2.b Combining functions

First we will show that given Riemann integrable $f$, its absolute value $|f|$ is Riemann integrable. Given $\epsilon > 0$, take $\delta > 0$ such that Riemann upper and lower sums of $f$ over $C$ with mesh size less than $\delta$ lie within $\epsilon$ of each other. For such a cover $C$, consider the difference:

$$U(|f|; C) - L(|f|; C) = \sum_{I \in C} \left( \sup_I |f| - \inf_I |f| \right) \text{vol}(I)$$

Fix a rectangle $I$. If $f$ has the same sign over all of $I$, then:

$$\sup_I (-f) - \inf_I (-f) = - \left( \inf_I f - \sup_I f \right) = \sup_I f - \inf_I f$$

If $f$ does not have the same sign over all of $I$, then:

$$\sup_I |f| - \inf_I |f| \leq \sup_I |f| \leq \sup I f - \inf I f$$

Hence if we sum over all $I \in C$, we have:

$$\sum_{I \in C} \left( \sup_I |f| - \inf_I |f| \right) \text{vol}(I) \leq \sum_{I \in C} \left( \sup_I f - \inf_I f \right) \text{vol}(I)$$

This is just equal to the difference between the Riemann upper and lower sums of $f$, which is less than $\epsilon$ by our choice of $C$. By choosing $\epsilon$ to be arbitrarily small we thus find upper and lower sums arbitrarily close to each other. Therefore (1.1.4) is satisfied:

$$\sup_C L(f; C) \geq \inf_C U(f; C)$$

Thus $|f|$ is Riemann integrable if $f$ is.

Linearity of the integral follows from linearity of Riemann sums; since Riemann sums are taken over finite covers, we are free to rearrange them:

$$\sum (\alpha f(\xi(I)) + \beta g(\xi(I))) \text{vol}(I) = \alpha \left( \sum f(\xi(I)) \right) + \beta \left( \sum g(\xi(I)) \right)$$

Let $A$ and $B$ be the Riemann integrals of $f$ and $g$, respectively. Apply the triangle inequality:

$$|\mathcal{R}(\alpha f + \beta g; C; \xi) - \alpha A + \beta B|$$

$$= |\alpha \mathcal{R}(f; C; \xi) + \beta \mathcal{R}(g; C; \xi) - \alpha A + \beta B|$$

$$\leq |\alpha| |\mathcal{R}(f; C; \xi) - A| + |\beta| |\mathcal{R}(g; C; \xi) - B|$$

For $\epsilon > 0$, if we take the mesh size so that $|\mathcal{R}(f; C; \xi) - A| < \frac{\epsilon}{2|\alpha|}$ and $|\mathcal{R}(g; C; \xi) - B| < \frac{\epsilon}{2|\beta|}$, this will be less than $\epsilon$. It follows from this that the integral of $\alpha f + \beta g$ is merely $\alpha A + \beta B$.

Now we can write $f \vee g$ as $\frac{f + g}{2} - \frac{|f - g|}{2}$ and see from the results above that it is Riemann integrable. Since $f \leq f \vee g$, for any $C$ we have the inequality of Riemann lower sums:

$$L(f; C) \leq L(f \vee g; C)$$