Vector fields and one forms

Definition 1. *A C^k vector field on M is a C^k map*

$$v : M \longrightarrow TM$$

so that for all $p \in M$, $v(p) \in T_pM$.

You should think of this as a C^k choice of vector in T_pM for all $p \in M$.

We can use the vector space structure on T_pM to add vector fields and multiply them by real valued functions. Explicitly, if v and w are vector fields, and $\lambda : M \longrightarrow \mathbb{R}$ is a function,

$$(v + w)(p) := v(p) + w(p)$$

Think of this as adding the vector fields pointwise.

$$(\lambda v)(p) := \lambda(p)v(p)$$

Think of this as scaling the vector field by $\lambda(p)$ at the point p.

This allows us to write vector fields on $\mathbb{R}^n$ in coordinates as follows:

$$v = \sum v_i \frac{\partial}{\partial x_i}$$

where

$$v_i : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ are } C^k \text{ functions if } v \text{ is } C^k$$

and $\frac{\partial}{\partial x_i}$ is the vector field which is the ith standard basis vector e_i at each point.

Definition 2. *The cotangent space of M at $p \in M$ is the dual vector space to T_pM, which we shall denote by T_p^*M.*

In other words, T_p^*M is given by linear maps

$$\alpha|_p : T_pM \longrightarrow \mathbb{R}$$

There is actually a vector bundle T^*M over M which has as it's fiber over p T_p^*M. T^*M is actually diffeomorphic to TM, but there is not a canonical choice of diffeomorphism.

Definition 3. *A C^k differential one form on M is a C^k map*

$$\alpha : TM \longrightarrow \mathbb{R}$$

so that the restriction to T_pM

$$\alpha|_p : T_pM \longrightarrow \mathbb{R}$$

is linear.

We can also regard α as a choice of $\alpha|_p \in T_p^*M$ for all p, which gives a C^k map $M \longrightarrow T^*M$. We shall often shorten 'differential one form' to 'one form'.

We can evaluate a one form α on M on any vector field v on M to give us a real valued function defined by

$$\alpha(v) : M \longrightarrow \mathbb{R}$$
$$\alpha(v)(p) := \alpha|_p(v(p))$$

this is just the composition

$$M \xrightarrow{v} TM \xrightarrow{\alpha} \mathbb{R}$$

Just as for vector fields, we can add one forms and multiply them by real valued functions on M, so we have the following identities if α and β are one forms, v, w are vector fields and λ is a real valued function,

$$(\alpha + \beta)(v) = \alpha(v) + \beta(v)$$
$$(\lambda\alpha)(v) = \lambda \times (\alpha(v))$$

The following identities are because $\alpha|_p$ is linear:

$$\alpha(v + w) = \alpha(v) + \alpha(w)$$

$$\alpha(\lambda v) = \lambda\alpha(v)$$

On $\mathbb{R}^n$, we can write one forms in a standard basis as follows:

$$\alpha = \sum \alpha_i dx_i$$

where

$$\alpha_i : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ are } C^k \text{ if } \alpha \text{ is } C^k$$

and

$$dx_i\left(\frac{\partial}{\partial x_i}\right) = 1 \qquad dx_i\left(\frac{\partial}{\partial x_j}\right) = 0 \text{ if } i \neq j$$

Definition 4. *Given a differentiable function $f : M \longrightarrow \mathbb{R}$ we have defined $Tf : TM \longrightarrow T\mathbb{R}$. Let $\pi_2 : T\mathbb{R} \longrightarrow \mathbb{R}$ be given by $\pi_2(x, v) := v$. This identifies $T_x\mathbb{R}$ with $\mathbb{R}$ itself using the fact that $\mathbb{R}$ is a vector space. Now we can define a differential one form*

$$df : TM \longrightarrow \mathbb{R}$$

by

$$df := \pi_2 \circ Tf$$

If f is a function $\mathbb{R}^n \longrightarrow \mathbb{R}$, this is simply

$$df(p, v) := Df(p)(v)$$

Note that the one forms dx_i above are simply given by taking d of the coordinate function $x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$.

For example, if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable,

$$df := \sum \frac{\partial f}{\partial x_i} dx_i$$

Definition 5. *The Lie derivative of a C^1 function $f : M \longrightarrow \mathbb{R}$ with respect to the vector field v on M is defined to be the function*

$$L_v f : M \longrightarrow \mathbb{R}$$

defined by

$$L_v f := df(v)$$

For example if $v = \sum v_i \frac{\partial}{\partial x_i}$,

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i}$$

(motivating our "derivation notation" for v).

Note that as the result of L_v is a function, we can compose Lie derivatives, for example,

$$L_{\frac{\partial}{\partial x_1}} L_{\frac{\partial}{\partial x_2}} f = D_1 D_2 f$$

Exercise.

Check that if $f_i \in C^1(M)$, $i = 1, 2$, then

$$L_v(f_1 f_2) = f_1 L_v f_2 + f_1 L_v f_2 \,. \tag{1}$$

Definition 6. *Given a differentiable map $f : M \longrightarrow N$ and a one form α on N, the pull back of α is a one form $f^*\alpha$ on M defined as follows*

$$f^*\alpha : TM \longrightarrow \mathbb{R}$$

$$f^*\alpha := \alpha \circ Tf$$

$$TM \xrightarrow{Tf} TN \xrightarrow{\alpha} \mathbb{R}$$

Exercise

1. Show that
$$f^*(\lambda_1\alpha + \lambda_2\beta) = (\lambda_1 \circ f)f^*\alpha + (\lambda_2 \circ f)f^*\beta$$

2. If $f = (x_1^2 + x_2^2, x_1^2 - x_2^2)$, compute
$$f^*(x_2dx_1 + 3dx_2)$$

3. Prove that
$$(f \circ g)^* = g^* \circ f^*$$

4. Prove that
$$f^*dg = d(g \circ f)$$

We can pull back one forms with any smooth map. There is an analogous push forward of vector fields which is defined for any diffeomorphism. This should be regarded as how vector fields change under a coordinate change.

Definition 7. *If $f : M \longrightarrow N$ is a diffeomorphism, and $v : M \longrightarrow TM$ is a vector field, the push forward of v is a vector field on N $f_*v : N \longrightarrow TN$. This is defined by*

$$f_*v = TF \circ v \circ f^{-1}$$

$$N \xrightarrow{f^{-1}} M \xrightarrow{v} TM \xrightarrow{Tf} TN$$

In other words,
$$f_*v(f(p)) = T_pf(v(p))$$

Exercise

1.
$$f_*(v + w) = f_*v + f_*w$$

2.
$$f_*(\lambda v) = (\lambda \circ f^{-1})f_*v$$

3. if $f(x_1, x_2) = (x_1, x_1 + x_1 x_2)$, on the set $\{x_1 > 0\}$ calculate $f_*(\frac{\partial}{\partial x_1})$ and $f_*(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2})$

4. $$(f \circ g)_* = f_* \circ g_*$$

5. $$f^* \alpha(v) = \alpha(f_* v) \circ f$$

6. $$L_{f_* v} g = L_v (g \circ f) \circ f^{-1}$$

Definition 8. *A C^1 curve $\gamma : (a, b) \to M$ is an integral curve of the vector field v if for all $a < t < b$ and $p = \gamma(t)$*

$$\left(p, \frac{d\gamma}{dt}(t)\right) = v(p)$$

In other words, if e_1 is the standard basis vector in $T_t(a, b)$,

$$T\gamma(t, e_1) = v(\gamma(t))$$

For example, if $v = \sum v_i \frac{\partial}{\partial x_i}$ is a vector field on $\mathbb{R}^n$, and $X : \mathbb{R} \to \mathbb{R}^n$ is the function $(v_1, \dots, v_n)$ the condition for $\gamma(t)$ to be an integral curve of v is that it satisfy the system of ODEs

$$\frac{d\gamma}{dt}(t) = X(\gamma(t)) \,. \tag{2}$$

The following theorem allows us to just consider the problem of finding integral curves in coordinate charts, as it tells us how the problem changes when we change coordinates using a diffeomorphism.

Theorem 1. *If $f : M \longrightarrow N$ is a diffeomorphism, v a vector field on M and $\gamma : (a, b) \longrightarrow M$ an integral curve of v, then*

$$f \circ \gamma : (a, b) \longrightarrow N$$

is an integral curve of the vector field $f_* v$ on N.

Proof.

$$T(f \circ \gamma)(t, e_1) = Tf \circ T\gamma(t, e_1) = Tf(v(\gamma(t))) = f_* v(f \circ \gamma(t))$$

□

Exercises on integrating one forms

1. Suppose that $\alpha := f dx$ is a continuous one form on $[a,b]$. Define the integral

$$\int_{[a,b]} \alpha := \int_{[a,b]} f dx$$

Show that given any C^1 map $g : [c,d] \longrightarrow [a,b]$ sending c to a and d to b,

$$\int_{[a,b]} \alpha = \int_{[c,d]} g^* \alpha$$

2. Let M be a smooth manifold and let $\gamma : [a,b] \to M$, be a C^1 curve. Given a one form ω, define the *line integral* of ω over γ to be the integral

$$\int_\gamma \omega = \int_{[a,b]} \gamma^* \omega$$

Show that if $\omega = df$ for some $f \in \mathcal{C}^\infty(U)$

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a)) \, .$$

In particular conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

3. Let

$$\omega = \frac{x_1 \, dx_2 - x_2 \, dx_1}{x_1^2 + x_2^2}$$

and let $\gamma : [0, 2\pi] \to \mathbb{R}^2 - \{0\}$ be the closed curve, $t \to (\cos t, \sin t)$. Compute the line integral, $\int_\gamma \omega$, and show that it's not zero. Conclude that ω can't be "d" of a function, $f \in \mathcal{C}^\infty(\mathbb{R}^2 - \{0\})$.

4. Let f be the function

$$f(x_1, x_2) = \begin{cases} \arctan \frac{x_2}{x_1} \, , x_1 > 0 \\ \frac{\pi}{2} \, , \; x_1 = 0 \, , \; x_2 > 0 \\ \arctan \frac{x_2}{x_1} + \pi \, , \; x_1 < 0 \end{cases}$$

where, we recall: $-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}$. Show that this function is $\mathcal{C}^\infty$ and that df is the 1-form, ω, in the previous exercise. Why doesn't this contradict what you proved earlier?