4.3 It suffices to show (1) \( V(\phi) = 0 \) and

(2) \( V \) has the countable additivity property.

1:

\[
V(\phi) = \int_E \phi \, dm = \int_E 1_{\phi = \text{true}} \, dm = \int_E 0 \, dm = 0
\]

2: Let \( \mathcal{F} = \{ A_n \} \) be a countable, disjoint collection of sets in \( B \). It is clear that:

\[
\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} 1_{A_n}
\]

By definition we have:

\[
V(\bigcup_{n=1}^{\infty} A_n) = \int_{E} \bigcup_{n=1}^{\infty} 1_{A_n} \, dm, \quad \int_{E} 1_{A_n} \, dm
\]

To show (2) we prove a more general lemma: If \( f_n : X \to [0, \infty] \) is measurable, for

\[ n = 1, 2, 3, \ldots \]

and \( f(x) = \sum_{n=1}^{\infty} f_n(x) \), for \( x \in \mathbb{R}^d \), then

\[
\int f \, dm = \sum_{n=1}^{\infty} \int f_n \, dm.
\]

pf: By the linearity lemma (Lemma 3.2.7 Stroock p. 46) we have that:

\[
\int f \, dm = \int f_1 \, dm + \int f_2 \, dm
\]

Now consider \( \Phi_M = f_1 + \cdots + f_M \). The sequence