HOMEWORK 3 FOR 18.125, SPRING 2010
DUE THURSDAY, FEBRUARY 25 AT THE BEGINNING OF LECTURE.

HW3.1 a) Let $E = \mathbb{R}$, let

$$\tau_0 = \{\text{half-open intervals } [a, b) : a < b\},$$

and let

$$\tau = \{\text{unions of sets in } \tau_0\}.$$

Show that $\tau$ is a topology on $E$.

b) Show that $\mathbb{R} \setminus \mathbb{Q}$ is a $G_\delta$ set.

c) This problem shows that $G_\delta$ is not closed under countable unions.

   Step 1: Show that if $\{U_n\} \subset \mathbb{R}$ and each $U_n$ is open and dense, then

$$\bigcap_{n=1}^{\infty} U_n$$

is dense. (This is a baby version of the Baire Category Theorem).

   Step 2: Show $\mathbb{Q}$ is not a $G_\delta$ set. (Hint: Assume the contrary, and use part (b) plus Step 1 to conclude $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$ is dense - a contradiction.)

   Step 3: Find a sequence of $\{B_n\}_{n=1}^{\infty} \subset G_\delta$ so that

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} B_n.$$

Conclude $G_\delta$ is not closed under countable unions.

HW3.2 a) Let $B = \mathcal{P}(\mathbb{R}^N)$ be the set of all sets in $\mathbb{R}^N$. Show $B$ is a $\sigma$-algebra.

   Let $\mu_1 : B \to [0, \infty]$ be defined by

$$\mu_1(A) = \begin{cases} 
\infty & \text{if } A \text{ is infinite,} \\
 m & \text{if } A \text{ has } m \text{ elements in it.}
\end{cases}$$

Show $\mu_1$ is a measure.

   Let $\mu_2 : B \to \{0, 1\}$ be defined by

$$\mu_2(A) = \begin{cases} 
1 & \text{if } 0 \in A, \\
0 & \text{if } 0 \notin A.
\end{cases}$$

Show $\mu_2$ is a measure.

b) Let $B$ be the collection of sets $E \subset \mathbb{R}^N$ such that either $E$ or $E^c$ is at most countable. Show $B$ is a $\sigma$-algebra.

   Let $\mu : B \to \{0, 1\}$ be defined by $\mu(E) = 0$ if $E$ is at most countable, and $\mu(E) = 1$ if $E^c$ is at most countable. Show $\mu$ is a measure.

HW3.3 Prove or disprove: There exists a $\sigma$-algebra $B$ which is countably infinite.

HW3.4 Stroock 3.1.9

HW3.5 Stroock 3.1.10