We subtract $\phi(a_m)d_m$ and rearrange terms:

$$\phi^+(\psi^+ - \psi^0) + \phi^-(\psi^0 - \psi^-) - \phi^0(\psi^+ - \psi^-) = \psi^+(\phi^+ - \phi^0) + \psi^-(\phi^0 - \phi^-) + \psi^0(\phi^- - \phi^+)$$

By our choice of $\delta$, each of the differences in values of $\phi$ is less than $2r\epsilon$. Having taken $M$ as a bound on $\psi$, the absolute value of this expression is less than $6Mr\epsilon = \frac{\epsilon}{2n}$.

At the endpoints $a_0$ and $a_n$ the arithmetic works out similarly; we end up with one of the following:

$$\psi^+(\phi^+ - \phi^0) + \psi^0(\phi^0 - \phi^+)$$
$$\psi^-(\phi^0 - \phi^-) + \psi^0(\phi^- - \phi^0)$$

Thus each endpoint also contributes less than $\frac{\epsilon}{2n}$. Adding everything together for all the $a_m$, we see:

$$\left| \mathcal{R}(\phi|\psi; C'; \xi') - \sum \phi(a_m)d_m \right| < \frac{\epsilon(n+1)}{2n}$$

Now what remains to be done is reconciling the Riemann sums over $C$ and $C'$. At each point $a_m$, if the interval $I \in C$ containing $a_m$ did not have $a_m$ as an endpoint, it became two intervals in $C'$. Fix $\xi \in \Xi(C)$ and choose $\xi' \in \Xi(C')$ by setting $\xi'(I) = \xi(I)$ for $I$ in both $C$ and $C'$. Then the Riemann sums differ only over intervals that were split.

At each point $a_m$ lying in a split interval $I \in C$, the relevant term in $\mathcal{R}(\phi|\psi; C; \xi)$ is $\phi(\xi(I))(\psi(a_m+) - \psi(a_m-))$, and the terms in $\mathcal{R}(\phi|\psi; C'; \xi')$ are as before. Adopting the same abbreviations as before, we subtract and rearrange terms:

$$\phi^+(\psi^+ - \psi^0) + \phi^-(\psi^0 - \psi^-) - \phi^0(\psi^+ - \psi^-) = \psi^+(\phi^+ - \phi^0) + \psi^-(\phi^0 - \phi^-) + \psi^0(\phi^- - \phi^+)$$

Again since all the points at which $\phi$ is evaluated lie within $\delta$ of $a_m$, the absolute value of this expression is less than $6Mr\epsilon = \frac{\epsilon}{2n}$. Since the intervals at the endpoints $a_0$ and $a_n$ couldn’t have been split, there are at most $n-1$ points which will contribute something to this difference. Hence:

$$\left| \mathcal{R}(\phi|\psi; C'; \xi') - \mathcal{R}(\phi|\psi; C; \xi) \right| < \frac{\epsilon(n-1)}{2n}$$

Combining these inequalities using the triangle inequality, we finally have:

$$\left| \mathcal{R}(\phi|\psi; C; \xi) - \sum \phi(a_m)d_m \right| < \frac{\epsilon(n+1)}{2n} + \frac{\epsilon(n-1)}{2n} = \epsilon$$