4.2 Suppose \( \phi \in L^1(M) \) and \( m(\{ \phi < 0 \}) = 0 \). Then it is clear that \( \int \phi \, dm \geq 0 \) for each \( \Gamma \in B \), because we may find a sequence \( \{ \psi_n \}_{n=1}^{\infty} \) of non-negative simple measurable functions such that \( \psi_n \to \phi \) pointwise as \( n \to \infty \).

So by monotone convergence Thm. we have:

\[
\int \phi \, dm = \lim_{n \to \infty} \int \psi_n \, dm \geq 0, \quad \text{conversely suppose that for each } \Gamma \in B \nabla \int \phi \, dm = 0.
\]

Then take \( \Gamma_n = \{ \phi \leq -\frac{1}{n} \} \) and note that

\[
0 \leq \int_{\Gamma_n} \phi \, dm \leq -\frac{1}{n} \mu(\Gamma_n)
\]

and thus \( 0 = \mu(\Gamma_n) \cap \mu(\{ \phi < 0 \}) \).

Now the preceding arguments reduce the problem at hand if we set \( g = f - \phi \). In this case, suppose that \( m(\{ \phi < 0 \}) = m(\{ g(f < 0) \}) = 0 \) iff \( \int_{\Gamma} (g-f) \, dm \geq 0 \) for each \( \Gamma \in B \). Then \( -\int_{\Gamma} (g-f) \, dm \leq 0 \) for each \( \Gamma \in B \) iff \( m(\{ \phi > 0 \}) = m(\{ g > 0 \}) = 0 \) iff \( \int_{\Gamma} f \, dm = \int_{\Gamma} g \, dm \), or equivalently, \( \int f - g \, dm = 0 = \int f \, dm = \int g \, dm \) iff \( m(\{ g-f < 0 \}) = 0 = m(\{ g-f > 0 \}) = m(\{ g-f = 0 \}) \) which by definition means that \( f \equiv g \). Q.E.D.