We also have \( L_1(x) \leq L_2(x) \leq \cdots \leq f(x) \leq \cdots \leq U_2(x) \leq U_1(x) \)

\( \forall x \in [0,1] \), from the construction of \( \mathcal{C}_{K_1} \), \( L(x) = \lim_{k \to \infty} L_k(x) \)

\( U(x) = \lim_{k \to \infty} U_k(x) \), because monotonic and bounded.

Sequences of functions converge pointwise.

It is clear that \( L \) and \( U \) are bounded (\( L(x) \leq f(x) \leq \infty \)

\( \forall x \in [0,1] \) and \( U(x) \leq U_1(x) \leq \infty \) \( \forall x \in [0,1] \)) and that

\( L \) and \( U \) are measurable by monotone convergence

Theorem to show \( U \) is measurable we note that

\(-U_k^{2n} \) is an increasing seq. of measurable functions

so \( \lim_{k \to \infty} -U_k = U'(x) \) is measurable. It follows that

\(-L_k \) is measurable.

\( \lim_{k \to \infty} 1 = \lim_{k \to \infty} \frac{1}{k} U_k = U'(x) = U(x) \) is measurable.

Therefore, we have \( \lim_{k \to \infty} \int_{[0,1]} L_k(x) = \int_{[0,1]} (R) \int_{[0,1]} f \, dx \) and

\( \lim_{k \to \infty} \int_{[0,1]} U_k(x) = \int_{[0,1]} (R) \int_{[0,1]} f \, dx \) by (1) and monotone convergence thm.

Now note that since \( f \) is Riemann integrable,

\( \int_{[0,1]} f \, dx \) is the Riemann integral of \( f \) on \([0,1]\)

We have (2) if \( f \) is Riemann integrable.

(2) \( \int_{[0,1]} f \, dx = \int_{[0,1]} L(x) = \int_{[0,1]} U(x) = \int_{[0,1]} f \, dx \).

Because \( L(x) \leq U(x) \) \( \forall x \in [0,1] \)

Thus we have (2) if \( L(x) = U(x) \) a.e.