18.103 Fourier Series: an outline

Consider a periodic function F on the real line of period 2π, integrable on $(-\pi, \pi)$ (or, equivalently, $F \in L^1(\mathbf{T})$ with $\mathbf{T} = \mathbf{R}/\mathbf{2}\pi\mathbf{Z}$). The *Fourier coefficients* of F are defined by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)e^{-inx} dx, \quad n \in \mathbf{Z}$$

Uniqueness

Theorem 1 *If $F \in L^1(\mathbf{T})$ and $c_n = 0$ for all $n \in \mathbf{Z}$, then $F(x) = 0$ for almost every x.*

In particular, taking differences, if two functions have the same Fourier coefficients, then they are the same (except on a set of measure zero).

We proved this uniqueness theorem first for functions $F \in L^2(\mathbf{T})$. For the more general case $F \in L^1(\mathbf{T})$, see lecture and remarks at the end of these notes. It is proved in the SS text for Riemann integrable functions, which are, in particular, bounded (and hence in $L^2(\mathbf{T})$).

Reconstruction of a function from its Fourier coefficients

Given that functions are determined by their Fourier coefficients, we wish to find formulas for them using their Fourier coefficients. Denote the partial sums, Cesáro means, and Abel means, respectively, of the Fourier series by

$$s_N(x) = \sum_{|n| \le N} c_n e^{inx} \tag{1}$$

$$\sigma_N(x) = \sum_{|n| \le N} \left(1 - \frac{|n|}{N}\right) c_n e^{inx} \tag{2}$$

$$A_r(x) = \sum_{n \in \mathbf{Z}} r^{|n|} c_n e^{inx} \quad (0 \le r < 1) \tag{3}$$

Theorem 2 *(Absolutely convergent series) If $\sum |c_n| < \infty$, then the (absolutely convergent) Fourier series converges to the function:*

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = F(x)$$

Moreover the convergence is uniform in x, that is,

$$\lim_{N \to \infty} \max_x |s_N(x) - F(x)| = 0$$

Proposition. $s_N = D_N * F$, $\sigma_N = K_N * F$ and $A_r = P_r * F$, where

$$D_N(x) = \sum_{|n|\le N} e^{inx} = \frac{\sin(N+1/2)x}{\sin(x/2)}$$

$$K_N(x) = \sum_{|n|\le N} \left(1 - \frac{|n|}{N}\right) e^{inx} = \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

$$P_r(x) = \sum_{n\in\mathbf{Z}} r^{|n|} e^{inx} = \frac{1-r^2}{1-2r\cos x + r^2} = \frac{1-r^2}{|1-re^{ix}|^2}$$

and convolution is defined by $f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy$

Theorem 3 *(Uniform convergence) If $F \in C(\mathbf{T})$, then* $\max_x |\sigma_N(x) - F(x)|$ *tends to* 0 *as* $N \to \infty$, *and* $\max_x |A(x,r) - F(x)|$ *tends to* 0 *as* $r \to 1$ *($r < 1$).*

Theorem 4 *(Pointwise convergence of Cesáro and Abel means) If $F \in L^1(\mathbf{T})$ is continuous at x_0, then $\sigma_N(x_0) \to F(x_0)$ as $N \to \infty$. If F has left and right limits at x_0, then $\sigma_N(x_0) \to (F(x_0^+) + F(x_0^-))/2$, and similarly for the Abel means.*

Theorem 5 *(Parseval formula and mean square convergence) If $F \in L^2(\mathbf{T})$, then*

$$\|F\|^2 := \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(x)|^2 dx = \sum_{n\in\mathbf{Z}} |c_n|^2$$

$$\|s_N - F\|^2 = \sum_{|n|>N} |c_n|^2$$

Continuity does not suffice for convergence every point of Fourier series, but a very modest rate of continuity (modulus of continuity) suffices.

Theorem 6 *(Pointwise convergence) If $F \in L^1(\mathbf{T})$ and $|F(x) - F(x_0)| \le C|x - x_0|^\alpha$ for some $\alpha > 0$ and all x near x_0, then*

$$\lim_{N\to\infty} s_N(x_0) = F(x_0)$$

This is proved using the following.

Lemma 1 *(Riemann-Lebesgue lemma) If $F \in L^1(\mathbf{T})$, then*

$$\lim_{|n|\to\infty} c_n = 0$$

Sine and Cosine series

There are analogues of each of these theorems for Fourier sine and cosine series which are corollaries of the theorems above, proved by taking odd and even parts of the Fourier series.

Define the Fourier cosine coefficients of $f \in L^1((0,\pi))$ by

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx, \quad n = 1,\, 2,\, \ldots$$

For $n = 0$, the definition is

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx$$

Define Fourier sine coefficients of $g \in L^1((0,\pi))$ by

$$b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx, \quad n = 1,\, 2,\, \ldots$$

These normalizations are chosen so that the series representations of f and g are

$$f(x) \sim \sum_{n=0}^\infty a_n \cos nx; \quad g(x) \sim \sum_{n=1}^\infty b_n \sin nx;$$

Indeed, if $\phi_n(x) = \sin nx$, $n = 1, \ldots$ and an inner product on $L^2((0,\pi))$ is defined by

$$\langle f, g\rangle = \int_0^\pi f(x)\overline{g(x)}dx$$

then $\langle \phi_n, \phi_n\rangle = \pi/2$ and

$$g \sim \sum_{n=1}^\infty \frac{\langle g, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}\phi_n = \sum_{n=1}^\infty b_n\phi_n$$

and similarly for cosines, the difference being that $\langle 1, 1\rangle = \pi$ rather than $\pi/2$.

Theorem 7 *If $f \in L^1((0,\pi))$ and $a_n = 0$, $n = 0, 1, \ldots,$ then $f = 0$ almost everywhere. If $g \in L^1((0,\pi))$ and $b_n = 0$, $n = 1, \ldots,$ then $g = 0$ almost everywhere.*

Consider a function F, periodic of period 2π, and write it as the sum of its even and odd parts:

$$F = f + g,$$

where $f(-x) = f(x)$ and $g(-x) = -g(x)$ are periodic of period 2π.

Then for $n > 0$,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^\pi F(x)e^{-inx}dx = \frac{1}{\pi}\int_0^\pi f(x)(\cos nx)dx + \frac{1}{\pi}\int_0^\pi g(x)(-i\sin nx)dx = (a_n - ib_n)/2$$

Similarly,

$$c_{-n} = (a_n + ib_n)/2, \quad n = 1,\, 2,\, \ldots; \quad c_0 = a_0$$

Thus

$$b_n = i(c_n - c_{-n}); \quad a_n = c_n + c_{-n}; \quad n = 1,\, 2,\, \ldots$$

The Fourier series representation of F is equivalent to the Fourier cosine and sine series representations of its even and odd parts:

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos nx = \sum_{n=0}^{\infty} a_n (e^{inx} + e^{-inx})/2$$
$$g(x) \sim \sum_{n=0}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} b_n (e^{inx} - e^{-inx})/2i$$

$$F = f + g \sim a_0 + \sum_{n=1}^{\infty} (a_n - ib_n) e^{inx}/2 + \sum_{n=1}^{\infty} (a_n + ib_n) e^{-inx}/2 \sim \sum_{n \in \mathbf{Z}} c_n e^{inx}$$

An Example. We will illustrate pointwise and norm convergence in the particular case of the sine series of the function $g(x) = 1$ on $0 < x < \pi$. Extend g to an odd function on the real line, periodic of period 2π, and, say, $g = 0$ at each multiple of π.

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = \begin{cases} 4/n & n \text{ odd;} \\ 0 & n \text{ even;} \end{cases}$$

Considering the case $f = 0$ above, the Fourier coefficients of $F = 0 + g$ are

$$c_n = -2i/n, \ n \text{ odd}; \quad c_n = 0, \ n \text{ even}$$

$$s_N(x) = \sum_{|n| \le N} c_n e^{inx} = \sum_{n\, odd,\, n=1}^{N} \frac{-2i}{n} (e^{inx} - e^{-inx}) = \sum_{n\, odd,\, n=1}^{N} \frac{4}{n} \sin nx$$

The square mean convergence of the sine series is a consequence of the formula

$$\int_0^{\pi} |1 - s_N(x)|^2 dx = \sum_{n \ odd, \ =N+1}^{\infty} \left(\frac{4}{n}\right)^2 \langle \sin nx, \sin nx \rangle = \sum_{n \ odd, \ =N+1}^{\infty} \frac{8\pi}{n^2}$$

Expressed in terms of the Parseval formula for exponential Fourier series,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(x) - s_N(x)|^2 dx = \sum_{|n| > N} |c_n|^2 = \sum_{n \ odd, \ =N+1}^{\infty} \frac{8}{n^2}$$

These two formulas are equivalent. Indeed, because g and s_N are odd,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(x) - s_N(x)|^2 dx = \frac{1}{\pi} \int_0^{\pi} |g(x) - s_N(x)|^2 dx = \frac{1}{\pi} \int_0^{\pi} |1 - s_N(x)|^2 dx$$

Moreover, our pointwise convergence theorem implies

$$\lim_{N\to\infty} s_N(x) = 1 \qquad \text{for all } 0 < x < \pi$$
$$\lim_{N\to\infty} s_N(x) = -1 \qquad \text{for all } -\pi < x < 0$$

The convergence fails at multiples of π because the function is discontinuous there. The convergence is only valid "on average" in square mean.

Exercise. Prove Gibbs's phenomenon for this series, namely,

$$\lim_{N\to\infty} \max_x s_N(x) = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t} dt \approx 1.18$$

See exercise 20 p. 94 of SS. No matter how large N is the Fourier series approximation overshoots 1 by a fixed percentage. Similarly, by symmetry, the minimum is approximately -1.18.

Exercise. Show that for any bounded measurable function F,

$$\max_x |\sigma_N(x)| \leq \text{esssup } |F|, \quad \text{and} \quad \max_x |A_r(x)| \leq \text{esssup } |F|$$

In other words, the Cesáro and Abel means cure Gibbs' phenomenon (the problem of overshooting at discontinuities).

Deeper waters

As mentioned earlier, the Fourier series of a continuous function may diverge at a point. The Fourier series of a Lebesgue integrable (L^1) function may diverge everywhere! On the other hand, in 1965 Lennart Carleson proved the following, which is the most delicate and difficult theorem in the theory of Fourier series.

Theorem 8 *(Carleson) If $F \in L^2(\mathbf{T})$, then $s_N(x)$ tends to $F(x)$ for almost every x.*

In particular, the Fourier series of a continuous function converges to the value of the function at almost every point. This is the best one can hope for: given any set of measure zero, there is a continuous function whose Fourier series diverges on that set.

Convergence in mean

We now return from the sometimes delicate issue of pointwise convergence to convergence "on average." We will now measure average convergence in terms of the L^p norm. The L^p norm of a function is given by

$$\|F\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(x)|^p dx\right)^{1/p}$$

Theorem 9 *If* $F \in L^p(\mathbf{T})$, $1 \leq p < \infty$, *then*

$$\lim_{N \to \infty} \|F - \sigma_N\|_p = 0; \quad \lim_{r \to 1} \|F - A_r\|_p = 0$$

Moreover, if $1 < p < \infty$,

$$\lim_{N \to \infty} \|F - s_N\|_p = 0$$

We will prove only the first half of this theorem. The assertions involving s_N are beyond the scope of this course.

The first part says in particular (in the case L^1) that the area between the graph of σ_N and F tends to zero as N tends to infinity. This yields as a corollary our first theorem (uniqueness of Fourier series for L^1 functions).

In contrast, $\|F - s_N\|_1$, the area between the graph of an L^1 function F and the partial sum of its Fourier series s_N, need not tend to zero. Finally, if F has a slightly better integrability property (class L^p for some $p > 1$) then the area between the graphs of F and s_N does tend to zero. (Indeed, $\|F - s_N\|_1 \leq \|F - s_N\|_p \to 0$ as $N \to \infty$.)