2. Problem 2

(a) Using Holder’s inequality and the fact that \( \mu(X) = 1 \),

\[
\|f\|_{L^r} = \left( \int |f|^r \, d\mu \right)^{1/r} = \left( \left( \int (|f|^r)^{r/s} \, d\mu \right)^{r/s} \left( \int d\mu \right)^{1-r/s} \right)^{1/r} = \left( \int |f|^s \, d\mu \right)^{1/s} = \|f\|_{L^s}, \text{ as required.}
\]

(b) Part (a) immediately establishes the claim since everything which has a finite \( L^s \) norm also has a finite \( L^r \) norm bounded by its \( L^s \) norm. Now, we claim that, given a measure space \( X \) with finite total measure \( \mu(X) < \infty \), every \( L^p \) space is equal if and only if there exists some “minimum measure” - i.e., there exists some number \( k \) such that for every measurable set \( E \), \( \mu(E) \neq 0 \) implies \( \mu(E) > k \). First, suppose that the space has this property.

Then, given arbitrary \( f \), there are two possibilities: either it is bounded, or it is not. If it's bounded, then by finiteness of the measure space, \( \int |f|^p \, d\mu \leq \sup |f|^p \mu(X) < \infty \). Thus, \( f \) is in every \( L^p \) space. If it is not bounded, then \( \mu(\{|f| = \infty\}) = 0 \) so \( \mu(\{|f| = \infty\}) > k \), so \( f \) does not belong to any \( L^p \) space because \( \int |f|^p \, d\mu \to \infty \) for every \( p > 0 \). Conversely, suppose that the space does not have the “minimum measure” property. Then, choose disjoint sets \( E_i \) such that \( \mu(E_i) < 1/2^i \). Then, given \( 0 < r < s < \infty \), the function \( f = \sum_{i=1}^{\infty} \mu(E_i)^{-1/s} \chi_{E_i} \) has \( L^p \) norm given by \( \sum \mu(E_i)^{1-p/s} \) and therefore will belong to \( L^r \) but not \( L^s \). If \( s = \infty \), then taking \( f = \sum_{i=1}^{\infty} \mu(E_i)^{-1/2^i} \chi_{E_i} \) will belong to \( L^r \) and not \( L^s \). This establishes the converse.

(c) We treat the \( \log f \in L_1 \) case first. Since the exponential function is convex, we may apply Jensen’s inequality:

\[
\exp \left( \int \log |f| \, d\mu \right) = \left( \exp \left( \int \log (|f|^p) \, d\mu \right) \right)^{1/p} \leq \left( \int |f|^p \, d\mu \right)^{1/p} = \|f\|_{L^p}.
\]

Letting \( p \) tend to 0, this establishes one direction. For the other direction, we use the fact that \( \log(y) \leq y - 1 \) for all positive \( y \) (this follows directly from Taylor expansion for sufficiently small \( y \) and monotonicity):

\[
\log \|f\|_{L^p} = \log \left( \int |f|^p \, d\mu \right)^{1/p} \leq \frac{1}{p} \left( \int |f|^p \, d\mu - 1 \right)
\]

But, \( \mu(X) = 1 \), so we can move the factor of 1 into the integral and obtain

\[
\log \|f\|_{L^p} \leq \frac{1}{p} \int (|f|^p - 1) \, d\mu.
\]

Now, the function \( g(p) = (|f|^p - 1)/p \) has derivative \( (p|f|^p \log |f| - |f|^p + 1)/p^2 > 0 \) near \( p = 0 \), so \( g(p) \) decreases as \( p \) tends to 0. Thus, given any \( r \) sufficiently small, \( g(p) \) is dominated by \( (|f|^r - 1)/r \) for all \( p < r \). Hence, we may take the limit as \( p \to 0 \) of both sides and apply Lebesgue’s dominated convergence theorem and L’Hospital’s rule:

\[
\lim_{p \to 0} \log \|f\|_{L^p} \leq \lim_{p \to 0} \int \frac{|f|^p - 1}{p} \, d\mu = \int \lim_{p \to 0} \frac{|f|^p - 1}{p} \, d\mu = \int \log |f| \, d\mu
\]

from which the required inequality is obtained by exponentiation (this is permissible since \( \exp \) is continuous everywhere).

Now, we deal with the case that \( \log f \not\in L_1 \) so that \( \int \log |f| \, d\mu = -\infty \) - i.e. the support of \( f \) has measure less than \( \mu(X) = 1 \). Then, we need to show that \( \lim_{p \to 0} \|f\|_{L^p} = 0 \). To this end, we approximate \( f \) by simple functions \( f_n \neq f \). Write \( f_n = \sum c_i \chi_{E_i} \), where the \( E_i \) are disjoint sets. Then, \( \lim_{p \to 0} \|f_n\|_{L^p} = \lim_{p \to 0} \sum (c_i^p \mu(E_i))^{1/p} \leq \lim_{p \to 0} \max(c_i) \mu(\bigcup E_i)^{1/p} \). But \( \mu(\bigcup E_i)^{1/p} < \mu(X) < 1 \), so this limit is just 0, as required. By non-negativity, however, we have \( \|f_n\| \geq 0 \), so \( \|f_n\| = 0 \). Now, by the monotone convergence theorem we may pass to the limit as \( n \to \infty \) so that \( \lim_{p \to 0} \|f\|_{L^p} = 0 \), as required.

3. Problem 3

Consider the measure space \( X = \mathbb{Z} \) with \( \mu \) the counting measure, and consider functions from \((X, \mu) \to \mathbb{R} \). These are countable sequences of real numbers \( \{a_i\} \) and the \( L_p \) norm of such a sequence is just \( \sum a_i^p \). If \( r < s \) and the