(1) (a) Prove $H_0 = \{ y \in \mathbb{R}^N : y_1 = 0 \}$ has Lebesgue measure 0.
(b) Prove if $H \subset \mathbb{R}^N$ is a hyperplane, $H$ has Lebesgue measure 0. ($H$ a hyperplane means there exists $c \in \mathbb{R}^N$ and $\ell \in \mathbb{R}^N \setminus \{0\}$ so that $H = \{ y \in \mathbb{R}^N : (y - c) \cdot \ell = 0 \}$.)

**Solution to a:** Let $\mu$ denote Lebesgue measure and let $A_{n,\epsilon} = \left[ \frac{-\epsilon}{2N+n} \right] \times \cdots \times \left[ -n, n \right] \subset \mathbb{R}^{N-1}$.

Then $H_0 \subset \bigcup_{n=1}^{\infty} A_{n,\epsilon}$ for every $\epsilon > 0$, so it follows that

$$\mu(H_0) \leq \sum_{n=1}^{\infty} \mu(A_{n,\epsilon}) = \sum_{n=1}^{\infty} \frac{\epsilon}{2n} = \epsilon$$

for every $\epsilon > 0$. Hence $\mu(H_0) = 0$.

**Solution to b:** Let $H' = \{ y \in \mathbb{R}^N : y \cdot \ell = 0 \}$. Then $\mu(H') = \mu(H)$ since $H'$ is obtained from $H$ by a translation and Lebesgue measure is translation invariant. There exists an orthogonal linear transformation corresponding to the matrix $A$ mapping $H_0$ onto $H'$ (just choose an orthonormal basis of $H'$ and complete it to $\mathbb{R}^N$ with the normal vector $\ell/\|\ell\|$). We have $|\det(A)| = 1$, and $\mu(H') = \mu(H_0) \cdot |\det(A)| = 0$, so $\mu(H) = 0$. 
(2) Let \((E, \mathcal{B}, \mu)\) be a measure space and assume \(\mu(E) < \infty\). Let \(\{f_n\}_{n=1}^{\infty}\) be a sequence of measurable functions on \(E\), and assume there exists a function \(f\) so that \(f_n \to f\) \(\mu\)-a.e. on \(E\). Prove for any \(\varepsilon > 0\), there exists a measurable set \(A \subset E\) and a number \(N\) such that

(i) \(\mu(A) < \varepsilon\), and

(ii) for all \(n \geq N\) and all \(x \in E \setminus A\), \(|f_n(x) - f(x)| < \varepsilon\).

(Hint: Consider a sequence of sets where the second conclusion does not hold, that is look at the sets \(G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}\).

Solution:

Fix \(\varepsilon > 0\). Let \(G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}\) and let \(H_n = \cup_{m \geq n} G_m\). If \(x \in H_n\) for infinitely many \(n\), then \(f_n(x)\) does not converge to \(f(x)\). Let \(H = \cap_{n=1}^{\infty} H_n\), so that \(H_n \setminus H\). Since \(f_n \to f\) \(\mu\)-a.e. on \(E\), it follows that \(\mu(H) = \mu(\cap_{n=1}^{\infty} H_n(x)) = 0\). Since \(H_n \setminus H\) and \(\mu(H_1) \leq \mu(E) < \infty\), it follows that \(\lim_{n \to \infty} \mu(H_n) = \mu(H) = 0\). Thus we can choose \(N\) so that \(\mu(H_N) < \varepsilon\). Letting \(A = H_N\) it follows that \(A\) satisfies (i). Then for \(n \geq N\) and \(x \in E \setminus A\), we have that \(|f_n(x) - f(x)| < \varepsilon\), so (ii) is satisfied.
Let \((E, \mathcal{B}, \mu)\) be a measure space, and suppose \(f : E \to [0, \infty] \) is an integrable function such that 
\[
\int f \, d\mu = c
\]
for some \(c \in (0, \infty)\). Prove for \(0 < \alpha < \infty\), 
\[
\lim_{n \to \infty} \int n \log(1 + (f/n)^\alpha) \, d\mu = \begin{cases} 
\infty & \text{if } 0 < \alpha < 1, \\
c & \text{if } \alpha = 1, \\
0 & \text{if } 1 < \alpha < \infty.
\end{cases}
\]

(Hint: Consider \(\alpha \geq 1\) and \(\alpha < 1\) separately. Then consider \(\{f > n\}\) and \(\{f \leq n\}\) and use an appropriate Taylor approximation.)

**Solution:** As we discussed in lecture, there were several ways to solve this problem. This solution is the one I had in mind for the hint.

**Case 1:** \(\alpha \geq 1\). The idea is to use Dominated Convergence Theorem, but we need to find an appropriate integrable dominant. On \(\{f > n\}\), \((f/n)^\alpha > 1\) implies we can expand 
\[
\log(1 + (f/n)^\alpha) = (\log((f/n)^\alpha) + \log(1 + (f/n)^{-\alpha})) \\
= n\alpha \log(f/n) + n(f/n)^{-\alpha} + nE_1,
\]
where \(|E_1| \leq (f/n)^{-2\alpha}/2\) by the alternating series remainder estimate. We have \(f/n > 1\) and \(\log\) is monotone increasing which implies \(\log(f/n) \leq f/n\).

Also, since \(f/n > 1\), we have \(n(f/n)^{-\alpha} \leq n < f\) and similarly for \(E_1\), so that 
\[
n \log(1 + (f/n)^\alpha) \leq n\alpha(f/n) + 2n \leq (\alpha + 2)f.
\]

Now if \(\{f \leq n\}\), we can expand 
\[
n \log(1 + (f/n)^\alpha) = n((f/n)^\alpha + E_2),
\]
where \(|E_2| \leq (f/n)^{2\alpha}/2\) by alternating series. Since \(\alpha \geq 1\) and \(f/n \leq 1\), \((f/n)^\alpha \leq f/n\). Hence we again have 
\[
n \log(1 + (f/n)^\alpha) \leq f \leq (\alpha + 2)f.
\]

That means the sequence of functions 
\[
g_n = n \log(1 + (f/n)^\alpha)
\]
is dominated by the integrable function \((\alpha + 2)f\) on all of \(E\). Then the Dominated Convergence Theorem implies 
\[
\lim_{n \to \infty} \int g_n \, d\mu = \int \lim_{n \to \infty} g_n \, d\mu.
\]

To compute the limit of \(g_n\), we recall if \(a \in [0, \infty]\), then 
\[
\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = \begin{cases} 
e^a & \text{if } \alpha = 1, \\
1 & \text{if } \alpha > 1, \\
\infty & \text{if } \alpha < 1.
\end{cases}
\]
Hence since \{f = \infty\} has \mu\text{-measure 0,

\[
\lim_{n \to \infty} g_n = \begin{cases} 
  f, & \text{if } \alpha = 1, \\
  0, & \text{if } \alpha > 1, \\
  \infty, & \text{if } \alpha < 1.
\end{cases}
\]

\mu\text{-a.e.}

This proves the formula for \(\alpha \geq 1\).

Case 2: For \(\alpha < 1\), we apply Fatou’s Lemma and this limit with \(\alpha < 1\):

\[
\lim_{n \to \infty} \int g_n d\mu \geq \int \liminf_{n \to \infty} g_n d\mu = \infty.
\]