm operators

DEFINITIONS. (i) A bounded linear operator $P : X_1 \to X_2$ is called a *Fredholm operator* if the kernel of P,

$$\ker P := \{u \in X_1 \mid Pu = 0\},$$

and the cokernel of P,

$$\operatorname{coker} P := X_2/\overline{PX_1}, \text{ where } PX_1 := \{Pu \mid u \in X_1\},$$

are both finite dimensional.

(ii) The *index* of a Fredholm operator is

$$\operatorname{ind} P := \dim \ker P - \dim \operatorname{coker} P.$$

EXAMPLE. Many important Fredholm operators have the form

$$P = I + K, \tag{D.4}$$

where K a compact operator mapping a Banach space X to itself.

Theorem D.3 below shows that the index does not change under continuous deformations of Fredholm operators (with respect to operator norm topology). Hence for operators of the form (D.4) the index is 0:

$$\operatorname{ind} P = \operatorname{ind}(I + tK) = \operatorname{ind} I = 0 \quad (0 \leq t \leq 1).$$

□

The connection between Grushin problems and Fredholm operators is this:

THEOREM D.2 (Grushin problem for Fredholm operators). (i) *Suppose that $P : X_1 \to X_2$ is a Fredholm operator.*

Then there exist finite dimensional spaces $X_\pm$ and operators $R_- : X_- \to X_2$, $R_+ : X_1 \to X_+$, for which the Grushin problem (D.2) *is well posed. In particular, $PX_1 \subset X_2$ is closed.*

(ii) *Conversely, suppose that that for some choice of spaces $X_\pm$ and operators $R_\pm$, the Grushin problem* (D.2) *is well posed.*

Then $P : X_1 \to X_2$ is a Fredholm operator if and only if $E_0 : X_+ \to X_-$ is a Fredholm operator; in which case

$$\operatorname{ind} P = \operatorname{ind} E_0. \tag{D.5}$$

Assertion (ii) is particularly useful when the spaces $X_\pm$ are finite dimensional.

Proof. 1. Assume $P : X_1 \to X_2$ is Fredholm. Let $n_+ := \dim \ker P$ and $n_- := \dim \operatorname{coker} P$, and write $X_+ := \mathbb{C}^{n_+}, X_- := \mathbb{C}^{n_-}$. Select then linear operators

$$R_- : X_- \to X_2, \quad R_+ : X_1 \to X_+,$$

of maximal rank such that

$$R_- X_- \cap PX_1 = \{0\}, \quad \ker(R_+|_{\ker P}) = \{0\}.$$

Then the operator

$$\begin{pmatrix} P & R_- \\ R_+ & O \end{pmatrix}$$

has a trivial kernel and is onto. Hence it is invertible, and by the Open Mapping Theorem the inverse is continuous.

In particular, consider P acting on the quotient space $X_1/\ker P$, which is a Banach space since $\ker P$ is closed. We have $n_+ = 0$, and

$$PX_1 = P(X_1/\ker P) = \begin{pmatrix} P & R_- \end{pmatrix} \begin{pmatrix} X_1/\ker P \\ \{0\} \end{pmatrix}$$

is a closed subspace.

2. Conversely, suppose that Grushin problem (D.2) is well-posed. According to Lemma D.1, the operators R_+, E_- are surjective, and the operators E_+, R_- are injective. We take $u_- = 0$. Then

$$\text{(D.6)} \qquad \begin{cases} \text{the equation } Pu = v \text{ is equivalent to} \\ u = Ev + E_+v_+, \ 0 = E_-v + E_0v_+. \end{cases}$$

This means that

$$E_- : \operatorname{Im} P \to \operatorname{Im} E_0,$$

and so we can define the induced map

$$E^{\#} : X_2/\operatorname{Im} P \to X_-/\operatorname{Im} E_0.$$

Since E_- is surjective, so is $E^{\#}$. Also, $\ker E^{\#} = \{0\}$. This follows since if $E_-v \in \operatorname{Im} E_0$, we can use (D.6) to deduce that $v \in \operatorname{Im} P$. Hence $E^{\#}$ is a bijection of the cokernels $X_2 \operatorname{Im} P$ and $X_-/\operatorname{Im} E_0$.

3. Next, we claim that

$$E_+ : \ker E_0 \to \ker P$$

is a bijection. Indeed, if $u \in \ker P$, then $u = E_+v_+$ and $E_0v_+ = 0$. Therefore E_+ is onto; and this is all we need check, since E_+ injective.

We conclude that

$$\dim \ker P = \dim \ker E_0, \quad \dim \operatorname{coker} P = \dim \operatorname{coker} E_0.$$

In particular, the indices of P and E_0 are equal. □

THEOREM D.3 (Invariance of the index under deformations). *The set of Fredholm operators is open in $L(X_1, X_2)$, and the index is constant in each component.*

Proof. When P is a Fredholm operator, we can use Theorem D.2 to obtain $E_0 : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}$, with

$$\operatorname{ind} E_0 = n_+ - n_-. \tag{D.7}$$

by the Rank-Nullity Theorem of linear algebra. The Grushin problem remains well-posed (with the same operators $R_\pm$) if P is replaced by P', provided $\|P - P'\| < \epsilon$ for some sufficiently small $\epsilon > 0$. Hence the set of Fredholm operators is open.

Using (D.7) we see that the index of P' is the same as the index of P. Consequently it remains constant in each connected component of the set of Fredholm operators. □

We refer to Hörmander [H2, Sect.19.1] for a comprehensive introduction to Fredholm operators

D.3 Meromorphic continuation of operators.

The Grushin problem framework provides an elegant proof of the following standard result:

THEOREM D.4 (Analytic Fredholm Theory). *Suppose $\Omega \subset \mathbb{C}$ is a connected open set and $\{A(z)\}_{z\in\Omega}$ is a family of Fredholm operators depending holomorphically on z.*

Then if $A(z_0)^{-1}$ exists at some point $z_0 \in \Omega$, the mapping $z \mapsto A(z)^{-1}$ is a meromorphic family of operators on Ω.

Proof. 1. Fix $z_1 \in \Omega$. We form a Grushin problem for $P = A(z_1)$, as described in the proof of Theorem D.2. The same operators $R_\pm^{z_1}$ also provide a well-posed Grushin problem for $P = A(z)$ for z in some sufficiently small neighborhood $V(z_1)$ of z_1.

According to Theorem D.3

$$\operatorname{ind} A(z) = \operatorname{ind} A(z_0) = 0.$$

Consequently

$$n_+ = n_- = n,$$

and $E_0^{z_1}(z)$ is an $n \times n$ matrix with holomorphic coefficients. The invertibility of $E_0^{z_1}(z)$ is equivalent to the invertibility of $A(z)$.

2. This shows that there exists a locally finite covering $\{\Omega_j\}$ of Ω, and a family of functions f_j, holomorphic in Ω_j, such that if $z \in \Omega_j$, then $A(z)$ is invertible precisely when

$$f_j(z) \neq 0.$$

Indeed, we can define $f_j := \det E_0^z$, where E_0^z exists for $z \in \Omega_j$ by the construction in Step 1. Since Ω is connected and since $A(z_0)$ is invertible for at least one $z_0 \in \Omega$, none of f_j's is identically zero.

So det $E_0(z)$ a non-trivial holomorphic function in $V(z_1)$; and consequently $E_0(z)^{-1}$ is a meromorphic family of matrices. Applying (D.1), we conclude that

$$A(z)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z)$$

is a meromorphic family of operators in the neighborhood $V(z_1)$. Since z_1 was arbitrary, $A(z)^{-1}$ is in fact meromorphic in all of Ω. $\square$

Appendix E. Symbol calculus on manifolds

E.1 Definitions. For reader's convenience we provide here some basic definitions.

DEFINITION. An n-dimensional *manifold* M is a Hausdorff topological space with a countable basis, each point of which has a neighbourhood homeomorphic to some open set in $\mathbb{R}^n$.

We say that M is a *smooth* (or C^∞) manifold if there exists a family $\mathcal{F}$ of homeomorphisms between open sets:

$$\gamma \ : \ U_\gamma \longrightarrow V_\gamma, \quad U_\gamma \subset M, \quad V_\gamma \subset \mathbb{R}^n,$$

satisfying the following properties:

(i)(Smooth overlaps) If $\gamma_1, \gamma_2 \in \mathcal{F}$ then

$$\gamma_2 \circ \gamma_1^{-1} \in C^\infty(V_{\gamma_2} \cap V_{\gamma_1}; V_{\gamma_1} \cap V_{\gamma_2}).$$

(ii)(Covering) The open sets U_γ cover M:

$$\bigcup_{\gamma \in \mathcal{F}} U_\gamma = M.$$

(iii) (Maximality) Let λ be a homeomorphism of an open set $U_\lambda \subset M$ onto an open set $V_\lambda \subset \mathbb{R}^n$. If for all $\gamma \in \mathcal{F}$,

$$\gamma \circ \lambda^{-1} \in C^\infty(V_\lambda \cap V_\gamma; V_\lambda \cap V_\gamma),$$

then $\lambda \in \mathcal{F}$.

We call $\{(\gamma, U_\gamma) \mid \gamma \in \mathcal{F}\}$ an *atlas* for M. The open set $U_\gamma \subset M$ is a *coordinate patch*.

DEFINITION. A C^∞ complex *vector bundle* over M with fiber dimension N consists of

(i) a C^∞ manifold V,

(ii) a C^∞ map $\pi : V \to M$, defining the *fibers* $V_x := \pi^{-1}(\{x\})$ for $x \in M$, and

(iii) local isomorphisms

$$V \supset \pi^{-1}(Y) \xrightarrow{\psi} Y \times \mathbb{C}^N, \qquad \psi(V_x) = \{x\} \times \mathbb{C}^N, \quad \psi|_{V_x} \in GL(N, \mathbb{C}), \tag{E.1}$$

where $GL(N, \mathbb{C})$ is the group of invertible linear transformations on $\mathbb{C}^N$.

REMARKS. (i) We can choose a covering $\{X_i\}_{i\in I}$ of M such that for each index i there exists

$$\psi_i : \pi^{-1}(X_i) \to X_i \times \mathbb{C}^N$$

with the properties listed in (iii) in the definition of a vector bundle.

Then

$$g_{ij} := \psi_i \circ \psi_j^{-1} \in C^\infty(X_i \cap X_j; GL(N, \mathbb{C})).$$

These maps are the *transition matrices.*

(ii) It is important to observe that we can recover the vector bundle V from the transition matrices. To see this, suppose that we are given functions g_{ij} satisfying the identities

$$\begin{cases} g_{ij}(x) \circ g_{ji}(x) = I & \text{for } \; x \in X_i \cap X_j, \\ g_{ij}(x) \circ g_{jk}(x) \circ g_{kj}(x) = I, & \text{for } \; x \in X_i \cap X_j \cap X_k. \end{cases}$$

Now form the set $V' \subset I \times M \times \mathbb{C}^N$, with the equivalence relation $(i, x, t) \sim (i', x', t')$ if and only if $x = x'$ and $t' = g_{i'i}(x)t$. Then

$$V = V'/\sim .$$

□

DEFINITION. A *section* of the vector bundle V is a smooth map

$$u : M \to V$$

such that

$$\pi \circ u(x) = x \quad (x \in M).$$

We write

$$u \in \mathbb{C}^\infty(M, V).$$

EXAMPLE 1: Tangent bundle. Let M be a C^∞ manifold and let N be the dimension of M. We define the *tangent bundle* of M, denoted

$$T(M),$$

by defining the transition functions

$$g_{\gamma_i\gamma_j}(x) := \partial(\gamma_i \circ \gamma_j^{-1})(x) \in GL(n, \mathbb{R}).$$

for $x \in U_{\gamma_i} \cap U_{\gamma_j}$. Its sections $C^\infty(M, T(M))$ are the smooth *vectorfields* on M. □

EXAMPLE 2: Cotangent bundle. For any vector bundle we can define its dual,

$$V^* := \bigcup_{x\in X} (V_x)^*,$$

since we can take

$$g_{\gamma_i\gamma_j} = (g^*_{\gamma_i\gamma_j})^{-1}.$$

If $V = T(M)$, we obtain the *cotangent bundle*, denoted

$$T^*(M).$$

Its sections $C^\infty(M, T^*(M))$ are the differential *one-forms* on M. □

EXAMPLE 3: S-density bundles. Let M be an n-dimensional manifold and let $(U_\gamma, \gamma_\alpha)$ form a set of coordinate patches of X.

We define the *s-density bundle* over X, denoted

$$\Omega^s(M),$$

by choosing the following transition functions:

$$g_{\gamma_i\gamma_j}(x) := |\det \partial(\gamma_i \circ \gamma_j^{-1})|^s \circ \gamma_j(x),$$

for $x \in U_{\gamma_i} \cap U_{\gamma_j}$.

This is a *line bundle* over M, that is, a bundle with with fibers of complex dimension one. □

E.2 Pseudodifferential operators on manifolds.

Pseudodifferential operators. In this section M denotes a smooth, n-dimensional compact Riemannian manifold without boundary. As above, we have $\{(\gamma, U_\gamma) \mid \gamma \in \mathcal{F}\}$ for the atlas of M, where each γ is a smooth diffeomorphism of the coordinate patch $U_\gamma \subset M$ onto an open subset $V_\kappa \subset \mathbb{R}^n$.

NOTATION. Recall from §9.3 that a class symbols for which we have invariance under coordinate chages is given by

$$S^{m,k} = \{a \in C^\infty(\mathbb{R}^{2n}) \ : \ |\partial_x^\alpha \partial_\xi^\beta a| \le C_{\alpha\beta} h^{-k} \langle \xi \rangle^{m-|\beta|}\}.$$

The index k records how singular the symbol a is as $h \to 0$, and m controls the growth rate as $|\xi| \to \infty$.

DEFINITION. A linear operator

$$A : C^\infty(M) \to C^\infty(M)$$

is called a *pseudodifferential operator* if there exist integers m, k such that for each coordinate patch U_γ, there exists a symbol $a_\gamma \in S^{m,k}$ such that for any $\varphi, \psi \in C^\infty_{\rm c}(U_\gamma)$ and for each $u \in C^\infty(M)$

$$\varphi A(\psi u) = \varphi \gamma^* a^{\rm w}_\gamma(x, hD)(\gamma^{-1})^*(\psi u). \tag{E.2}$$

NOTATION. (i) In this case, we write

$$A \in \Psi^{m,k}(M),$$

and sometimes call A a *quantum observable*.

(ii) To simplify notation, we also put

$$\Psi^k(M) := \Psi^{0,k}(M), \quad \Psi(M) := \Psi^{0,0}(M).$$

The symbol of a pseudodifferential operator. Our goal is to associate with a pseudodifferential operator A a symbol a defined on T^*M, the cotangent space of M.

The first lemma is a direct consequence of Lemmma 9.14:

LEMMA E.1 (More on disjoint support). *Let $b \in S^{m,k}$ and suppose $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$. If*

$$\operatorname{spt}(\varphi) \cap \operatorname{spt}(\psi) = \emptyset, \tag{E.3}$$

then

$$\|\varphi\, b^w(x, hD)\, \psi\|_{H^{-N} \to H^N} = O(h^\infty) \tag{E.4}$$

for all N.

THEOREM E.2 (Symbol of a pseudodifferential operator). *There exist linear maps*

$$\sigma : \Psi^{m,k}(M) \to S^{m,k}/S^{m,k-1}(T^*M) \tag{E.5}$$

and

$$\operatorname{Op} : S^{m,k}(T^*M) \to \Psi^{m,k}(M) \tag{E.6}$$

such that

$$\sigma(A_1 A_2) = \sigma(A_1)\sigma(A_2) \tag{E.7}$$

and

$$\sigma(\operatorname{Op}(a)) = [a] \in S^{m,k}/S^{m,k-1}(T^*M). \tag{E.8}$$

We call $a = \sigma(A)$ the *symbol* of the pseudodifferential operator A.

REMARK. In the identity (E.8) "$[a]$" denotes the equivalence class of a in $S^{m,k}/S^{m,k-1}(T^*M)$. This means that

$$[a] = [\hat{a}] \quad \text{if and only if} \quad a - \hat{a} \in S^{m,k-1}(T^*M).$$

The symbol is therefore uniquely defined in $S^{m,k}$, up to a lower order term which is less singular as $h \to 0$. □

Proof. 1. Let U be an open subset of $\mathbb{R}$. Suppose that $B : C_c^\infty(U) \to C^\infty(U)$ and that for all $\varphi, \psi \in C^\infty$ the mapping $u \longmapsto \varphi B \psi u$ belongs to $\Psi^{m,k}(\mathbb{R}^n)$, for all $u \in \mathcal{S}$.

We claim that there then exists a symbol $a \in S^k_{\rm loc}(U, \langle\xi\rangle^m)$ such that

$$B = a(x, D) + B_0, \tag{E.9}$$

where for all m

$$B_0 : H_c^{-m}(U) \to H^m_{\rm loc}(U) \quad \text{is } O(h^\infty). \tag{E.10}$$

To see this, first choose a locally finite partition of unity $\{\psi_j\}_{j\in J} \subset C_c^\infty(U)$:

$$\sum_{j\in J} \psi_j(x) \equiv 1 \qquad (x \in U).$$

Then

$$\psi_j B \psi_k = a^{\rm w}_{jk}(x, hD),$$

where $a_{jk} \in S^k(\langle\xi\rangle)$ and $a_{jk}(x,\xi) = 0$ if $x \notin {\rm spt}\psi_j$. Now put

$$a := {\sum_{j,k}}' a_{jk}(x,\xi) \in S^k_{\rm loc}(\langle\xi\rangle^m),$$

where we are sum over those indices j, k's for which ${\rm spt}\psi_j \cap {\rm spt}\psi_k \neq \emptyset$. This sum is consequently locally finite.

2. We must next verify (E.10) for

$$B_0 := B - a(x, hD) = {\sum_{j,k}}'' \psi_j B \psi_k,$$

the sum over j, k's for which

$${\rm spt}\psi_j \cap {\rm spt}\psi_k = \emptyset.$$

Let $K_B(x, y)$ be the Schwartz kernel of B. Then the Schwartz kernel of B_0 is

$$K_{B_0}(x, y) = {\sum_{j,k}}'' \psi_j(x) K_B(x, y) \psi_k(y), \tag{E.11}$$

with the sum locally finite in $U \times U$. The operators $\psi_j B \psi_k$ satisfy the assumptions of Lemma 7.4, and hence have the desired mapping property. Because of the local finiteness of (E.11) we get the global mapping property from $H^{-m}_{\rm loc}$ to $H^m_{\rm loc}$.

3. For each coordinate chart (γ, U_γ), where $\gamma : U_\gamma \to V_\gamma$, we can now use (E.9) with $X = V_\gamma$ and $B = (\gamma^{-1})^* A \gamma^*$, to define $a_\gamma \in T^*(U_\gamma)$.

The second part of Theorem 9.12 shows that if $U_{\gamma_1} \cap U_{\gamma_2} \neq \emptyset$, then

$$(a_{\gamma_1} - a_{\gamma_2})|_{U_{\kappa_1} \cap U_{\gamma_2}} \in S^{k-1}(T^*(U_{\gamma_1} \cap U_{\gamma_2}), \langle\xi\rangle^m). \tag{E.12}$$

Suppose now that we choose a covering of M by coordinate charts, $\{U_\alpha\}_{\alpha\in J}$, and a locally finite partition of unity $\{\varphi_\alpha\}_{\alpha\in J}$:

$$\mathrm{spt}\varphi_\alpha \subset U_\gamma, \quad \sum_{\alpha\in J}\varphi_j(x)\equiv 1,$$

and define

$$a := \sum_{\alpha\in J}\varphi_\alpha a_\alpha.$$

We see from (E.12) that $a \in S^k(T^*M, \langle\xi\rangle^m)$ is invariantly defined up to terms in $S^{k-1}(T^*M, \langle\xi\rangle^m)$. We consequently can define

$$\sigma(A) := [a] \in S^k(T^*M, \langle\xi\rangle^m)/S^{k-1}(T^*M, \langle\xi\rangle^m).$$

4. It remains to show the existence of

$$\mathrm{Op} \ : \ S^k(T^*M, \langle\xi\rangle^m) \longrightarrow \Psi^{m,k}(M), \quad \sigma(\mathrm{Op}(a)) = [a].$$

Suppose that for our covering of M by coordinate charts, $\{U_\alpha\}_{\alpha\in J}$, we choose $\{\psi_\alpha\}_{\alpha\in J}$ such that

$$\mathrm{spt}\psi_\alpha \subset U_\gamma, \quad \sum_{\alpha\in J}\psi_j^2(x)\equiv 1,$$

a sum which is locally finite. Define

$$A := \sum_{\alpha\in J}\psi_\alpha\gamma_\alpha^*\mathrm{Op}(\tilde{a}_\alpha)(\gamma_\alpha^{-1})^*\psi_\alpha,$$

where $\tilde{a}_\alpha(x,\xi) := a(\gamma_\alpha^{-1}(x), (\partial\gamma(x)^T)^{-1}\xi)$. Theorem 9.12 demonstrates that $\sigma(A)$ equals $[a]$. □

Pseudodifferential operators acting on half-densities. We now apply the full strength of Theorem 9.12 by making the pseudodifferential operators act on half-densities.

DEFINITION. A linear operator

$$A : C^\infty(M, \Omega^{\frac{1}{2}}(M)) \to C^\infty(M, \Omega^{\frac{1}{2}}(M))$$

is called a *pseudodifferential operator on half-densities* if there exist integers m, k such that for each coordinate patch U_α, and there exists a symbol $a_\alpha \in S^k(\langle\xi\rangle^m)$ such that for any $\varphi, \psi \in C_{\mathrm{c}}^\infty(U_\gamma)$

$$\varphi A(\psi u) = \varphi\gamma_\alpha^* a_\alpha^{\mathrm{w}}(x, hD)(\gamma_\alpha^{-1})^*(\psi u) \tag{E.13}$$

for each $u \in C^\infty(M, \Omega^{\frac{1}{2}}(M))$.

NOTATION. In this case, we write

$$A \in \Psi^{m,k}(M, \Omega^{\frac{1}{2}}(M)).$$

By adapting the proof of Theorem E.2 to the case of half-densities using the first part of Theorem 9.12 we obtain

THEOREM E.3 (Symbol on half-densities). *There exist linear maps*

$$\sigma : \Psi^{m,k}(M, \Omega^{1/2}(M)) \to S^{m,k}/S^{m,k-2}(T^*M) \tag{E.14}$$

and

$$\mathrm{Op} : S^{m,k}(T^*M) \to \Psi^{m,k}(M, \Omega^{1/2}(M))) \tag{E.15}$$

such that

$$\sigma(A_1 A_2) = \sigma(A_1)\sigma(A_2) \tag{E.16}$$

and

$$\sigma(\mathrm{Op}(a)) = [a] \in S^{m,k}/S^{m,k-2}(T^*M). \tag{E.17}$$

E.3 PDE on manifolds.

We revisit in this last section some of our theory from Chapters 5–7, replacing the flat spaces $\mathbb{R}^n$ and $\mathbb{T}^n$ by an arbitrary compact Riemannian manifold (M, g), for the metric

$$g := \sum_{i,j=1}^{n} g_{ij} dx_i dx_j.$$

Write

$$((g^{ij})) := ((g_{ij}))^{-1}, \quad \bar{g} := \det((g_{ij})).$$

E.3.1 Notation.

Tangent, cotangent bundles. We can use the metric to build an identification of the tangent and cotangent bundles of M. We identify

$$\xi \in T_x^*M \quad \text{with} \quad X \in T_xM,$$

written $\xi \sim X$, provided

$$\xi(Y) = g_x(Y, X)$$

for all $Y \in T_xM$.

Flows. Under the identification $X \sim \xi$, the flow of H_p on T^*M, generated by the symbol

$$p := |\xi|_g^2 = \sum_{i,j=1}^n g^{ij}\xi_i\xi_j = \sum_{i,j=1}^n g_{ij}X_iX_j = g(X,X), \tag{E.18}$$

is the geodesic flow on TM.

Laplace-Beltrami operator. The *Laplace-Beltrami operator* Δ_g on M is defined in local coordinates by

$$\Delta_g := \frac{1}{\sqrt{\bar{g}}}\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(g^{ij}\sqrt{\bar{g}}\frac{\partial}{\partial x_j}\right). \tag{E.19}$$

The function p defined by (E.18) is the symbol of the Laplace-Beltrami operator $-h^2\Delta_g$.

PDE on manifolds. Given then a potential $V \in C^\infty(M)$, we can define the Schrödinger operator

$$P(h) := -h^2\Delta_g + V(x). \tag{E.20}$$

The flat wave equation from Chapter 5 is replaced by an equation involving the Laplace-Beltrami operator:

$$(\partial_t^2 + a(x)\partial_t - \Delta_g)u = 0. \tag{E.21}$$

The unknown u is a function of $x \in M$ and $t \in \mathbb{R}$.

Half-densities. Half-densities on M can be identified with functions using the Riemannian density:

$$u = u(x)|dx|^{\frac{1}{2}} = \tilde{u}(x)\left(\bar{g}^{\frac{1}{2}}dx\right)^{\frac{1}{2}}.$$

E.3.2 Damped wave equation on manifolds. We consider this initial-value problem for the wave equation:

$$\begin{cases} (\partial_t^2 + a(x)\partial_t - \Delta)u = 0 & \text{on } M \times \mathbb{R} \\ u = 0,\ u_t = f & \text{on } M \times \{t = 0\}, \end{cases} \tag{E.22}$$

where $a \geq 0$; and, as in Chapter 6, define the energy of a solution at time t to be

$$E(t) := \frac{1}{2}\int_M (\partial_t u)^2 + |\partial_x u|^2\, dx.$$

It is then straightforward to adapt the proofs in §5.3 to establish

THEOREM E.4 (Exponential decay on manifold). *Suppose u solves the wave equation with damping* (E.21)*, with the initial conditions*

Assume also that there exists a time $T > 0$ such that each geodesic of length greater than or equal to T intersects the set $\{a > 0\}$.

Then there exist constants $C, \beta > 0$ such that

$$E(t) \leq Ce^{-\beta t}\|f\|_{L^2} \tag{E.23}$$

for all times $t \geq 0$.

E.3.3 Weyl's Law for compact manifolds. More work is needed to generalize Weyl's Law from Chapter 6 to manifolds. We will prove it using a different approach, based on the Spectral Theorem.

First, we need to check that the spectrum is discrete and that follows from the compactness of the resolvent:

LEMMA E.5 (Resolvent on manifold). *If P is defined by* (E.20)*, then*

$$(P + i)^{-1} = O(1) \ : \ L^2(M) \to H^2_h(M),$$

where the semiclassical Sobolev spaces are defined as in §7.1.

We prove this by the same method as that for Lemma 7.1.

Eigenvalues and eigenfunctions. According to Riesz's Theorem on the discreteness of the spectrum of a compact operator, we conclude that the spectrum of $(P + i)^{-1}$ is discrete, with an accumulation point at 0.

Hence we can write

$$P(h) = \sum_{j=1}^{\infty} E_j(h)u_j(h) \otimes \overline{u_j(h)}, \tag{E.24}$$

where $\{u_j(h)\}_{j=1}^{\infty}$ is an orthonormal set of all eigenfunctions of $P(h)$:

$$P(h)u_j(h) = E_j(h)u_j(h), \quad \langle u_k(h), u_l(h)\rangle = \delta_{lk},$$

and

$$E_j(h) \to \infty.$$

THEOREM E.6 (Functional calculus). *Suppose that f is a holomorphic function, such that for $|\mathrm{Im} z| \le 2$ and any N:*

$$f(z) = O(\langle z \rangle^{-N}).$$

Define

$$f(P) := \tag{E.25}$$
$$\frac{1}{2\pi i}\int_{\mathbb{R}} (t-i-P)^{-1} f(t-i) - (t+i-P)^{-1} f(t+i)\, dt.$$

Then $f(P) \in \Psi^{-\infty}(M)$, with

$$\sigma(f(P)) = f(|\xi|_g^2 + V(x)).$$

Furthermore,

$$f(P) = \sum_{j=1}^{\infty} f(E_j(h)) u_j(h) \otimes \overline{u_j(h)} \tag{E.26}$$

in L^2.

Proof. 1. The statement (E.26) follows from (E.24), which shows that

$$(P-z)^{-1} = \sum_{j=1}^{\infty} \frac{u_j \otimes \overline{u_j}}{E_j(h) - z}.$$

Since f decays rapidly as $t \to \infty$, we can compute residues in (E.25) to conclude that

$$f(E_j(h)) =$$
$$\frac{1}{2\pi i}\int_{\mathbb{R}} \big((t-i-E_j(h))^{-1} f(t-i) - (t+i-E_j(h))^{-1} f(t+i)\big)\, dt.$$

2. We now use Beals's Theorem 9.8, to deduce that $f(P)$ is a pseudodifferential operator. As discussed in Appendix E all we need to show is that for $\varphi, \psi \in C_c^\infty(M)$, with supports in arbitrary coordinate patches, $\varphi f(P) \psi$ is a pseudodifferential operator. As described there it can be considered as an operator on $\mathbb{R}^n$ and, by Theorem 9.8, it suffices to check that for any linear $l_j(x,\xi)$ we have

$$\|\mathrm{ad}_{l_1(x,hD)} \circ \cdot \circ \mathrm{ad}_{l_N(x,hD)} f(P)\|_{L^2 \to L^2} = O(h^N).$$

To show this, note that according to Lemma E.5,

$$\|(P-t\pm i)^{-1}(\mathrm{ad}_{L_1} \circ \cdots \circ \mathrm{ad}_{L_k} P(P-t\pm i)^{-1}\|_{L^2\to L^2} = O(h^k),$$

where $L_j \in \Psi^{0,0}(M)$. Now for a linear function l on $\mathbb{R}^{2n}$,

$$\begin{aligned}&\mathrm{ad}_{l(x,hD)}(\varphi(P-t\pm i)^{-1}\psi) = \\ &\quad -(P-t\pm i)^{-1}(\mathrm{ad}_L P)(P-t\pm i)^{-1} + O_{L^2\to L^2}(h),\end{aligned}$$

where $L \in \Psi^{0,0}(M)$. The rapid decay of f gives

$$\|\mathrm{ad}_L \int_{\mathbb{R}} f(t)(t\pm i-P)^{-1}dt\| \leq$$

$$\int_{\mathbb{R}} |f(t)| \|(P-t\pm i)^{-1}\mathrm{ad}_L P(P-t\pm i)^{-1}\|_{L^2\to L^2} dt = O(h),$$

and this argument can be easily iterated.

3. Since

$$\mathrm{Op}(|\xi|_g^2 + V(x) - t \pm i)^{-1})(P-t\pm i) = I + O_{L^2\to L^2}(h),$$

it follows that

$$\mathrm{Op}(|\xi|_g^2 + V(x) - t\pm i)^{-1}) = (P-t\pm i)^{-1} + O_{L^2\to L^2}(h).$$

Hence the symbol of $(P+t\pm i)^{-1}$ (which we already know is a pseudodifferential operator) is given by $(|\xi|_g^2 + V(x) - t \pm i)^{-1}$.

A residue calculation now shows us that

$$f(P) = \mathrm{Op}(f(|\xi|_g^2 + V(x) - t\pm i)) + O_{L^2\to L^2}(h);$$

that is, the symbol of $f(P)$ is $f(|\xi|_g^2 + V(x))$. □

THEOREM E.7 (Weyl's asymptotics on compact manifolds). *For any $a < b$, we have*

$$\begin{aligned}&\#\{E(h) \mid a \leq E(h) \leq b\} = \\ &\qquad \frac{1}{(2\pi h)^n}(\mathrm{Vol}_{T^*M}\{a \leq |\xi|_g^2 + V(x) \leq b\} + o(1))\end{aligned} \tag{E.27}$$

as $h \to 0$.

Proof. 1. Let f_1, f_2 be two functions satisfying the assumptions of Theorem E.6 such that for real x

$$f_1(x) \leq \mathbf{1}_{[a,b]}(x) \leq f_2(x), \tag{E.28}$$

where $\mathbf{1}_{[a,b]}(x)$ is the characteristic function of the interval $[a,b]$.

It follows that

$$\mathrm{tr} f_1(P) \leq \#\{E(h) \mid a \leq E(h) \leq b\} \leq \mathrm{tr} f_2(P).$$

2. Theorem C.9 now shows that for $j = 1, 2$

$$\mathrm{tr} f_j(P) = \frac{1}{(2\pi h)^n} \left(\int_{T^*M} f_j(|\xi|_g^2 + V(x)) dx d\xi + O(h) \right).$$

We note that since $f_j(P) \in \Psi^{-\infty}(M)$, the errors in the symbolic computations are all $O(h\langle\xi\rangle^{-\infty})$, and hence can be integrated.

3. The final step is to construct f_1^ϵ and f_2^ϵ satisfying the hypotheses of Theorem E.6 and (E.28), and such that for $j = 1, 2$, we have

$$\int_{T^*M} f_j^\epsilon(|\xi|_g^2 + V(x)) dx d\xi \to \mathrm{Vol}_{T^*M}\{a \leq |\xi|_g^2 + V(x) \leq b\},$$

as $\epsilon \to 0$. This is done as follows. Define

$$\chi_1^\epsilon := (1-\epsilon)\mathbf{1}_{[a+\epsilon, b-\epsilon]} - \epsilon(\mathbf{1}_{[a-\epsilon, a+\epsilon]} + \mathbf{1}_{[b-\epsilon, b+\epsilon]}), \quad \chi_2^\epsilon := (1+\epsilon)\mathbf{1}_{[a-\epsilon, b+\epsilon]},$$

and then put

$$f_j^\epsilon(z) := \frac{1}{2\pi\epsilon} \int_{\mathbb{R}} \chi_j^\epsilon(x) \exp\left(-\frac{(x-z)^2}{2\epsilon^2} \right) dx.$$

We easily check that all the assumptions are satisfied. □

REMARKS. (i) If $V \equiv 0$, we recover the leading term in the usual Weyl asymptotics of the Laplacian on a compact manifold: let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ be the complete set of eigenvalues of $-\Delta_g$ on M. Then

$$\text{(E.29)} \qquad \#\{j \,:\, \lambda_j \leq r\} \sim \frac{\mathrm{Vol}(B_{\mathbb{R}^n}(0,1))}{(2\pi)^n} \mathrm{Vol}(M) r^{n/2}, \quad r \to \infty.$$

In fact, we can take $a = 0$, $b = 1$, and $h = 1/\sqrt{r}$, and apply Theorem E.7: the eigenvalues $-\Delta_g$ are just rescaled eigenvalues of $-h^2\Delta_g$ and the $\mathrm{Vol}(B_{\mathbb{R}^n}(0,1))$ term comes from integrating out the ξ variables.

We note also that (E.29) implies that

$$\text{(E.30)} \qquad j^{2/n}/C_M \leq \lambda_j \leq C_M j^{2/n}.$$

(ii) Also, upon rescaling and applying Theorem C.8, we obtain estimates for counting *all* the eigenvalues of $P(h) = -h^2\Delta_g + V(x)$. Let $E_0(h) < E_1(h) \leq \cdots \leq E_j(h) \to \infty$ be all the eigenvalues of the self-adjoint operator $P(h)$. Then for $r > 1$,

$$\text{(E.31)} \qquad \#\{j \,:\, E_j(h) \leq r\} \leq C_{M,V} h^{-n} r^{n/2}.$$

This crude estimate will be useful in §9.3. □

Sources and further reading

Chapter 1: The book of Griffiths [G] provides a nice elementary introduction to quantum mechanics. For a modern physical perspective, consult Heller–Tomsovic [H-T] or Stöckmann [St].

Chapter 2: The proof of Theorem 2.11 is from Moser [Mo]; see also Cannas da Silva [CdS]. A PDE oriented introduction to symplectic geometry may be found in Hörmander [H3, Chapter 21].

Chapter 3: Good references are Friedlander and M. Joshi [F-J] and Hörmander [H1]. The PDE example in §3.1 is from [H1, Section 7.6].

Chapter 4: The presentation of semiclassical calculus is based uponM. Dimassi–Sjöstrand [D-S, Chapter 7]. See also Martinez [M], in particular for the Fefferman-Cordoba proof of the sharp Gårding inequality. The argument presented here followed the proof of [D-S, Theorem 7.12].

Chapter 5: Semiclassical defect measures were introduced independently in Gérard [Ge] and Lions–Paul [L-P]; see also Tartar [T]. The basic results presented here come from [Ge]. Theorem 5.8 comes from [R-T], but the proof here follows [L] and uses also some ideas of Morawetz.

Chapter 6: The proof of Weyl asymptotics is a semiclassical version of the classical Dirichlet-Neumann bracketting proof for the bounded domains.

Chapter 7: Estimates in the classically forbidden region in §7.1 are known as Agmon or Lithner-Agmon estimates. They play a crucial role in the analysis of spectra of multiple well potential and of the Witten complex: see [D-S, Chapter 6] for an introduction and references. Here we followed an argument of [N], but see also [**?**, Proposition 3.2]. The presentation of Carleman estimates in §7.2 is based on discussions with N. Burq and D. Tataru.

Chapter 8: The Quantum Ergodicity Theorem 8.4 is from a 1974 paper of Shnirelman, and it is sometimes referred to as Shnirelman's Theorem. The first complete proof, in a different setting, was provided by Zelditch. We have followed his more recent proof, as presented in [**?**]. The same proof applied with finer spectral asymptotics gives a stronger semiclassical version, first presented in [**?**].

Chapter 9: For $h = 1$, $\Psi^{m,k}$ form the class of Kohn-Nirenberg pseudodifferential operators: see [H2, §18.1] or [G-S] for a thorough presentation. Much can be said about the properties of semiclassical wave front sets and we refer to Alexandrova [A] for a recent discussion.

The proof of symbol invariance is from the appendix to [?]. The semiclassical wavefront set is an analog of the usual wavefront set in microlocal analysis – see [H2] and is closely related to the frequency set introduced in [?]. [?] presented the semiclassical pointwise bounds reproduced here.

Our presentation of Beals's Theorem follows [D-S, Chapter 8], where it was based on [?]. Theorem 9.9, in a much greater generality, was proved in [?]. The self-contained proof in the simple case considered here comes from the appendix to [?].

Section 9.4 a special of a general result in [?, Théoreme 6.4]. See [?], [?] for examples of conjugation techniques, and [M] for a slightly different perspective.

Chapter 10: The definition of quantization of symplectomorphisms using deformation follows the *Heisenberg picture of quantum mechanics*. The proof of Theorem 10.2 comes from [Ch, Section 3] where a stronger version of the result is also given. The construction of $U(t)$ borrows from the essentially standard presentation in [?, Section 7]. For the discussion of the Maslov index see [?] and [?]. Fourier integral operators which are closely related to our discussion of quantization and of propagators are discussed in detail in [?] and [H2, Chapter 25].

Semiclassical Strichartz estimates for $P = -h^2\Delta_g - 1$ appeared explicitely in [?] who used them to prove existence results for non-linear Schrödinger equations on two and three dimensional compact manifolds. We refer to that paper for pointers to the vast literature on Strichartz estimates and their applications. The adaptation of Sogge's L^p estimates to the semiclassical setting comes from [?] and was inspired by discussions with N. Burq, H. Koch, C.D. Sogge, and D. Tataru, see [?] and [S].

The proofs for the theorems cited in §10.4 are in [H2, Theorem 21.1.6] and [H2, Theorem 21.1.6]. Theorem 10.18 is a semiclassical analog of the standard C^∞ result of Duistermaat-Hörmander [H2, Proposition 26.1.3′]. Theorem 10.19 is a semiclassical adaptation of a microlocal result of Duistermaat-Sjöstrand [H2, Proposition 26.3.1].

Theorem 10.20 was proved in one dimension in [?]. See also [?] for more on quasimodes and pseudospectra and for further references.

Appendices: Ilan Hirshberg provided us with Theorem C.4 and its proof.

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