18.156 LECTURE 3

Partitions of unity.

Proposition 1. *If O_a, $a \in A$, is any open cover of a manifold M then there exists $\chi_i \in \mathcal{C}^\infty_c(M)$, with the properties*

1. *For any compact set $K \subset M$ there are at most a finite number of i such that* $\operatorname{supp}(\chi_i) \cap K \neq \emptyset$.
2. *For each i, there exists $a = a(i)$ such that* $\operatorname{supp}(\chi_i) \subset O_a$.
3. $\sum_i \chi_i = 1$.

The first condition implies that the sum in the last, partition of unity, condition only involves a finite sum. The second condition is that the partition of unity be 'subordinate to the open cover O_a.' The first condition is that the supports are locally finite.

Linear differential operators of order m (with smooth coefficient). We may think of these as operators

$$P : \mathcal{C}^\infty(M;\mathbb{C}^N) \longrightarrow \mathcal{C}^\infty(M;\mathbb{C}^{N'}) \tag{1}$$

with two properties. First, locality, that $u = 0$ on any open subset of M implies $Pu = 0$. Secondly we demand that for each coordinate set $F : U \longrightarrow V \subset \mathbb{R}^n$, the local representative, defined by

$$P_F v = (F^{-1})^* P F^* v,\ v \in \mathcal{C}^\infty_{\mathrm{c}}(V) \text{ is a differential operator of order } m. \tag{2}$$

We denote the space of these operators by $\operatorname{Diff}^m(M;\mathbb{C}^N,\mathbb{C}^{N'})$.

Proposition 2. *The symbol of a differential operator of order m is a well-defined 'function' on T^*M with values in $N' \times N$ matrices which is a homogeneous polynomial of degree m on each fibre T^*_pM. If $P_m(T^*M;\mathbb{C}^{N'},\mathbb{C}^N)$ denotes the space of such maps then there is a short exact sequence*

$$\operatorname{Diff}^{m-1}(M;\mathbb{C}^N,\mathbb{C}^{N'}) \longleftrightarrow \operatorname{Diff}^m(M;\mathbb{C}^N,\mathbb{C}^{N'}) \longrightarrow P_m(T^*M;\mathbb{C}^{N'},\mathbb{C}^N). \tag{3}$$

Proof. □

Densities, distributions, Sobolev spaces.

Next, recall the proof of local elliptic regularity. We have an elliptic system of variable coefficient operators on an open set $\Omega \subset \mathbb{R}^n$,

$$P(x, D_x) = \sum_{|\alpha| \le m} P_\alpha(x) D_x^\alpha \tag{4}$$

where $P_\alpha \in \mathcal{C}^\infty(\Omega; M(N)$ is a smooth family of $N \times N$ matrices and ellipticity means that

$$p(x,\xi) = \sum_{|\alpha|=m} P_\alpha(x)\xi^\alpha \neq 0 \ \text{ for } \xi \in \mathbb{R}^n \setminus \{0\},\ x \in \Omega. \tag{5}$$

Then we want to show that if $u \in H^s_{\mathrm{loc}}(\Omega;\mathbb{C}^N)$ and $f \in H^t_{\mathrm{loc}}(\Omega;\mathbb{C}^N)$ then

$$P(x, D_x)u = f \Longrightarrow u \in H^r_{\mathrm{loc}}(\Omega;\mathbb{C}^N),\ r = max(s, t + m). \tag{6}$$

Of course, if $s > t+m$ then it follows in fact that $f \in H^{s-m}_{\text{loc}}(\Omega;\mathbb{C}^N)$ so we might as well assume that $t < s-m$.

In fact it suffices to assume that $t = s-m+1$, i.e. that $u \in H^s_{\text{loc}}(\Omega;\mathbb{C}^N)$, that $f \in H^{s-m+1}_{\text{loc}}(\Omega;\mathbb{C}^N)$ and to conclude that $u \in H^{s+1}_{\text{loc}}(\Omega;\mathbb{C}^N)$. Indeed, we can recover (6) by repeated application of this.

So now we suppose

$$u \in H^s_{\text{loc}}(\Omega;\mathbb{C}^N),\ P(x,D_x)u = f \in H^{s-m+1}_{\text{loc}}(\Omega;\mathbb{C}^N). \tag{7}$$

Now, consider $\chi \in \mathcal{C}^\infty_{\text{c}}(\Omega)$, that $\chi(p) \neq 0$ and eventually that the support of χ is in a small ball around p. We can commute χ through the operator and write

$$P(x,D_x)\chi u = \chi f + [P(x,D_x),\chi]u = f' \in H^{s-m+1}_{\text{c}}(\Omega;\mathbb{C}^N), \tag{8}$$

since the commutator is a differential operators of order at most $m-1$.

Next we consider the differential operator $P_m(p;D_x) = \sum\limits_{|\alpha|=m} P_\alpha(p)D^\alpha_x$ obtained by 'freezing' the leading part at $x = p$. Then (8) can be written

$$P_m(p,D_x)\chi u + \sum_{|\alpha|=m}(P_\alpha(x)-P_\alpha(p))\chi' D^\alpha_x \chi u = f'' = f' - \sum_{|\alpha<m} P_\alpha(x)\chi u \in H^{s-m+1}_{\text{c}}(\Omega;\mathbb{C}^N) \tag{9}$$

Here, $\chi' \in \mathcal{C}^\infty_{\text{c}}(\Omega)$ is equal to one on the support of χ so inserting it makes no difference.

As discussed earlier, this almost looks like the end, except for a small problem to do with *a prioir* regularity. Namely from (9) we can use the existence and boundedness properties of a parametrix for $P_m(p,D)$ – which is a constant coefficient elliptic operator – to see that

$$\|\chi u\|_{H^s} \le \delta \|\chi u\|_{H^{s+1}} + C' \tag{10}$$

where C' incorporates the norms of f'' etc in H^{s-m+1} and $\delta > 0$ is small. The little problem with this is that (10) *does not* tell us that $\chi u \in H^{s+1}(\mathbb{R}^n)$ so both sides may be infinite! That is why we need to do some regularization.

So, back to the drawing-board, we choose $0 \le \phi \in \mathcal{C}^\infty_{\text{c}}(\mathbb{R}^n)$ with compact support and integral one and consider the 'approximate identity' $\phi_\epsilon *$ where $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. We know that if $v \in H^s$ then $\phi_\epsilon * v \to v$ in $H^s(\mathbb{R}^n)$ as $\epsilon \to 0$. Now, apply $\phi_\epsilon *$ to (9). It commutes with constant coefficient differential operators so we can write out the resulting identity as

$$\begin{aligned} P_m(p,D_x)(\phi_\epsilon * \chi u) + \sum_{|\alpha|=m}(P_\alpha(x)-P_\alpha(p))\chi' D^\alpha_x(\phi_\epsilon * \chi u) \\ = \phi_\epsilon * f'' + \sum_{|\alpha|=m}[\chi' P_\alpha(x), \phi_\epsilon *]D^\alpha_x(\chi u). \end{aligned} \tag{11}$$

Now apply the convolution parametrix, Q, for $P_m(p,D)$, which satisfies

$$Q * P_m(p,D) = \operatorname{Id} + R*,\ R \in \mathcal{S}(\mathbb{R}^n) \tag{12}$$

and we find that

$$\phi_\epsilon * \chi u + B\phi_\epsilon * \chi u = Qg_\epsilon + R * \phi_\epsilon * \chi u,\ B = Q * \sum_{|\alpha|=m}(P_\alpha(x)-P_\alpha(p))\chi' D^\alpha_x. \tag{13}$$

Now, if we choose χ, and hence χ' to have support sufficiently near p then the Schwartz (compactly supported) functions $(P_\alpha(x) - P_\alpha(p))\chi'$ in (13) have small supremum norm. It follows from Lemma **??** above, that

$$B : H^{s+1}(\mathbb{R}^n) \longrightarrow H^{s+1}(\mathbb{R}^n),\ \|Bv\|_{H^{s+1}} \le \delta\|v\|_{H^{s+1}} + C\|v\|_{H^s} \tag{14}$$

where we can arrange that $\delta < 1/2$. Applying this to (13) we find that

$$(1-\delta)\|\phi_\epsilon * \chi u\|_{H^{s+1}} \le C\|\phi_\epsilon * \chi u\| + C'\|\chi' u\|_{H^s} + \|f''\|_{H^{s-m+1}} \tag{15}$$

where C' represents all the bounded norms on the right, including the term which from $[\chi' P_\alpha(x), \phi_\epsilon *]$ which is uniformly bounded (as $\epsilon \downarrow 0$) by Lemma **??**. So it follows that we do indeed get a uniform bound on the (finite) norm $\|\phi_\epsilon * \chi u\|_{H^{s+1}}$. This in turn means that along a sequence $\epsilon_n \to 0$, $\phi_\epsilon * \chi u$ is weakly convergent in $H^{s+1}(\mathbb{R}^n)$, but since $\phi_\epsilon * \chi u \to u \in H^s(\mathbb{R}^n)$ it follows that the weak limit is $u \in H^{s+1}(\mathbb{R}^n)$ which proves the desired regularity result.

In fact using a partition of unity result it is straightforward to check that for any $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ any $\chi' \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ with $\chi' = 1$ in a neighbourhood of $\mathrm{supp}(\chi)$ and any t, $s \in \mathbb{R}$ there exist constants such that

$$\|\chi u\|_{H^{t+m}} \le C\|\chi' u\|_{H^s} + C'\|\chi' P(x,D)u\|_{H^t}. \tag{16}$$

The first term on the right cannot be dispensed with in general, because the operator might have null space – on an open set it may well be infinite dimensional but consists of smooth functions.