CHAPTER 10

THE INTEGRAL CALCULUS ON MANIFOLDS

In this chapter we shall study integration on manifolds. In order to develop the integral calculus, we shall have to restrict the class of manifolds under consideration. In this chapter we shall assume that all manifolds M that arise satisfy the following two conditions:

1) M is finite-dimensional.
2) M possesses an atlas $\mathcal{A}$ containing (at most) a countable number of charts; that is, $\mathcal{A} = \{(U_i, \alpha_i)\}_{i=1,2,\ldots}$.

Before getting down to the business of integration, there are several technical facts to be established. The first two sections will be devoted to this task.

1. COMPACTNESS

A subset A of a manifold M is said to be *compact* if it has the following property:

i) If $\{U_\iota\}$ is any collection of open sets with
$$A \subset \bigcup_\iota U_\iota,$$
there exist finitely many of the U_ι, say $U_{\iota_1}, \ldots, U_{\iota_r}$, such that
$$A \subset U_{\iota_1} \cup \cdots \cup U_{\iota_r}.$$

Alternatively, we can say:

ii) A set A is compact if and only if for any family $\{F_\iota\}$ of closed sets such that
$$A \cap \bigcap_\iota F_\iota = \emptyset,$$
there exist finitely many of the F_ι such that
$$A \cap F_{\iota_1} \cap \cdots \cap F_{\iota_r} = \emptyset.$$

The equivalence of (i) and (ii) can be seen by taking U_ι equal to the complement of F_ι.

In Section 5 of Chapter 4 we established that if $M = U$ is an open subset of $\mathbb{R}^n$, then $A \subset U$ *is compact if and only if A is a closed bounded subset of* $\mathbb{R}^n$.

We make some further trivial remarks about compactness:

iii) If $A_1, \ldots, A_r$ are compact, so is $A_1 \cup \cdots \cup A_r$.

In fact, if $\{U_\iota\}$ covers $A_1 \cup \cdots \cup A_r$, it certainly covers each A_j. We can thus choose for each j a finite subcollection which covers A_j. The union of these subcollections forms a finite subcollection covering $A_1 \cup \cdots \cup A_r$.

iv) If $\psi: M_1 \to M_2$ is continuous and $A \subset M_1$ is compact, then $\psi[A]$ is compact.

In fact, if $\{U_\iota\}$ covers $\psi[A]$, then $\{\psi^{-1}(U_\iota)\}$ covers A. If the U_ι are open, so are the $\psi^{-1}(U_\iota)$, since ψ is continuous. We can thus choose $\iota_1, \ldots, \iota_r$ so that

$$A \subset \psi^{-1}(U_{\iota_1}) \cup \cdots \cup \psi^{-1}(U_{\iota_r}),$$

which implies that $\psi[A] \subset U_{\iota_1} \cup \cdots \cup U_{\iota_r}$.

We see from this that if $A = A_1 \cup \cdots \cup A_r$, where each A_j is contained in some W_i, where (W_i, β_i) is a chart, and $\beta_i(A_j)$ is a compact subset of $\mathbb{R}^n$, then A is compact. In particular, the manifold M itself may be compact. For instance, we can write S^n as the union of the upper and lower hemispheres: $S^n = \{x : x^{n+1} \geq 0\} \cup \{x : x^{n+1} \leq 0\}$. Each hemisphere is compact. In fact, the upper hemisphere is mapped onto $\{y : \|y\| \leq 1\}$ by the map φ_1 of Section 8.1, and the lower hemisphere is mapped onto the same set by φ_2. Thus the sphere is compact.

On the other hand, an open subset of $\mathbb{R}^n$ is not compact. However, it can be written as a *countable* union of compact sets. In fact, if $U \subset \mathbb{R}^n$ is an open set, let

$$A_n = \{x \in U : \|x\| \leq n \text{ and } \rho(x, \partial U) \geq 1/n\}.$$

It is easy to check that A_n is compact and that

$$\bigcup A_n = U.$$

In view of condition (2), we can say the same for any manifold M under consideration:

Proposition 1.1. Any manifold M satisfying (1) and (2) can be written as

$$M = \bigcup_{i=1}^{\infty} A_i,$$

where each $A_i \subset M$ is compact.

Proof. In fact, by (2)

$$M = \bigcup_{j=1}^{\infty} U_j,$$

and by the preceding discussion each U_j can be written as the countable union of compact sets. Since the countable union of a countable union is still countable, we obtain Proposition 1.1. □

An immediate corollary is:

Proposition 1.2. Let M be a manifold [satisfying (1) and (2)], and let $\{U_\iota\}$ be an open covering of M. Then we can select a countable subcollection $\{U_j\}$ such that

$$\bigcup U_j = M.$$

Proof. Write $M = \bigcup A_r$, where A_r is compact. For each r we can choose finitely many $U_{r,1}, U_{r,2}, \ldots, U_{r,k_r}$ so that

$$A_r \subset U_{r,1} \cup \cdots \cup U_{r,k_r}.$$

The collection

$$\{U_{r,k}\}_{\substack{r=1,\ldots,\infty \\ k=1,\ldots,k_r}}$$

is a countable subcollection covering M. □

2. PARTITIONS OF UNITY

In the following discussion it will be convenient for us to have a method of "breaking up" functions, vector fields, etc., into "little pieces". For this purpose we introduce the following notation:

Definition 2.1. A collection $\{g_j\}$ of C^∞-functions is said to be a *partition of unity* if

i) $g_i \geq 0$ for all i;

ii) supp g_i† is compact for all i;

iii) each $x \in M$ has a neighborhood V_x such that $V_x \cap \operatorname{supp} g_i = \emptyset$ for all but a finite number of i; and

iv) $\sum g_i(x) = 1$ for all $x \in M$.

Note that in view of (iii) the sum occurring in (iv) is actually finite, since for any x all but a finite number of the $g_i(x)$ vanish. Note also that:

Proposition 2.1. If A is a compact set and $\{g_j\}$ is a partition of unity, then

$$A \cap \operatorname{supp} g_i = \emptyset$$

for all but a finite number of i.

Proof. In fact, each $x \in A$ has a neighborhood V_x given by (iii). The sets $\{V_x\}_{x \in A}$ form an open covering of A. Since A is compact, we can select a finite subcollection $\{V_1, \ldots, V_r\}$ with $A \subset V_1 \cup \cdots \cup V_r$. Since each V_k has a nonempty intersection with only finitely many of the supp g_i, so does their union, and so *a fortiori* does A. □

† Recall that supp g is the closure of the set $\{x : g(x) \neq 0\}$.

Definition 2.2. Let $\{U_\iota\}$ be an open covering of M, and let $\{g_j\}$ be a partition of unity. We say that $\{g_j\}$ is *subordinate* to $\{U_\iota\}$ if for every j there exists an $\iota(j)$ such that

$$\operatorname{supp} g_j \subset U_{\iota(j)}. \tag{2.1}$$

Theorem 2.1. Let $\{U_\iota\}$ be any open covering of M. There exists a partition of unity $\{g_j\}$ subordinate to $\{U_\iota\}$.

The proof that we shall present below is due to Bonic and Frampton.†
First we introduce some preliminary notions.
The function f on $\mathbb{R}$ defined by

$$f(u) = \begin{cases} e^{-1/u} & \text{if} \quad u > 0, \\ 0 & \text{if} \quad u \leq 0 \end{cases}$$

is C^∞. For $u \neq 0$ it is clear that f has derivatives of all orders. To check that f is C^∞ at 0, it suffices to show that $f^{(k)}(u) \to 0$ as $u \to 0$ from the right. But $f^{(k)}(u) = P_k(1/u)e^{-1/u}$, where P_k is a polynomial of degree $2k$. So

$$\lim_{u \to 0} f^{(k)}(u) = \lim_{s \to \infty} P_k(s)e^{-s} = 0,$$

since e^s goes to infinity faster than any polynomial.

Note that $f(u) > 0$ if and only if $u > 0$. Now consider the function g_a^b on $\mathbb{R}$ defined by

$$g_a^b(x) = f(x - a)f(b - x).$$

Then g_a^b is C^∞ and nonnegative and

$$g_a^b(x) \neq 0 \qquad \text{if and only if} \qquad a < x < b.$$

More generally, if $\mathbf{a} = \langle a^1, \ldots, a^k \rangle$ and $\mathbf{b} = \langle b^1, \ldots, b^k \rangle$, define the function $g_\mathbf{a}^\mathbf{b}$ on $\mathbb{R}^k$ by setting

$$g_\mathbf{a}^\mathbf{b}(x) = g_{a^1}^{b^1}(x)g_{a^2}^{b^2}(x^2) \cdots g_{a^k}^{b^k}(x^k),$$

where $x = \langle x^1, \ldots, x^k \rangle$. Then $g_\mathbf{a}^\mathbf{b} \geq 0$, $g_\mathbf{a}^\mathbf{b} \in C^\infty$, and

$$g_\mathbf{a}^\mathbf{b}(x) > 0 \qquad \text{if and only if} \qquad a^1 < x^1 < b^1, \ldots, a^k < x^k < b^k. \tag{2.2}$$

Lemma. Let $f_1, \ldots, f_k$ be C^∞-functions on a manifold M, and let $W = \{x : a^1 < f_1(x) < b^1, \ldots, a^k < f_k(x) < b^k\}$. There exists a nonnegative C^∞-function g such that $W = \{x : g(x) > 0\}$.

In fact, if we define g by

$$g(x) = g_\mathbf{a}^\mathbf{b}(f_1(x), \ldots, f_k(x)),$$

then it is clear that g has the desired properties.

---

† Smooth functions on Banach manifolds, *J. Math and Mech.* **15,** 877–898 (1966).

We now turn to the proof of Theorem 2.1.

Proof. For each $x \in M$ choose a U_ι containing x and a chart (U, α) about x. Then $\alpha(U \cap U_\iota)$ is an open set containing $\alpha(x)$ in $\mathbb{R}^n$. Choose $\mathbf{a}$ and $\mathbf{b}$ such that

$$\alpha(x) \in \operatorname{int} \square_{\mathbf{a}}^{\mathbf{b}} \qquad \text{and} \qquad \overline{\square_{\mathbf{a}}^{\mathbf{b}}} \subset \alpha(U \cap U_\iota).$$

Let $W_x = \alpha^{-1}(\operatorname{int} \square_{\mathbf{a}}^{\mathbf{b}})$. Then

$$\overline{W_x} \subset U_\iota \qquad \text{and} \qquad \overline{W_x} \text{ is compact.} \tag{2.3}$$

Also if $x^1, \ldots, x^n$ are the coordinates given by α,

$$W_x = \{y : a^1 < x^1(y) < b^1, \ldots, a^n < x^n(y) < b^n\}.$$

By our lemma we can find a nonnegative C^∞-function f_x such that

$$W_x = \{y : f_x(y) > 0\}.$$

Since $x \in W_x$, the $\{W_x\}$ cover M. By Proposition 1.2 we can select a countable subcovering $\{W_i\}$. Let us denote the corresponding functions by f_i; that is, if $W_i = W_x$, we set $f_i = f_x$.

Let

$$\begin{aligned} V_1 &= W_1 = \{x : f_1(x) > 0\}, \\ V_2 &= \{x : f_2(x) > 0, f_1(x) < \tfrac{1}{2}\}, \\ &\vdots \\ V_r &= \{x : f_r(x) > 0, f_1(x) < 1/r, \ldots, f_{r-1}(x) < 1/r\}. \end{aligned}$$

It is clear that V_j is open and that $V_j \subset W_j$, so that, by (2.3),

$$\overline{V_j} \text{ is compact} \qquad \text{and} \qquad \overline{V_j} \subset U_\iota \tag{2.4}$$

for some $\iota = \iota(j)$.

For each $x \in M$ let $q(x)$ denote the first integer q for which $f_q(x) > 0$. Thus $f_p(x) = 0$ if $p < q(x)$ and $f_{q(x)}(x) > 0$.

Let $V_x = \{y : f_{q(x)}(y) > \frac{1}{2}f_{q(x)}(x)\}$. Since $f_{q(x)}(x) > 0$, it follows that $x \in V_x$ and V_x is open. Furthermore,

$$V_x \cap \overline{V_r} = \varnothing \qquad \text{if} \quad r > q(x) \quad \text{and} \quad 1/r < \tfrac{1}{2}f_{q(x)}(x). \tag{2.5}$$

According to the lemma, each set V_i can be given as $V_i = \{x : \bar{g}_i(x) > 0\}$, where $\bar{g}_i$ is a suitable C^∞-function. Let $g = \sum \bar{g}_i$. In view of (2.5) this is really a finite sum in the neighborhood of any x. Thus g is C^∞. Now $\bar{g}_{q(x)}(x) > 0$, since $x \in V_{q(x)}$. Thus $g > 0$. Set

$$g_j = \frac{\bar{g}_j}{g}.$$

We claim that $\{g_j\}$ is the desired partition of unity. In fact, (i) holds by our construction, (ii) and (2.1) follow from (2.4), (iii) follows from (2.5), and (iv) holds by construction. □

3. DENSITIES

If we regard $\mathbb{R}^n$ as a differentiable manifold, then the law for change of variables for an integral shows that the integrand does not have the same transition law as that of a function under change of chart. For this reason we cannot expect to integrate functions on a manifold. We now introduce the type of object that we can integrate.

Definition 3.1. A *density* ρ is a rule which assigns to each chart (U, α) of M a function ρ_α defined on $\alpha(U)$ subject to the following transition law: If (W, β) is a second chart of M, then

$$\rho_\alpha(v) = \rho_\beta(\beta \circ \alpha^{-1}(v))|\det J_{\beta\circ\alpha^{-1}}(v)| \qquad \text{for} \quad v \in \alpha(U \cap W). \tag{3.1}$$

If $\mathfrak{A}$ is an atlas of M and functions ρ_{α_i} are given for all $(U_i, \alpha_i) \in \mathfrak{A}$ satisfying (3.1), then the ρ_{α_i} define a density ρ on M. In fact, if (U, α) is any chart of M (not necessarily belonging to $\mathfrak{A}$), define ρ_α by

$$\rho_\alpha(v) = \rho_{\alpha_i}(\alpha_i \circ \alpha^{-1}(v))|\det J_{\alpha_i\circ\alpha^{-1}}(v)| \qquad \text{if} \quad v \in \alpha(U \cap U_i).$$

This definition is consistent: If $v \in \alpha(U \cap U_i) \cap \alpha(U \cap U_j)$, then by (3.1),

$$\begin{aligned}
&\rho_{\alpha_j}(\alpha_j \circ \alpha^{-1}(v))|\det J_{\alpha_j\circ\alpha^{-1}}(v)| \\
&\qquad = \rho_{\alpha_i}(\alpha_i \circ \alpha_j^{-1}(\alpha_j \circ \alpha^{-1}(v)))|\det J_{\alpha_i\circ\alpha_j^{-1}}(\alpha_j \circ \alpha^{-1}(v))||\det J_{\alpha_j\circ\alpha^{-1}}(v)| \\
&\qquad = \rho_{\alpha_i}(\alpha_i \circ \alpha^{-1}(v))|\det J_{\alpha_i\circ\alpha^{-1}}(v)|
\end{aligned}$$

by the chain rule and the multiplicative property of determinants.

In view of (3.1) it makes sense to talk about local smoothness properties of densities. We will say that a density ρ is C^k if for any chart (U, α) the function ρ_α is C^k. As usual, it suffices to verify this for all charts (U, α) belonging to some atlas. Similarly, we say that a density ρ is *locally absolutely integrable* if for any chart (U, α) the function ρ_α is absolutely integrable. By the last proposition of Chapter 8 this is again independent of the choice of atlases.

Let ρ be a density on M, and let x be a point of M. It does not make sense to talk about the value of ρ at x. However, (3.1) shows that it does make sense to talk about the sign of ρ at x, More precisely, we say that

$$\rho > 0 \text{ at } x \qquad \text{if} \quad \rho_\alpha(\alpha(x)) > 0 \tag{3.2}$$

for a chart (U, α) about x. Equation (3.1) shows that if $\rho_\alpha(\alpha(x)) > 0$, then $\rho_\beta(\beta(x)) > 0$ for any other chart (W, β) about x. Similarly, it makes sense to say that $\rho < 0$ at x, $\rho > 0$ at x, or $\rho \neq 0$ at x.

Definition 3.2. Let ρ be a density on M. By the *support* of ρ, denoted by $\operatorname{supp} \rho$, we shall mean the closure of the set of points of M at which ρ does not vanish. That is,

$$\operatorname{supp} \rho = \overline{\{x : \rho \neq 0 \text{ at } x\}}.$$

Let ρ_1 and ρ_2 be densities. We define their sum by setting

$$(\rho_1 + \rho_2)_\alpha = \rho_{1\alpha} + \rho_{2\alpha} \tag{3.3}$$

for any chart (U, α). It is immediate that the right-hand side of (3.3) satisfies the transition law (3.1), and so defines density on M.

Let ρ be a density, and let f be a function. We define the density $f\rho$ by

$$(f\rho)_\alpha = f_\alpha \rho_\alpha. \tag{3.4}$$

Again, the verification of (3.1) is immediate in view of the transition laws for functions.

It is clear that

$$\text{supp } (\rho_1 + \rho_2) \subset \text{supp } \rho_1 \cup \text{supp } \rho_2 \tag{3.5}$$

and

$$\text{supp } (f\rho) = \text{supp } f \cap \text{supp } \rho. \tag{3.6}$$

We shall write

$$\rho_1 \leq \rho_2 \text{ at } x \qquad \text{if} \quad \rho_2 - \rho_1 \geq 0 \text{ at } x$$

and

$$\rho_1 \leq \rho_2 \qquad \text{if} \quad \rho_1 \leq \rho_2 \text{ at all } x \in M.$$

Let P denote the space of locally absolutely integrable densities of compact support. We observe that P is a vector space and that the product $f\rho$ belongs to P if f is a (bounded) locally contented function and $\rho \in \text{P}$.

Theorem 3.1. There exists a unique linear function $\int$ on P satisfying the following condition: If $\rho \in \text{P}$ is such that $\text{supp } \rho \subset U$, where (U, α) is a chart of M, then

$$\int \rho = \int_{\alpha(U)} \rho_\alpha. \tag{3.7}$$

Proof. We first show that there is at most one linear function satisfying (3.7). Let $\mathfrak{A}$ be an atlas of M, and let $\{g_j\}$ be a partition of unity subordinate to $\mathfrak{A}$. For each j choose an $i(j)$ so that

$$\text{supp } g_j \subset U_{i(j)}.$$

Write $\rho = 1 \cdot \rho = \sum g_j \cdot \rho$. Since $\text{supp } \rho$ is compact, only finitely many of the terms $g_j\rho$ are not identically zero. Thus the sum is finite. Since $\int$ is linear,

$$\int \rho = \int \sum g_j \rho = \sum \int g_j \rho.$$

By (3.7),

$$\int g_j \rho = \int_{\alpha_{i(j)}(U_{i(j)})} (g_j \rho)_{\alpha_{i(j)}}.$$

Thus

$$\int \rho = \sum_j \int_{\alpha_{i(j)}(U_{i(j)})} (g_j \rho)_{\alpha_{i(j)}}. \tag{3.8}$$

Thus $\int$, if it exists, must be given by (3.8). To establish the existence of $\int$, we must show that (3.8) defines a linear function on P satisfying (3.7). The linearity is obvious; we must verify (3.7).

Suppose supp $\rho \subset U$ for some chart (U, α). We must show that

$$\int_{\alpha(U)} \rho_\alpha = \sum_j \int_{\alpha_{i(j)}(U_{i(j)})} (g_j\rho)_{\alpha_{i(j)}}.$$

Since $\rho = \sum g_j\rho$ and therefore $\rho_\alpha = \sum (g_j\rho)_\alpha$, it suffices to show that

$$\int_{\alpha(U)} (g_j\rho)_\alpha = \int_{\alpha_i(U_i)} (g_j\rho)_{\alpha_i}, \tag{3.9}$$

where supp $g_j\rho \subset U \cap U_i$. By (3.1),

$$(g_j\rho)_\alpha = (g_j\rho)_{\alpha_i} \circ (\alpha_i \circ \alpha^{-1}) \cdot |\det J_{\alpha_i \circ \alpha^{-1}}|,$$

so that (3.9) holds by the transformation law for integrals in $\mathbb{R}^n$. □

We can derive a number of useful properties of the integral from the formula (3.8):

$$\text{if} \quad \rho_1 \leq \rho_2, \qquad \text{then} \quad \int \rho_1 \leq \int \rho_2. \tag{3.10}$$

In fact, since $g_j \geq 0$, we have $(g_j\rho_1)_\alpha \leq (g_j\rho_2)_\alpha$ for any chart (U, α). Thus (3.10) follows from the corresponding fact on $\mathbb{R}^n$ if we use (3.8).

Let us say that a set A has content zero if $A \subset A_1 \cup \cdots \cup A_p$ where each A_i is compact, $A_i \subset U_i$ for some chart (U_i, α_i), and $\alpha_i(A_i)$ has content zero in $\mathbb{R}^n$. It is easy to see that the union of any finite number of sets of content zero has content zero. It is also clear that the function e_A is contented.

Let us call a set $B \subset M$ contented if the function e_B is contented. For any $\rho \in$ P we define $\int_B \rho$ by

$$\int_B \rho = \int e_B\rho. \tag{3.11}$$

It follows from (3.8) that

$$\int_A \rho = 0$$

for any $\rho \in$ P if A has content zero. We can thus ignore sets of content zero for the purpose of integration. In practice, one usually takes advantage of this when computing integrals, rather than using (3.8). For instance, in computing an integral over S^n, we can "ignore" any meridian: for example, if

$$A = \{x \in S^n : x = (t, 0, \ldots, \pm\sqrt{1 - t^2}) \in \mathbb{R}^{n+1}\},$$

then

$$\int_{S^n} \rho = \int_{S^n - A} \rho \qquad \text{for any } \rho.$$

This means that we can compute $\int_{S^n} \rho$ by introducing polar coordinates (Fig. 10.1) and expressing ρ in terms of them. Thus in S^2, if $U = S^2 - A$ and α is the polar coordinate chart on U, then

$$\int_{S^2} \rho = \int_0^{2\pi} \int_0^{\pi} \rho_\alpha \, d\theta \, d\varphi.$$

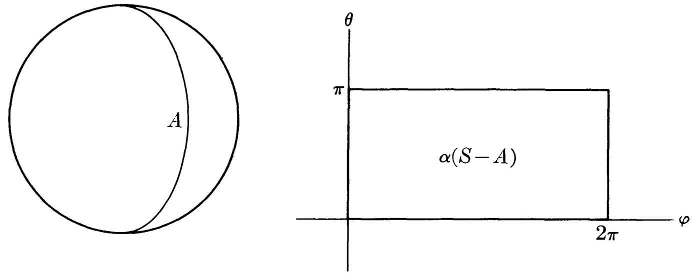

Fig. 10.1

It is worth observing that if N is a differentiable manifold of dimension less than dim M and ψ is a differentiable map of $N \to M$, then Proposition 7.3 of Chapter 8 implies that if A is any compact subset of N, then $\psi(A)$ has content zero in M. In this sense, one can ignore "lower-dimensional sets" when integrating on M.

4. VOLUME DENSITY OF A RIEMANN METRIC

Let M be a differentiable manifold with a Riemann metric $\mathfrak{a}$. We define the density σ $[=\sigma(\mathfrak{a})]$ as follows. For each chart (U, α) with coordinates $x^1, \ldots, x^n$ let

$$\sigma_\alpha(\alpha(x)) = \left|\det\left[\left(\frac{\partial}{\partial x^i}(x), \frac{\partial}{\partial x^j}(x)\right)\right]\right|^{1/2} = |\det(g_{ij}(x))|^{1/2}. \tag{4.1}$$

Here

$$\left[\left(\frac{\partial}{\partial x^i}(x), \frac{\partial}{\partial x^j}(x)\right)\right]$$

is the matrix whose ijth entry is the scalar product of the vectors

$$\frac{\partial}{\partial x^i}(x) \quad \text{and} \quad \frac{\partial}{\partial x^j}(x),$$

so that (in view of Exercise 8.1 of Chapter 8)

$\sigma_\alpha(\alpha(x))$ = volume of the parallelepiped spanned by $(\partial/\partial x^i)(x)$ with respect to the Euclidean metric $(\ ,\ )_{\mathfrak{a},x}$ on $T_x(M)$.

It is easy to see that (4.1) actually defines a density. Let (W, β) be a second chart about x with coordinates $y^1, \ldots, y^n$. Then

$$\frac{\partial}{\partial y^k} = \sum_i \frac{\partial x^i}{\partial y^k}\frac{\partial}{\partial x^i},$$

so that

$$\sigma_\beta(\beta(x)) = \left|\det\left[\left(\frac{\partial}{\partial y^k}(x), \frac{\partial}{\partial y^l}(x)\right)\right]\right|^{1/2}.$$

Now

$$\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l}\right) = \sum_{i,j} \frac{\partial x^i}{\partial y^k} \cdot \frac{\partial x^j}{\partial y^l} \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

for all k and l. We can write this as the matrix equation

$$\left[\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l}\right)\right] = \left[\frac{\partial x^i}{\partial y^k}\right]\left[\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right)\right]\left[\frac{\partial x^j}{\partial y^l}\right],$$

so that

$$\begin{aligned}\sigma_\beta(\beta(x)) &= \left|\det\left[\left(\frac{\partial}{\partial x^i}(x), \frac{\partial}{\partial x^j}(x)\right)\right]\det\left[\frac{\partial x^i}{\partial y^k}\right]\det\left[\frac{\partial x^j}{\partial y^l}\right]\right|^{1/2} \\ &= \left|\det\left[\left(\frac{\partial}{\partial x^i}(x), \frac{\partial}{\partial x^j}(x)\right)\right]\right|^{1/2}\left|\det\left[\frac{\partial x^i}{\partial y^k}\right]\right| \\ &= \sigma_\alpha(\alpha(x))\left|\det\left[\frac{\partial x^i}{\partial y^k}\right]\right|(x).\end{aligned}$$

If M is an open subset of Euclidean space with the Euclidean metric, then the volume density, when integrated over any contented set, yields the ordinary Euclidean volume of that set. In fact, if $x^1, \ldots, x^n$ are orthonormal coordinates corresponding to the identity chart, then $g_{ij}(x) = 0$ if $i \neq j$ and $g_{ii} = 1$, so that $\sigma_{\mathrm{id}} \equiv 1$ and thus

$$\int_A \sigma = \int_A 1 = \mu(A).$$

More generally, let φ be an immersion of a k-dimensional manifold M into $\mathbb{R}^n$ such that $\varphi(M)$ is an open subset of a k-dimensional hyperplane in $\mathbb{R}^n$, and let m be the Riemann metric induced on M by φ. Then, if σ denotes the corresponding volume density, $\int_A \sigma$ is the k-dimensional Euclidean volume of $\varphi(A)$. In fact, by a Euclidean motion, we may assume that φ maps M into $\mathbb{R}^k \subset \mathbb{R}^n$. Then, since φ is an immersion and M is k-dimensional, we can use $x^1, \ldots, x^k$ as coordinates on M and conclude, as before, that σ is given by the function in terms of these coordinates, and hence that $\int_A \sigma = \mu(\varphi(A))$.

Now let φ_1 and φ_2 be two immersions of $M \to \mathbb{R}^n$. Let (U, α) be a coordinate chart on M with coordinates $y^1, \ldots, y^k$. If m_1 is the Riemann metric induced by φ_i, then

$$\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)_{\mathsf{m}_1} = \left(\frac{\partial \varphi_1}{\partial y^i}, \frac{\partial \varphi_1}{\partial y^j}\right)$$

and

$$\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)_{\mathsf{m}_2} = \left(\frac{\partial \varphi_2}{\partial y^i}, \frac{\partial \varphi_2}{\partial y^j}\right),$$

where the scalar product on the right is the Euclidean scalar product. Let σ_1

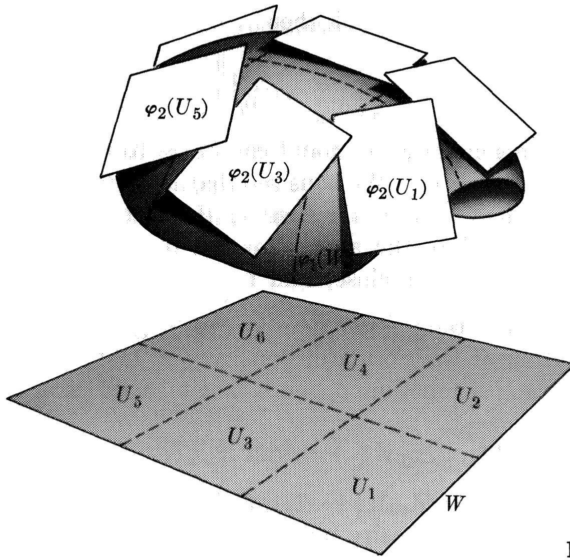

Fig. 10.2

and σ_2 be the volume densities corresponding to m_1 and m_2. Then

$$\sigma_{1\alpha} = \left|\det\left[\left(\frac{\partial\varphi_1}{\partial y^i}, \frac{\partial\varphi_1}{\partial y^j}\right)\right]\right|^{1/2}$$

and

$$\sigma_{2\alpha} = \left|\det\left[\left(\frac{\partial\varphi_2}{\partial y^i}, \frac{\partial\varphi_2}{\partial y^j}\right)\right]\right|^{1/2}$$

In particular, given an $L > 0$, there is a $K = K(k, n, L)$ such that if

$$\left\|\frac{\partial\varphi_1}{\partial y^i}\right\| < L \quad \text{and} \quad \left\|\frac{\partial\varphi_2}{\partial y^i}\right\| < L \quad \text{for all} \quad i = 1, \ldots, k,$$

then, by the mean-value theorem,

$$|\sigma_{1\alpha} - \sigma_{2\alpha}| \leq K\left(\left\|\frac{\partial\varphi_2}{\partial y^1} - \frac{\partial\varphi_1}{\partial y^1}\right\| + \cdots + \left\|\frac{\partial\varphi_2}{\partial y^k} - \frac{\partial\varphi_1}{\partial y^k}\right\|\right).$$

Roughly speaking, this means that if φ_1 and φ_2 are close, in the sense that their derivatives are close, then the densities they induce are close.

We apply this remark to the following situation. We let φ_1 be an immersion of M into $\mathbb{R}^n$ and let (W, α) be some chart of M with coordinates $y^1, \ldots, y^k$. We let $U = W - C = \bigcup U_l$, where C is some closed set of content zero and such that $U_l \cap U_{l'} = \emptyset$ if $l \neq l'$. For each l let z_l be a point of U_l whose coordinates are $\langle y_l^1, \ldots, y_l^k \rangle$, and for $z = \langle y^1, \ldots, y^k \rangle$ define φ_2 by setting,

$$\varphi_2(y^1, \ldots, y^k) = \varphi_1(z_l) + \sum (y^i - y_l^i)\frac{\partial\varphi_1}{\partial y^i}(z_l)$$

if $z \in U_l$. (See Fig. 10.2.)

If the U_l's are sufficiently small, then

$$\left\|\frac{\partial\varphi_2}{\partial y^i} - \frac{\partial\varphi_1}{\partial y^i}\right\|$$

will be small. More generally, we could choose φ_2 to be any affine linear map approximating φ_1 on each U_l. We thus see that *the volume of W in terms of the Riemann metric induced by φ is the limit of the (surface) volume of polyhedra approximating $\varphi(W)$.* Here the approximation must be in the *sense of slope* (i.e., the derivatives must be close) and not merely in the sense of position.

The construction of the volume density can be generalized and suggests an alternative definition of the notion of density. In fact, let ρ be a rule which assigns to each x in M a function, ρ_x, on n tangent vectors in $T_x(M)$ subject to the rule

$$\rho_x(A\,\xi_1, \ldots, A\,\xi_n) = |\det A|\rho_x(\xi_1, \ldots, \xi_n), \tag{4.2}$$

where $\xi_i \in T_x(M)$ and $A\colon T_x(M) \to T_x(M)$ is a linear transformation. Then we see that ρ determines a density by setting

$$\rho_\alpha(\alpha(x)) = \rho\left(\frac{\partial}{\partial u^1}(x), \ldots, \frac{\partial}{\partial u^n}(x)\right) \tag{4.3}$$

if (U, α) is a chart with coordinates $u^1, \ldots, u^n$. The fact that (4.3) defines a density follows immediately from (4.2) and the transformation law for the $\partial/\partial u^i$ under change of coordinates.

Conversely, given a density ρ in terms of the ρ_α, define $\rho(\partial/\partial u^1, \ldots, \partial/\partial u^n)$ by (4.3). Since the vectors $\{\partial/\partial u^i\}_{i=1,\ldots,n}$ form a basis at each x in U, any $\xi_1, \ldots, \xi_n$ in $T_x(M)$ can be written as

$$\xi_i = B\frac{\partial}{\partial u^i}(x),$$

where B is a linear transformation of $T_x(M)$ into itself. Then (4.2) determines $\rho(\xi_1, \ldots, \xi_n)$ as

$$\rho(\xi_1, \ldots, \xi_n) = |\det B|\rho_\alpha(\alpha(x)). \tag{4.4}$$

That this definition is consistent (i.e., doesn't depend on α) follows from (4.2) and the transformation law (3.1) for densities.

EXERCISES

4.1 Let $M = S^1 \times S^1$ be the torus, and let $\varphi\colon M \to \mathbb{R}^4$ be given by

$$\begin{aligned} x^1 \circ \varphi(\theta_1, \theta_2) &= \cos\theta_1, \\ x^2 \circ \varphi(\theta_1, \theta_2) &= \sin\theta_1, \\ x^3 \circ \varphi(\theta_1, \theta_2) &= 2\cos\theta_2, \\ x^4 \circ \varphi(\theta_1, \theta_2) &= 2\sin\theta_2, \end{aligned}$$

where $x^1, \dots, x^4$ are the rectangular coordinates on $\mathbb{R}^4$ and θ^1, θ^2 are angular coordinates on M.

a) Express the Riemann metric induced on M by φ (from the Euclidean metric on $\mathbb{R}^4$) in terms of the coordinates θ^1, θ^2. [That is, compute the $g_{ij}(\theta^1, \theta^2)$.]
b) What is the volume of M relative to this Riemann metric?

4.2 Consider the Riemann metric induced on $S^1 \times S^1$ by the immersion φ into $\mathbb{E}^3$ by

$$x \circ \varphi(u, v) = (a - \cos u) \cos v,$$
$$y \circ \varphi(u, v) = (a - \cos u) \sin v,$$
$$z \circ \varphi(u, v) = \sin u,$$

where u and v are angular coordinates and $a > 2$. What is the total surface area of $S^1 \times S^1$ under this metric?

4.3 Let φ map a region U of the xy-plane into $\mathbb{E}^3$ by the formula

$$\varphi(x, y) = (x, y, F(x, y)),$$

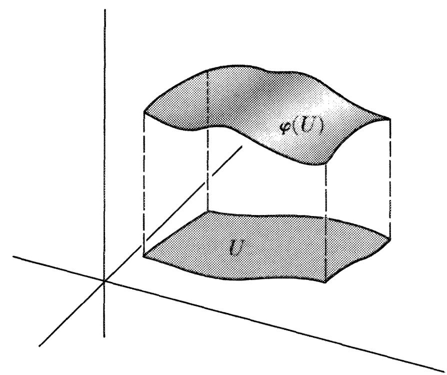

Fig. 10.3

so that $\varphi(U)$ is the surface $z = F(x, y)$. (See Fig. 10.3.) Show that the area of this surface is given by

$$\int_U \sqrt{1 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}.$$

4.4 Find the area of the paraboloid

$$z = x^2 + y^2 \qquad \text{for} \quad x^2 + y^2 \leq 1.$$

4.5 Let $U \subset \mathbb{R}^2$, and let $\varphi: U \to \mathbb{E}^3$ be given by

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y, z are rectangular coordinates on $\mathbb{E}^3$. Show the area of the surface $\varphi(U)$ is given by

$$\int_U \sqrt{\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2}.$$

4.6 Compute the surface area of the unit sphere in $\mathbb{E}^3$.

4.7 Let M_1 and M_2 be differentiable manifolds, and let σ be a density on M_2 which is nowhere zero. For each density ρ on $M_1 \times M_2$, each product chart $(U_1 \times U_2, \alpha_1 \times \alpha_2)$, and each $x_2 \in U_2$, define the function $\rho_{1\alpha_1}(\cdot, x_2)$ by

$$\rho_{1\alpha_1}(v_1, x_2)\sigma_\alpha(\alpha_2(x_2)) = \rho_{\alpha_1 \times \alpha_2}(v_1, \alpha_2(x_2))$$

for all $v_1 \in \alpha_1(U_1)$.

a) Show that $\rho_{1\alpha_1}(v_1, x_2)$ is independent of the chart (U_2, α_2).
b) Show that for each fixed $x_2 \in M_2$ the functions $\rho_{1\alpha_1}(\cdot, x_2)$ define a density on M_1. We shall call this density $\rho_1(x_2)$.

c) Show that if ρ is a smooth density of compact support on $M_1 \times M_2$ and σ is smooth, then $\rho_1(x_2)$ is a smooth density of compact support on M_1.

d) Let ρ be as in (c). Define the function F_ρ on M_2 by

$$F_\rho(x_2) = \int_{M_1} \rho_1(x_2).$$

Sketch how you would prove the fact that F_ρ is a smooth function of compact support on M_2 and that

$$\int_{M_1 \times M_2} \rho = \int_{M_2} F_\rho \cdot \sigma.$$

5. PULLBACK AND LIE DERIVATIVES OF DENSITIES

Let $\varphi: M_1 \to M_2$ be a diffeomorphism, and let ρ be a density on M_2. Define the density $\varphi^*\rho$ on M_1 by

$$\varphi^*\rho(\xi_1, \ldots, \xi_n) = \rho(\varphi_*\xi_1, \ldots, \varphi_*\xi_n) \tag{5.1}$$

for $\xi_i \in T_x(M_1)$ and $\varphi_* = \varphi_{*x}$. To show that $\varphi^*\rho$ is actually a density, we must check that (4.2) holds for any linear transformation A of $T_x(M_1)$. But

$$\begin{aligned}\varphi^*\rho(A\xi_1, \ldots, A\xi_n) &= \rho(\varphi_*A\xi_1, \ldots, \varphi_*A\xi_n)\\ &= \rho(\varphi_*A\varphi_*^{-1}\varphi_*\xi_1, \ldots, \varphi_*A\varphi_*^{-1}\varphi_*\xi_n)\\ &= |\det \varphi_*A\varphi_*^{-1}|\, \rho(\varphi_*\xi_1, \ldots, \varphi_*\xi_n)\\ &= |\det A|\varphi^*\rho(\xi_1, \ldots, \xi_n),\end{aligned}$$

which is the desired identity.

Let (U, α) and (W, β) be compatible charts on M_1 and M_2 with coordinates $u^1, \ldots, u^n$ and $w^1, \ldots, w^n$, respectively. Then for all points of U we have, by (4.3),

$$\begin{aligned}(\varphi^*\rho)_\alpha(\alpha(\cdot)) &= \rho\left(\varphi_*\frac{\partial}{\partial u^1}, \ldots, \varphi_*\frac{\partial}{\partial u^n}\right) = \left|\det\left(\frac{\partial w^j}{\partial u^i}\right)\right| \rho\left(\frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^n}\right)\\ &= \left|\det\left(\frac{\partial w^j}{\partial u^i}\right)\right| \rho_\beta(\beta \circ \varphi(\cdot)).\end{aligned}$$

In other words, we have

$$(\varphi^*\rho)_\alpha = |\det J_{\beta\circ\varphi\circ\alpha^{-1}}|\rho(\beta \circ \varphi \circ \alpha^{-1}(\cdot)). \tag{5.2}$$

The density $\varphi^*\rho$ is called the pullback of ρ by φ^*. It is clear that

$$\varphi^*(\rho_1 + \rho_2) = \varphi^*(\rho_1) + \varphi^*(\rho_2)$$

and that

$$\varphi^*(f\rho) = \varphi^*(f)\varphi^*(\rho)$$

for any function f.

It follows directly from the definition that

$$\operatorname{supp} \varphi^*\rho = \varphi^{-1}[\operatorname{supp} \rho].$$

Proposition 5.1. Let $\varphi\colon M_1 \to M_2$ be a diffeomorphism, and let ρ be a locally absolutely integrable density with compact support on M_2. Then

$$\int \varphi^*\rho = \int \rho. \tag{5.3}$$

Proof. It suffices to prove (5.3) for the case

$$\operatorname{supp} \rho \subset \varphi(U)$$

for some chart (U, α) of M_1 with $\varphi(U) \subset W$, where (W, β) is a chart of M_2. In fact, the set of all such $\varphi(U)$ is an open covering of M_2, and we can therefore choose a partition of unity $\{g_j\}$ subordinate to it. If we write $\rho = \sum g_j\rho$, then the sum is finite and each $g_j\rho$ has the observed property. Since both sides of (5.3) are linear, we conclude that it suffices to prove (5.3) for each term.

Now if $\operatorname{supp} \rho \subset \varphi(U)$, then

$$\int \rho = \int_{\beta(W)} \rho_\beta = \int_{\beta\circ\varphi(U)} \rho_\beta$$

and

$$\begin{aligned}\int \varphi^*\rho &= \int_{\alpha(U)} (\varphi^*\rho)_\alpha \\ &= \int_{\alpha(U)} \rho_\beta(\beta\circ\varphi\circ\alpha^{-1})|\det J_{\beta\circ\varphi\circ\alpha^{-1}}| \\ &= \int_{\beta\circ\varphi(U)} \rho_\beta,\end{aligned}$$

thus establishing (5.3). □

Now let φ_t be a one-parameter group on M with infinitesimal generator X. Let ρ be a density on M, let (U, α) be a chart, and let W be an open subset of U such that $\varphi_t(W) \subset U$ for all $|t| < \epsilon$. Then

$$(\varphi_t^*\rho)_\alpha(v) = \rho_\alpha(\Phi_\alpha(v, t)) \left|\det\left(\frac{\partial\Phi_\alpha}{\partial v}\right)_{(v,t)}\right| \qquad \text{for} \quad v \in \alpha(W),$$

where $\Phi_\alpha(v, t) = \alpha\circ\varphi_t\circ\alpha^{-1}(v)$ and $(\partial\Phi_\alpha/\partial v)_{(v,t)}$ is the Jacobian of $v \mapsto \Phi_\alpha(v, t)$. We would like to compute the derivative of this expression with respect to t at $t = 0$. Now $\Phi_\alpha(v, 0) = v$, and so

$$\det\left(\frac{\partial\Phi_\alpha}{\partial v}\right)_{(v,0)} = 1.$$

Consequently, we can conclude that

$$\det\left(\frac{\partial\Phi_\alpha}{\partial v}\right)_{(v,t)} > 0$$

for t close to zero. We can therefore omit the absolute-value sign and write

$$\left.\frac{d(\varphi_t^*\rho)_\alpha}{dt}\right|_{t=0} = \left.\frac{d\rho_\alpha(\Phi_\alpha)}{dt}\right|_{t=0} + \rho_\alpha(v)\left.\frac{d}{dt}\left(\det\frac{\partial\Phi_\alpha}{\partial v}\right)\right|_{t=0}.$$

We simply evaluate the first derivative on the right by the chain rule, and get

$$d\rho_\alpha\left(\frac{\partial\Phi_\alpha}{\partial t}\right) = d\rho_\alpha(X_\alpha(v)).$$

In terms of coordinates $x^1, \ldots, x^n$, we can write

$$\frac{d\rho_\alpha(\Phi_\alpha(v, t))}{dt} = \sum \frac{\partial\rho_\alpha}{\partial x^i} X^i_\alpha$$

if $X_\alpha = \langle X^1_\alpha, \ldots, X^n_\alpha \rangle$.

To evaluate the second term on the right, we need to make a preliminary observation. Let $A(t) = (a_{ij}(t))$ be a differentiable matrix-valued function of t with $A(0) = \mathrm{id} = (\delta^j_i)$. Then

$$\frac{d(\det A(t))}{dt} = \lim \frac{1}{t}(\det A(t) - 1).$$

Now $a_{ii}(0) = 1$ and $a_{ij}(0) = 0$ $(i \neq j)$. To say that A is differentiable means that each of the functions $a_{ij}(t)$ is differentiable. We can therefore find a constant K such that $|a_{ij}(t)| \leq K|t|$ $(i \neq j)$ and $|a_{ii}(t) - 1| \leq K|t|$. In the expansion of $\det A(t)$, the only term which will not vanish at least as t^2 is the diagonal product $a_{11}(t) \cdots a_{nn}(t)$. In fact, any other term in $\sum \pm a_{1i_1}(t) \cdots a_{ni_n}(t)$ involves at least two off-diagonal terms and thus vanishes at least as t^2. Thus

$$\begin{aligned}\frac{d}{dt}(\det A(t)) &= \lim_{t=0} \frac{1}{t}(a_{11}(t) \cdots a_{nn}(t) - 1) \\ &= a'_{11}(0) + \cdots + a'_{nn}(0) \\ &= \operatorname{tr} A'(0).\end{aligned}$$

If we take $A = \partial\Phi_\alpha/\partial v$, we conclude that

$$\frac{d}{dt}\left(\det \frac{\partial\Phi_\alpha}{\partial v}\right) = \operatorname{tr} \frac{\partial X_\alpha}{\partial v} = \sum \frac{\partial X^i_\alpha}{\partial x^i}.$$

Thus

$$\frac{d(\overset{*}{\varphi_t}\rho)_\alpha}{dt} = \sum \frac{\partial\rho_\alpha}{\partial x^i} X^i_\alpha + \rho_\alpha \frac{\partial X^i_\alpha}{\partial x^i} = \sum \frac{\partial}{\partial x^i}(\rho_\alpha X^i_\alpha).$$

We repeat:

Proposition 5.2. Let φ_t be a one-parameter group of diffeomorphisms of M with infinitesimal generator X, and let ρ be a differentiable density on M. Then

$$D_X\rho = \lim_{t=0} \frac{\overset{*}{\varphi_t}\rho - \rho}{t}$$

exists and is given locally by

$$(D_X\rho)_\alpha = \sum \frac{\partial(\rho_\alpha X^i_\alpha)}{\partial x^i}$$

if $X = \langle X^1_\alpha, \ldots, X^n_\alpha \rangle$ on the chart (U, α).

The density $D_X\rho$ is sometimes called the *divergence* of $\langle X, \rho\rangle$ and is denoted by div $\langle X, \rho\rangle$. Thus div $\langle X, \rho\rangle = D_X\rho$ is the density given by

$$(\text{div} \langle X, \rho\rangle)_\alpha = \sum \frac{\partial}{\partial x^i}(X^i_\alpha \rho_\alpha) \qquad \text{on} \quad (U, \alpha).$$

Now let ρ be a differentiable density, and let A be a compact contented set. Then

$$\begin{aligned}\int_{\varphi_t(A)} \rho &= \int_M e_{\varphi_t(A)}\rho \\ &= \int_M \varphi_t^*(e_{\varphi_t(A)}\rho) \\ &= \int (\varphi_t^* e_{\varphi_t(A)})(\varphi_t^*\rho) \\ &= \int e_A \varphi_t^*(\rho) \\ &= \int_A \varphi_t^*\rho.\end{aligned}$$

Thus

$$\frac{1}{t}\left(\int_{\varphi_t(A)} \rho - \int_A \rho\right) = \int_A \frac{1}{t}(\varphi_t^*\rho - \rho).$$

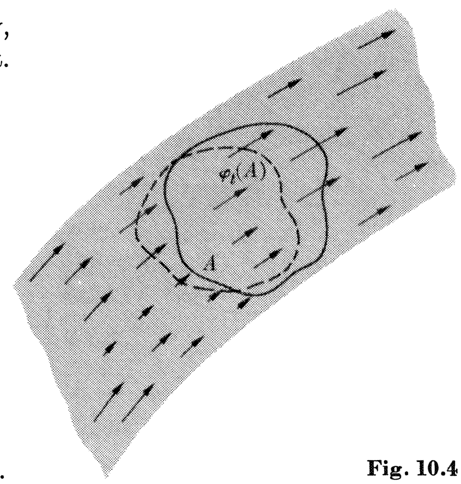

Fig. 10.4

Using a partition of unity, we can easily see that the limit under the integral sign is uniform, and we thus have the formula

$$\frac{d}{dt}\left(\int_{\varphi_t(A)} \rho\right)\Bigg|_{t=0} = \int_A D_X\rho = \int_A \text{div} \langle X, \rho\rangle.$$

6. THE DIVERGENCE THEOREM

Let φ be a flow on a differentiable manifold M with infinitesimal generator X. Let ρ be a density belonging to P, and let A be a contented subset of M. Then for small values of t, we would expect the difference $\int_{\varphi_t(A)} \rho - \int_A \rho$ to depend only on what is happening near the boundary of A (Fig. 10.4). In the limit, we would expect the derivative of $\int_{\varphi_t(A)} \rho$ at $t = 0$ (which is given by $\int_A$ div $\langle X, \rho\rangle$) to be given by some integral over ∂A. In order to formulate such a result, we must first single out a class of sets whose boundaries are sufficiently nice to allow us to integrate over them. We therefore make the following definition:

Definition. Let M be a differentiable manifold, and let D be a subset of M. We say that D is a *domain with regular boundary* if for every $x \in M$ there is a chart (U, α) about x, with coordinates $x^1_\alpha, \ldots, x^n_\alpha$, such that one of the following three possibilities holds:

i) $U \cap D = \emptyset$;

ii) $U \subset D$;

iii) $\alpha(U \cap D) = \alpha(U) \cap \{\mathbf{v} = \langle v^1, \ldots, v^n\rangle \in \mathbb{R}^n : v^n \geq 0\}$.

Note that if $x \notin \overline{D}$, we can always find a (U, α) about x such that (i) holds. If $x \in \operatorname{int} D$, we can always find a chart (U, α) about x such that (ii) holds. This imposes no restrictions on D. The crucial condition is imposed when $x \in \partial D$. Then we cannot find charts about x satisfying (i) or (ii). In this case, (iii) implies that $\alpha(U \cap \partial D)$ is an open subset of $\mathbb{R}^{n-1}$ (Fig. 10.5). In fact, $\alpha(U \cap \partial D) = \{\mathbf{v} \in \alpha(U) : v^n = 0\} = \alpha(U) \cap \mathbb{R}^{n-1}$, where we regard $\mathbb{R}^{n-1}$ as the subspace of $\mathbb{R}^n$ consisting of those vectors with last component zero.

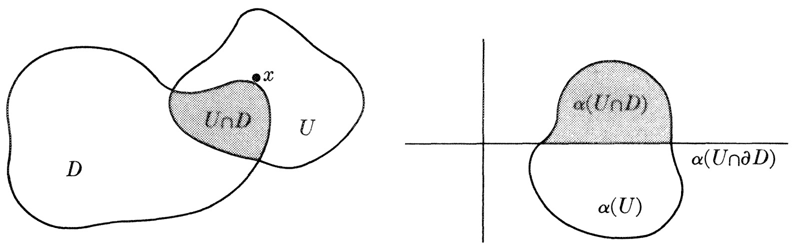

Fig. 10.5

Let $\mathcal{A}$ be an atlas of M such that each chart of $\mathcal{A}$ satisfies either (i), (ii), or (iii). For each $(U, \alpha) \in \mathcal{A}$ consider the map $\alpha \restriction \partial D \colon U \cap \partial D \to \mathbb{R}^{n-1} \subset \mathbb{R}^n$. [Of course, the maps $\alpha \restriction \partial D$ will have a nonempty domain of definition only for charts of type (iii).] We claim that $\{(U \cap \partial D, \alpha \restriction \partial D)\}$ is an atlas on ∂D. In fact, let (U, α) and (W, β) be two charts in $\mathcal{A}$ such that $U \cap W \cap \partial D \neq \emptyset$. Let $x^1, \ldots, x^n$ be the coordinates of (U, α), and let $y^1, \ldots, y^n$ be those of (W, β). The map $\beta \circ \alpha^{-1}$ is given by

$$\langle x^1, \ldots, x^n \rangle \mapsto \langle y^1(x^1, \ldots, x^n), \ldots, y^n(x^1, \ldots, x^n) \rangle.$$

On $\alpha(U \cap W \cap \partial D)$, we have $x^n = 0$ and $y^n = 0$. In particular,

$$y^n(x^1, \ldots, x^{n-1}, 0) \equiv 0,$$

and the functions $y^1(x^1, \ldots, x^{n-1}, 0), \ldots, y^{n-1}(x^1, \ldots, x^{n-1}, 0)$ are differentiable. This shows that $(\beta \restriction \partial D) \circ (\alpha \restriction \partial D)^{-1}$ is differentiable on $\alpha(U \cap \partial D)$. We thus get a manifold structure on ∂D.

It is easy to see that this manifold structure is independent of the particular atlas of M that was chosen. We shall denote by ι the map of $\partial D \to M$ which sends each $x \in \partial D$, regarded as an element of M, into itself. It is clear that ι is a differentiable map. (In fact, $(U \cap \partial D, \alpha \restriction \partial D)$ and (U, α) are compatible charts in terms of which $\alpha \circ \iota \circ (\alpha \restriction \partial D)^{-1}$ is just the map of $\mathbb{R}^{n-1} \to \mathbb{R}^n$.)

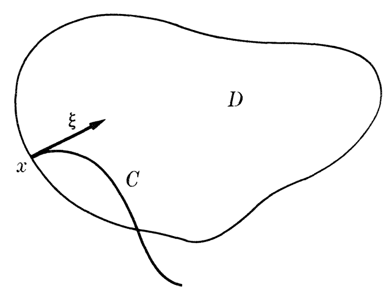

Fig. 10.6

Let x be a point of ∂D regarded as a point of M, and let ξ be an element of $T_x(M)$. We say that ξ points into D if for every curve C with $C'(0) = \xi$, we have $C(t) \in D$ for sufficiently small positive t (Fig. 10.6). In

terms of a chart (U, α) of type (iii), let $\xi_\alpha = \langle \xi^1, \ldots, \xi^n \rangle$. Then it is clear that ξ points into D if and only if $\xi^n > 0$. Similarly, a tangent vector ξ points out of D (obvious definition) if and only if $\xi^n < 0$. If $\xi^n = 0$, then ξ is tangent to the boundary—it lies in $\iota_* T_x(\partial D)$.

Let ρ be a density on M and X a vector field on M. Define the density ρ_X on ∂D by

$$\rho_X(\xi_1, \ldots, \xi_{n-1}) = \rho(\iota_*\xi_1, \ldots, \iota_*\xi_{n-1}, X(x)) \qquad \text{for} \quad \xi_i \in T_x(\partial D). \tag{6.1}$$

It is easy to check that (6.1) defines a density. (This is left as an exercise for the reader.) If (U, α) is a chart of type (iii) about x and $X_\alpha = \langle X^1, \ldots, X^n \rangle$, then applying (4.3) to the chart $(U \cap \partial D, \alpha \restriction \partial D)$ and the density ρ_X, we see that

$$(\rho_X)_{\alpha \restriction \partial D} = \rho\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}}, X\right).$$

Let A be the linear transformation of $T_x(M)$ given by

$$A \frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^1}, \qquad A \frac{\partial}{\partial x^{n-1}} = \frac{\partial}{\partial x^{n-1}}, \qquad A \frac{\partial}{\partial x^n} = X.$$

The matrix of A is

$$\begin{bmatrix} 1 & 0 & & \cdots & X^1 \\ 0 & 1 & 0 & \cdots & X^2 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & & X^n \end{bmatrix},$$

and therefore $|\det A| = |X^n|$. Thus we have

$$(\rho_X)_{\alpha \restriction \partial D} = |X^n| \rho_\alpha \qquad \text{at all points of} \quad \alpha(U \cap \partial D). \tag{6.2}$$

We can now state our results.

Theorem 6.1 (*The divergence theorem*).† Let D be a domain with regular boundary, let $\rho \in \mathrm{P}$, and let X be a smooth vector field on M. Define the function ϵ_X on ∂D by

$$\epsilon_X(x) = \begin{cases} 1 & \text{if} \quad X(x) \text{ points out of } D, \\ 0 & \text{if} \quad X(x) \text{ is tangent to } \partial D, \\ -1 & \text{if} \quad X(x) \text{ points into } D. \end{cases}$$

Then

$$\int_D \operatorname{div} \langle X, \rho \rangle = \int_{\partial D} \epsilon_X \rho_X. \tag{6.3}$$

Remark. In terms of a chart of type (iii), the function ϵ_X is given by

$$\epsilon_X = -\operatorname{sgn} X^n. \tag{6.4}$$

† This formulation and proof of the divergence theorem was suggested to us by Richard Rasala.

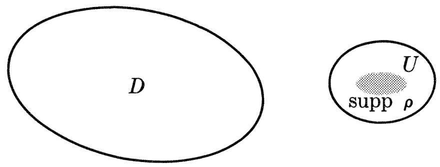

Fig. 10.7

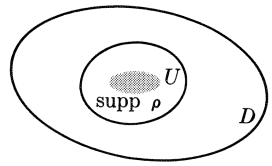

Fig. 10.8

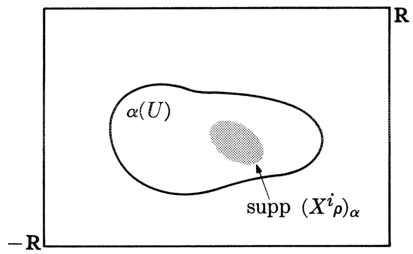

Fig. 10.9

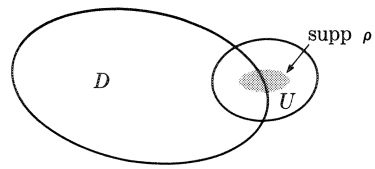

Fig. 10.10

Proof. Let $\mathcal{A}$ be an atlas of M each of whose charts is one of the three types. Let $\{g_i\}$ be a partition of unity subordinate to $\mathcal{A}$. Write $\rho = \sum g_i\rho$. This is a finite sum. Since both sides of (6.3) are linear functions of ρ it suffices to verify (6.3) for each of the summands $g_i\rho$. Changing our notation (replacing $g_i\rho$ by ρ), we reduce the problem to proving (6.3) under the additional assumption $\text{supp}\,\rho \subset U$, where (U, α) is a chart of type (i), (ii), or (iii). There are therefore three cases to consider.

CASE I. $\text{supp}\,\rho \subset U$ and $U \cap \overline{D} = \varnothing$. (See Fig. 10.7.) Then both sides of (6.3) vanish, and so (6.3) is correct.

CASE II. $\text{supp}\,\rho \subset U$ with $U \subset \text{int}\, D$. (See Fig. 10.8.) Then the right-hand side of (6.3) vanishes. We must show that the left-hand side does also. But

$$\int_D \text{div} \langle X, \rho\rangle = \int_U \text{div} \langle X, \rho\rangle = \int_{\alpha(U)} \sum \frac{\partial(X^i\rho_\alpha)}{\partial x^i} = \sum \int_{\alpha(U)} \frac{\partial(X^i\rho_\alpha)}{\partial x^i}\,.$$

Now each of the functions $X^i\rho_\alpha$ has its support lying inside $\alpha(U)$. Choose some large R so that $\alpha(U) \subset \square_{-\mathbf{R}}^{\mathbf{R}}$. We can then replace $\int_{\alpha(U)}$ by $\int_{\square_{-\mathbf{R}}^{\mathbf{R}}}$. We extend its domain of definition to all of $\mathbb{R}^n$ by setting it equal to zero outside $\alpha(U)$. (See Fig. 10.9.) Writing the integral as an iterated integral and integrating with respect to x^i first, we see that

$$\int_{\alpha(U)} \frac{\partial X^i\rho_\alpha}{\partial x^i}$$

$$= \int X^i\rho_\alpha(\ldots, R, \ldots) - X^i\rho_\alpha(\ldots, -R, \ldots)\, dx^1\, dx^2\, dx^{i-1}\, dx^i \cdots dx^n = 0.$$

This last integral vanishes, because the function $X^i\rho_\alpha$ vanishes outside $\alpha(U)$.

CASE III. supp ρ is contained in a chart of type (iii). (See Fig. 10.10.) Then

$$\int_D \operatorname{div} \langle X, \rho\rangle = \int_{D\cap U} \operatorname{div} \langle X, \rho\rangle = \sum \int_{\alpha(D\cap U)} \frac{\partial X^i \rho_\alpha}{\partial x^i}.$$

Now

$$\alpha(U \cap D) = \alpha(U) \cap \{\mathbf{v} : v^n \geq 0\}.$$

We can therefore replace the domain of integration by the rectangle $\square^{\langle R,\ldots,R\rangle}_{\langle -R,\ldots,-R,0\rangle}$. (See Fig. 10.11.) For $1 \leq i < n$ all the integrals in the sum vanish as before. For $i = n$ we obtain

$$\int_D \operatorname{div} \langle X, \rho\rangle = -\int_{\mathbb{R}^{n-1}} X^n \rho_\alpha.$$

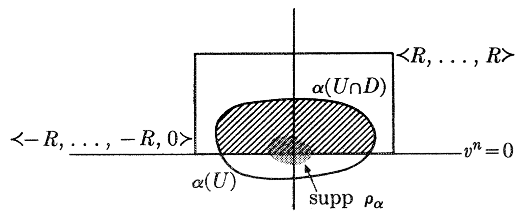

Fig. 10.11

If we compare this with (6.2) and (6.4), we see that this is exactly the assertion of (6.3). □

If the manifold M is given a Riemann metric, then we can give an alternative version of the divergence theorem. Let dV be the volume density of the Riemann metric, so that

$$dV(\xi_1, \ldots, \xi_n) = |\det((\xi_i, \xi_j))|^{1/2}, \quad \xi_i \in T_x(M),$$

is the volume of the parallelepiped spanned by the ξ_i in the tangent space (with respect to the Euclidean metric given by the scalar product on the tangent space).

Now the map ι is an immersion, and therefore we get an induced Riemann metric on ∂D. Let dS be the corresponding volume density on ∂D. Thus, if $\{\xi_i\}_{i=1,\ldots,n-1}$ are $n-1$ vectors in $T_x(\partial D)$, $dS(\xi_1, \ldots, \xi_{n-1})$ is the $(n-1)$-dimensional volume of the parallelepiped spanned by $\iota_*\xi_1, \ldots, \iota_*\xi_{n-1}$ in $\iota_*T_x(\partial D) \subset T_x(M)$. For any $x \in \partial D$ let $\mathbf{n} \in T_x(M)$ be the vector of unit length which is orthogonal to $\iota_*T_x(\partial D)$ and which points out of D (Fig. 10.12). We

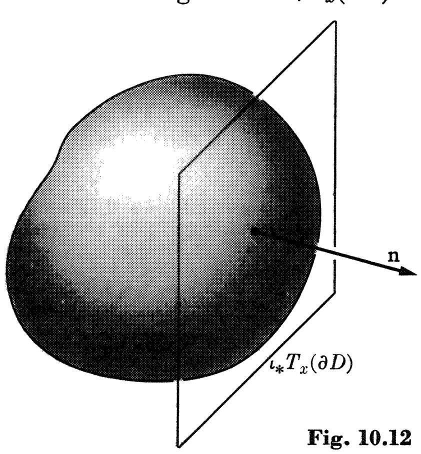

Fig. 10.12

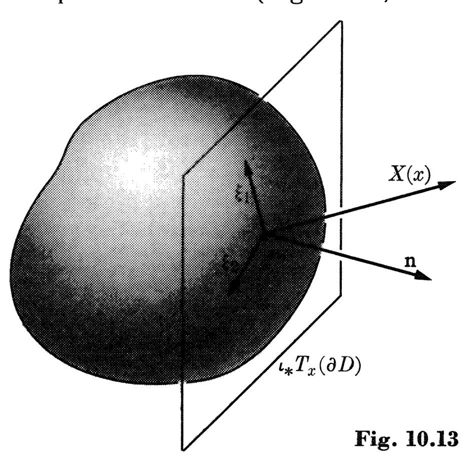

Fig. 10.13

clearly have

$$dS(\xi_1, \ldots, \xi_{n-1}) = dV(\iota_*\xi_1, \ldots, \iota_*\xi_{n-1}, n).$$

For any vector $X(x) \in T_x(M)$ (Fig. 10.13) the volume of the parallelepiped spanned by $\xi_1, \ldots, \xi_{n-1}, X(x)$ is $|(X(x), \mathbf{n})| dS(\xi_1, \ldots, \xi_{n-1})$. [In fact, write

$$X(x) = (X(x), \mathbf{n})\mathbf{n} + \mathbf{m},$$

where $\mathbf{m} \in \iota_* T(\partial D)$.] If we compare this with (6.1), we see that

$$dV_X = |(X, \mathbf{n})| dS. \tag{6.5}$$

Furthermore, it is clear that

$$\epsilon(x) = \operatorname{sgn}\,(X(x), \mathbf{n}).$$

Let ρ be any density on M. Then we can write

$$\rho = f\,dV,$$

where f is a function. Furthermore, we clearly have $\rho_X = f\,dV_X$ and

$$\operatorname{div} \langle X, \rho \rangle = \operatorname{div} \langle X, f\,dV \rangle.$$

We can then rewrite (6.3) as

$$\int_D \operatorname{div} \langle X, f\,dV \rangle = \int_{\partial D} f \cdot (X, \mathbf{n})\,dS. \tag{6.6}$$

7. MORE COMPLICATED DOMAINS

For many purposes, Theorem 6.1 is not quite sufficiently broad. The trouble is that we would like to apply (6.3) to domains whose boundaries are not completely smooth. For instance, we would like to apply it to a rectangle in $\mathbb{R}^n$. Now the boundary of a rectangle is regular at all points except those lying on an edge (i.e., the intersection of two faces). Since the edges form a set "of dimension $n - 2$", we would expect that their presence does not invalidate (6.3). This is in fact the case.

Let M be a differentiable manifold, and let D be a subset of M. We say that D is a domain with *almost regular boundary* if to every $x \in M$ there is a chart (U, α) about x, with coordinates $x^1, \ldots, x^n_\alpha$, such that one of the following four possibilities holds:

i) $U \cap D = \emptyset$;

ii) $U \subset D$;

iii) $\alpha(U \cap D) = \alpha(U) \cap \{\mathbf{v} = \langle v^1, \ldots, v^n \rangle \in \mathbb{R}^n : v^n \geq 0\}$;

iv) $\alpha(U \cap D) = \alpha(U) \cap \{\mathbf{v} = \langle v^1, \ldots, v^n \rangle \in \mathbb{R}^n : v^k \geq 0, \ldots, v^n \geq 0\}$.

The novel point is that we are now allowing for possibility (iv) where $k < n$. This, of course, is a new possibility only if $n > 1$. Let us assume $n > 1$ and see what (iv) allows. We can write $\alpha(U \cap \partial D)$ as the union of certain open subsets lying in $(n - 1)$-dimensional subspaces of $\mathbb{R}^{n-1}$, together with a union of portions lying in subspaces of dimension $n - 2$.

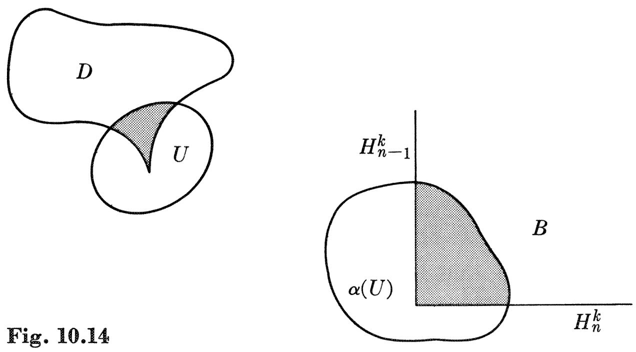

Fig. 10.14

In fact, for $k \leq p \leq n$ let

$$H^k_p = \{\mathbf{v} : v^k > 0, \ldots, v^p = 0, v^{p+1} > 0, \ldots, v^n > 0\}.$$

Thus H^k_p is an open subset of the $(n-1)$-dimensional subspace given by $v^p = 0$. (See Fig. 10.14.) We can write

$$\alpha(U \cap \partial D) \subset \alpha(U) \cap \{(H^k_k \cup H^k_{k+1} \cup \cdots \cup H^k_n) \cup S\},$$

where S is the union of the subspaces (of dimension $n-2$) where at least two of the v^p vanish.

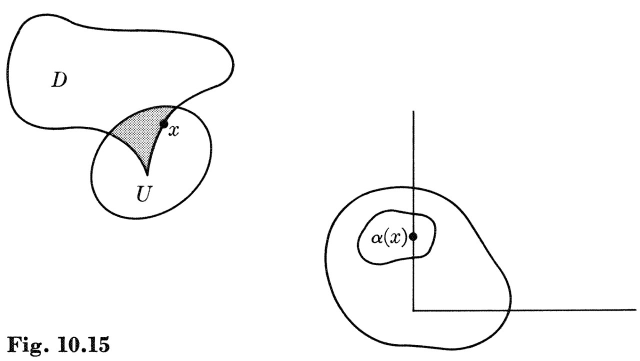

Fig. 10.15

Observe that if $x \in U \cap \partial D$ is such that $\alpha(x) \in H^k_p$ for some p, then there is a chart about x of type (iii). In fact, simply renumber the coordinates so that v^p becomes v^n, that is, map $\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^n$ by sending $\langle v^1, \ldots, v^n \rangle \to \langle w^1, \ldots, w^n \rangle$, where

$$\begin{aligned} w^i &= v^i && \text{for } i < p, \\ w^i &= v^{i+1} && \text{for } p \leq i < n, \\ w^n &= v^p. \end{aligned}$$

Then in a sufficiently small neighborhood U^1 of x the chart $(U^1, \varphi \circ \alpha)$ is of type (iii). (See Fig. 10.15.)

We next observe the set of $x \in \partial D$ having a neighborhood of type (iii) forms a differentiable manifold. The argument is just as before. The only difference is that this time these points do not exhaust all of ∂D. We shall denote this manifold by $\breve{\partial D}$. Thus $\breve{\partial D}$ is a manifold which, as a set, is not ∂D but only the "regular" points of ∂D, that is, those having charts of type (iii).

Theorem 7.1 (*The divergence theorem*). Let M be an n-dimensional manifold, and let $D \subset M$ be a domain with almost regular boundary. Let $\breve{\partial D}$ be as above, and let i be the injection of $\breve{\partial D} \to M$. Then for any $\rho \in \mathrm{P}$ we have

$$\int_D \operatorname{div} \langle X, \rho \rangle = \int_{\breve{\partial D}} \epsilon_X \rho_X. \tag{7.1}$$

Proof. The proof proceeds as before. We choose a connecting atlas of charts of types (i) through (iv) and a partition of unity $\{g_j\}$ subordinate to the atlas. We write $\rho = \sum g_j \rho$ and now have four cases to consider. The first three cases have already been handled.

The new case arises when ρ has its support in U, where (U, α) is a chart of type (iv). We must evaluate

$$\int_{\alpha(U \cap D)} \sum \frac{\partial X^i \rho_\alpha}{\partial x^i}.$$

The terms in the sum corresponding to $i < k$ make no contribution to the integral, as before. Let us extend $X^i\rho_\alpha$ to be defined on all of $\mathbb{R}^n$ by setting it equal to zero outside $\alpha(U)$, just as before. Then, for $k \leq i \leq n$ we have

$$\int_{\alpha(U \cap D)} \frac{\partial X^i \rho_\alpha}{\partial x^i} = \int_B \frac{\partial X^i \rho_\alpha}{\partial x^i},$$

where $B = \{\mathbf{v} : v^k \geq 0, \ldots, v^n \geq 0\}$. Writing this as an iterated integral and integrating first with respect to x^i, we obtain

$$\int_B \frac{\partial X^i \rho_\alpha}{\partial x^i} = \int_{A_i} X^i \rho_\alpha,$$

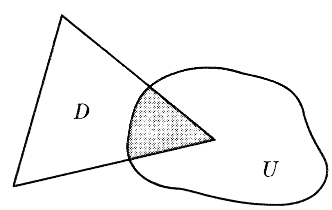

Fig. 10.16

where the set $A_i \subset \mathbb{R}^{n-1}$ is given by

$$A_i = \{\langle v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^n \rangle : v^k \geq 0, \ldots, v^n \geq 0\}.$$

Note that A_i differs from H_i^k by a set of content zero in $\mathbb{R}^{n-1}$ (namely, where at least one of the $v^l = 0$ for $k = l \leq n$). Thus we can replace the A_i by the H_i^k in the integral. Summing over $k \leq i \leq n$, we get

$$\int_{\alpha(D \cap U)} \sum \frac{\partial X^i \rho_\alpha}{\partial x^i} = \sum_{i=k}^{n} \int_{H_i^k} X^i \rho_\alpha,$$

which is exactly the assertion of Theorem 7.1 for case (iv). □

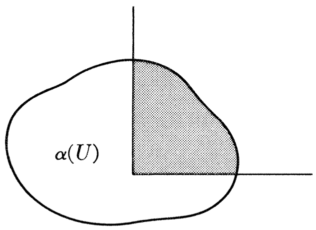

Fig. 10.17

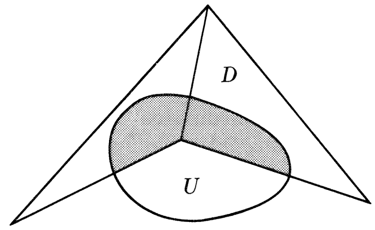

Fig. 10.18

We should point out that even Theorem 7.1 does not cover all cases for which it is useful to have a divergence theorem. For instance, in the plane, Theorem 7.1 does apply to the case where D is a triangle. (See Fig. 10.16.) This is because we can "stretch" each angle to a right angle (in fact, we can do this by a linear change of variables of $\mathbb{R}^2$). (See Fig. 10.17.)

However Theorem 7.1 does not apply to a quadrilateral such as the one in Fig. 10.18, since there is no C^1-transformation that will convert an angle greater than π into one smaller than π (since its Jacobian at the corner must carry lines into lines). Thus Theorem 7.1 doesn't apply directly. However, we can write the quadrilateral as the union of two triangles, apply Theorem 7.1 to each triangle, and note that the contributions of each triangle coming from the common boundary cancel each other out. Thus the divergence theorem does apply to our quadrilateral.

This procedure works in a quite general context. In fact, it works for all cases where we shall need the divergence theorem in this book, whether Theorem 7.1 applies directly or we can reduce to it by a finite subdivision of our domain, followed by a limiting argument. We shall not, however, formulate a general theorem covering all such cases; it is clear in each instance how to proceed.

EXERCISES

In Euclidean space we shall write div X instead of div $\prec X, \rho \succ$ when ρ is taken to be the Euclidean volume density.

7.1 Let x, y, z be rectangular coordinates on $\mathbb{E}^3$. Let the vector field X be given by

$$X = r^2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right),$$

where $r^2 = x^2 + y^2 + z^2$. Show directly that

$$\int_S (X, n)\, dA = \int_B \operatorname{div} X$$

by integrating both sides. Here B is a ball centered at the origin and S is its boundary.

7.2 Let the vector field Y be given by

$$Y = Y_r n_r + Y_\theta n_\theta + Y_\varphi n_\varphi$$

in terms of polar "coordinates" r, θ, φ on $\mathbb{E}^3$, where n_r, n_θ and n_φ are the unit vectors in the directions $\partial/\partial r$, $\partial/\partial\theta$ and $\partial/\partial\varphi$ respectively. Show that

$$\operatorname{div} Y = \frac{1}{r^2 \sin\varphi}\left\{\frac{\partial}{\partial r}(r^2 \sin\varphi\; Y_r) + \frac{\partial}{\partial\theta}(rY_\theta) + \frac{\partial}{\partial\varphi}(r\sin\varphi\; Y_\varphi)\right\}.$$

7.3 Compute the divergence of a vector field in terms of polar coodrinates in the plane.

7.4 Compute the divergence of a vector field in terms of cylindrical coordinates in $\mathbb{E}^3$.

7.5 Let σ be the volume (area) density on the unit sphere S^2. Compute $\operatorname{div} \sigma X$ in terms of the coordinates θ, φ (polar coordinates) on the sphere.