4.1 cont.

Now suppose that \( Ef, \alpha \in B \), for each \( \alpha \in \mathcal{A} \). Given \( b \in \mathbb{R} \), choose a sequence \( \{a_n\} \subseteq \mathcal{A}(b, 0) \) such that \( a_n \to b \). It is clear that \( Ef > b^2 = Ef, \alpha \in B \), hence the current case reduces to the previous.

We may consider the cases "\( \ast \)" and "\( \ast \ast \)" by considering \( -f \) in place of \( f \).

Suppose that \( g \) is a second \( \mathbb{R} \)-valued function on \( (E, B) \). First we show \( Ef, g \in B \).

Let \( \mathbb{Q} \) denote the set of rationals in \( \mathbb{R} \), then we have:

\[
Ef, g \in B = \bigcup_{p \in \mathbb{Q}} \{Ef, p^2 \land Ef, g \geq p^2\} \subseteq B
\]

The correctness of this assertion is clear. If \( x \in Ef, g \), then \( x \in \{Ef, p^2 \land Ef, g \geq p^2\} \) where \( f(x) \leq p \leq g(x) \), the existence of \( p \) follows from the Archimedean principle and the density of \( \mathbb{Q} \). Conversely, if \( x \in \{Ef, p^2 \land Ef, g \geq p^2\} \) we have \( f(x) \leq p \leq g(x) \), so \( x \in Ef, g \). It is clear that \( \bigcup_{p \in \mathbb{Q}} \{Ef, p^2 \land Ef, g \geq p^2\} \subseteq B \) from the preceding arguments. \( Ef, g \in B \) by symmetry.