18.101 Mid-term Exam 2, 2007 **Name____________________**

(50 minutes. Please write your name at the top of each page.)

1. (30 points: yes or no answers only.) Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers on the interval, $(0, 1)$.

   a. Does the set $\{q_i \in \mathbb{R} \quad i = 1, 2, \ldots\}$ have measure zero?

   Yes, it is countable.

   b. Is this set rectifiable? (In other words, is the indicator function of this set Riemann integrable?)

   No, the boundary of this set is $[0, 1]$, which does not have measure 0.

   c. Does the set $\{\frac{1}{n} \in \mathbb{R} \quad n = 1, 2, \ldots\}$ have measure zero?

   Yes, countable.

   d. Is this set rectifiable?

   Yes, the boundary of this set is the set itself, which has measure 0.

   e. Does the set

   $$\{(q_n, \frac{1}{n}) \in \mathbb{R}^2\,, \quad n = 1, 2, \ldots\}$$

   have measure zero?

   Yes, countable.

   f. Is this set rectifiable?

   Yes, the boundary of this set is the union of the set with the set $[0, 1] \times 0$, which has measure 0.

2. (20 points) Let $v = x^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

   a. Compute

   $$L_v x = dx(x^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}) = x^2$$

   b. Is v complete? No. Note that the x coordinate of integral curves satisfy the equation $\frac{dx}{dt} = x^2$. Solutions of this are of the form $x(t) = \frac{1}{c-t}$. These solutions do not exist for all time.

   c. Let $w = y\frac{\partial}{\partial x}$. Compute

   $$L_v w = [x^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, y\frac{\partial}{\partial x}] = \frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial x}$$

   d. Let $\alpha = xdy$. Compute

   $$L_v \alpha = (L_v x)dy + x(L_v dy) = (L_v x)dy + xd(L_v y) = x^2 dy + xd(1) = x^2 dy$$

3. (30 points) Let $f = (x_1, x_2, e^{x_1 x_2})$. Use coordinates (y_1, y_2, y_3) on $\mathbb{R}^3$. Note that the image of f is a manifold $M \subset \mathbb{R}^3$.

   a. Compute

$$\begin{aligned} f^*(y_3 dy_1 + y_2 dy_3) &= e^{x_1 x_2} dx_1 + x_2 d(e^{x_1 x_2}) \\ &= e^{x_1 x_2} dx_1 + x_2(x_1 e^{x_1 x_2} dx_2 + x_2 e^{x_1 x_2} dx_1) \\ &= e^{x_1 x_2}(1 + x_2^2) dx_1 + x_1 x_2 e^{x_1 x_2} dx_2 \end{aligned}$$

   b. Consider the vector field v on $\mathbb{R}^2$ given by

$$v = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$$

   What are the integral curves of $f_* v$? Show that $f_* v$ is complete. Integral curves of $f_* v$ are equal to f composed with integral curves of v. Solutions of v are of the form $(r\cos(t-c), r\sin(t-c))$, therefore, the solutions of $f_* v$ are of the form $(r\cos(t-c), r\sin(t-c), e^{r^2 \cos(t-c)\sin(t-c)})$. These exist for all time, so $f_* v$ is complete.

   c. Let $\Phi_{tf_* v} : M \to M$ be the flow for time t of the vector field $f_* v$, and define the vector field

$$w = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$$

   Show that

$$(\Phi_{tf_* v})_*(f_* w) = f_* w$$

   In other words, show that the flow of $f_* v$ preserves $f_* w$.

   The flow of $f_* v$ preserves $f_* w$ if and only if $L_{f_* v} f_* w = 0$. We have that

$$\begin{aligned} L_{f_* v} f_* w &= [f_* v, f_* w] = f_*[v, w] \\ &= f_*[x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}, x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}] \\ &= f_*(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1}) = 0 \end{aligned}$$

4. (20 points) Prove that if $f_i : Q \longrightarrow \mathbb{R}$ is a sequence of Riemann integrable functions on a rectangle Q, which converge uniformly to $f : Q \longrightarrow \mathbb{R}$, then f is integrable, and

$$\int_Q f = \lim_{i\to\infty} \int_Q f_i$$

Given $\epsilon > 0$, Choose some N so that for any $i > N$,

$$\|f_i - f\| < \epsilon$$

Then for any given $i > N$ choose a partition P so that

$$U(P, f_i) - L(P, f_i) < \epsilon$$

Then we have

$$\begin{aligned}\left|U(P,f) - \int_Q f_i\right| &\leq |U(P,f) - U(P,f_i)| + \left|\int_Q f_i - U(P,f_i)\right| \\ &\leq \epsilon v(Q) + \epsilon\end{aligned}$$

Similarly,

$$\left|L(P,f) - \int_Q f_i\right| < \epsilon(v(Q) + 1)$$

Note that this means that we can find partitions P so that $U(P, f)$ is arbitrarily close to $L(P, f)$, so f is Riemann integrable. Once we know that f is integrable, we have

$$L(P,f) \leq \int_Q f \leq U(P,f)$$

so the above two estimates give

$$\left|\int_Q f - \int_Q f_i\right| < \epsilon(v(Q) + 1)$$

for $i > N$. We therefore have our conclusion,

$$\int_Q f = \lim_{i\to\infty} \int_Q f_i$$