Exercise 1 (Constructive Urysohn's Lemma). Let $X$ be a metric space with metric $\rho$. For any non-empty subset $E \subset X$, define

$$\rho_E(x) = \inf \{\rho(x, y) | y \in E\}.$$ 

Claim 1.1. The function $\rho_E$ is uniformly continuous on $X$.

Proof. Fix $\epsilon > 0$. Then, for $\delta = \epsilon > 0$, for any $x, y \in X$ such that $\rho(x, y) < \delta$,

$$|\rho_E(x) - \rho_E(y)| = |\inf_{e \in E} \rho(x, e) - \inf_{e \in E} \rho(y, e)| \leq |\inf_{e \in E} [\rho(e, y) + \rho(x, y)] - \inf_{e \in E} \rho(y, e)|$$

$$= |\rho(x, y) + \inf_{e \in E} \rho(e, y) - \inf_{e \in E} \rho(y, e)|$$

$$= \rho(x, y) < \delta = \epsilon$$

Hence, $f$ is uniformly continuous. $\Box$

Claim 1.2. Let $E \subset X$. Then $\rho_E(e) = 0 \iff e \in \overline{E}$

Proof. Note that

$$\rho_E(e) = 0 \iff \forall \epsilon \in \mathbb{R}, \exists \epsilon > 0, \exists x \in E \text{ s.t. } \rho(x, e) < \epsilon$$

$$\iff \forall n \in \mathbb{N}, \exists x_n \in E \text{ s.t. } \rho(x_n, e) < \frac{1}{n}$$

$$\iff \forall n \in \mathbb{N}, \exists x_i \in E \text{ s.t. the sequence } x_i \text{ converges to } e$$

$$\iff e \in \overline{E} \quad \Box$$

If $A, B \subset X$ are disjoint closed subsets, define

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

Claim 1.3. The function $f$ defined above is such that

(1) The function $f$ is continuous, and $f(X) \subset [0,1]$.

(2) For any $a \in A$, $f(a) = 0$.

(3) For any $b \in B$, $f(b) = 1$.

That is, the function $f$ separates $A$ and $B$.

Proof. Note that $\rho_A(x) + \rho_B(x) \neq 0$ for all $x \in X$. Indeed, $\rho_A(x) + \rho_B(x) = 0 \implies \rho_A(x) = 0$ and $\rho_B(x) = 0 \implies x \in \overline{A \cap B} = A \cap B = \emptyset$, contradiction!

Hence, the function $x \mapsto \rho_A(x) + \rho_B(x)$ is continuous and non-vanishing. Therefore, the map $x \mapsto (\rho_A(x) + \rho_B(x))^{-1}$ is also continuous. Finally, the map

$$x \mapsto f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

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