Flows of Vector fields on manifolds

We have proved in class the following theorems for integral curves of vector fields on manifolds.

**Theorem 1 (Existence).** If $v$ is a $C^1$ vector field on a smooth manifold $M$, for any point $p \in M$, there exists some $\epsilon > 0$ and an integral curve of $v$

$$\gamma : (-\epsilon, \epsilon) \longrightarrow M$$

so that $\gamma(0) = p$.

**Theorem 2 (Uniqueness).** If $v$ is a $C^1$ vector field on $M$, let $\gamma_i : I_i \rightarrow M$, $i = 1, 2$, be integral curves of $v$. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$ then $\gamma_1 \equiv \gamma_2$ on $I_1 \cap I_2$ and the curve $\gamma : I_1 \cup I_2 \rightarrow M$ defined by

$$\gamma(t) = \begin{cases} 
\gamma_1(t), & t \in I_1 \\
\gamma_2(t), & t \in I_2 
\end{cases}$$

is an integral curve.

We also have the following theorem that says the there is a unique maximal integral curve passing through the point $p$ which either exists for all time, or runs off to infinity, or off the edge of $M$.

**Theorem 3.** If $v$ is a $C^1$ vector field on $M$, for any point $p$ there exists a unique maximal integral curve

$$\gamma : [0, b) \longrightarrow M \quad \gamma(0) = p$$

so that at least one of the following is true

1. either

$$b = \infty$$

2. or:

For every compact set $K \subset M$ there exists some time $T < b$ so that $\gamma(t) \notin K$ for all $t \geq T$.

**Theorem 4 (Smooth dependence on initial data).** Let $v$ be a $C^k$-vector field on a smooth manifold $M$, and $U \subset M$ an open subset. Suppose that $\Phi(u, t) : (-\epsilon, \epsilon) \times U \longrightarrow M$ has the following properties:

(i) $\Phi(p, 0) = p$. 
(ii) For all \( p \in U \) the curve
\[
\gamma_p : (-\epsilon, \epsilon) \to M \quad \gamma_p(t) = \Phi(p, t)
\]
is an integral curve of \( v \).

Then \( \Phi \) is \( C^k \) (if \( k \geq 1 \)).

**Definition 1.** A \( C^1 \) vector field \( v \) on \( M \) is complete if for all \( p \in M \) there exists an integral curve of \( v \),
\[
\gamma_p : \mathbb{R} \longrightarrow M \text{ so that } \gamma_p(0) = p
\]
In such a case, the flow of \( v \) is defined by
\[
\Phi_{tv}(p) := \gamma_p(t)
\]

For example, Theorem 3 tells us that \( v \) is complete if \( M \) is compact, or if \( v \) is compactly supported.

Note that if \( \gamma_p(t) \) is an integral curve, \( \gamma_p(t + t_0) \) is also an integral curve. This tells us the following important identity:
\[
\Phi_{t_1 v} \circ \Phi_{t_2 v} = \Phi_{(t_1 + t_2)v}
\]
Note that as \( \Phi_{0v}(x) = x \), this in particular tells us that
\[
\Phi_{t_0}^{-1} = \Phi_{-tv}
\]

If \( v \) is smooth, then Theorem 4 tells us that that each of these must be smooth, so \( \Phi_{tv} \) is a diffeomorphism.

If \( v \) is not complete, then \( \Phi_{tv}(x) \) is not defined for all \( t \) and \( x \). We can, however still make the following definition:

**Definition 2.** If \( v \) is a \( C^1 \) vector field on \( M \) define the flow \( \Phi_{tv} \) as follows: If there exists an integral curve \( \gamma_p \) so that \( \gamma_p(0) = p \), and \( \gamma_p(t) \) is defined, then
\[
\Phi_{tv}(p) = \gamma_p(t)
\]
If there does not exist such a vector field, then \( \Phi_{tv}(p) \) is not defined.

**Theorem 5.** If \( v \) is \( C^1 \), then for all \( p \in M \), there exists some open set \( U \) containing \( p \), and \( \epsilon > 0 \) so that for all \( t < \epsilon \), \( \Phi_{tv} : U \to M \) is defined, and \( p \in \Phi_{tv}(U) \).
Theorem 4 tells us $\Phi_{tv}: U \rightarrow M$ is $C^k$ if $v$ is. Note also that if this is the case, the inverse of $\Phi_{tv}$ is given by

$$\Phi_{tv}^{-1} : \Phi_{tv}(U) \rightarrow U$$

$$\Phi_{tv}^{-1} = \Phi_{-tv}$$

In the following, we shall use the following notation for the pull back of a function

$$f^*g := g \circ f$$

**Theorem 6.** If $v$ is a $C^1$ vector field, then

$$L_v f(p) = \frac{d}{dt}|_{t=0}(\Phi_{tv}^*f)(p)$$

**Proof.**

$$\frac{d}{dt}|_{t=0}(\Phi_{tv}^*f)(p) = \frac{d}{dt}|_{t=0}f(\Phi_{tv}(p)) = df(\frac{d}{dt}|_{t=0}\Phi_{tv}(p)) = df(v(p)) = L_v f(p)$$

This says that $L_v f$ measures how $f$ changes in the direction of the flow of $v$. Recall that we defined a conserved quantity of $v$ to be a function which is constant on all integral curves of $v$. The following theorem was proved in class:

**Theorem 7.** If $v$ is a $C^1$ vector field on $M$, and $f: M \rightarrow \mathbb{R}$ is a differentiable function, $f$ is a conserved quantity of $v$ if and only if $L_v f = 0$.

Now, let us define the Lie derivative of a vector field. We have defined the push forward of a vector field $w$ by

$$f_*w := Tf \circ w \circ f^{-1}$$

Define the pull back of a vector field by

$$f^*w := (f^{-1})_*w := Tf^{-1} \circ w \circ f$$

**Definition 3.** If $v$ and $w$ are two $C^1$ vector field on $M$, define the Lie derivative of $w$ with respect to $v$ as follows:

$$L_v w(p) := \frac{d}{dt}|_{t=0}(\Phi_{tv}^*w) := \frac{d}{dt}|_{t=0}(\Phi_{-tv})_*w$$
Note that theorem 5 says that $\Phi^*_{tv} w(p) \in T_p M$ will be defined for $t$ small enough, so this definition is valid even if $v$ is not complete.

Note that if $L_v w = 0$, $\Phi^*_{tv} w = w$, in other words, the flow defined by $v$ preserves $w$. For example, this means that if $\gamma$ is an integral curve of $w$, and $L_v w = 0$, then $\Phi_{tv} \circ \gamma$ is also an integral curve.

You should think of $L_{-v} w$ as measuring how you see $w$ change as you flow it by $v$. $L_v w$ measures how you see $w$ change if you flow yourself by $v$. Unless you happen to be particularly diffuse and flexible, this is somewhat less intuitive than the interpretation of $L_{-v} w$, however it has the advantage of coinciding with the nice formula $L_v f = df(v)$.

**Theorem 8.** If $v$ and $w$ are $C^1$ vector fields,

$$L_v w = [v, w]$$

**Proof.** Recall that the Lie bracket $[v, w]$ is the unique vector field so that

$$L_{[v, w]} f = L_v L_w f - L_w L_v f$$

for all smooth $f$.

Let us compute

$$L_{L_v w} f = \frac{d}{dt} \bigg|_{t=0} df(\Phi^*_{tv} w)$$

$$= \frac{d}{dt} \bigg|_{t=0} df \circ T \Phi^{-1}_{tv} \circ w \circ \Phi_{tv}$$

$$= \frac{d}{dt} \bigg|_{t=0} df \circ T \Phi^{-1}_{tv} \circ w + \frac{d}{dt} \bigg|_{t=0} df \circ w \circ \Phi_{tv}$$

$$= \frac{d}{dt} \bigg|_{t=0} d(\Phi^*_{-tv} f)(w) + \frac{d}{dt} \bigg|_{t=0} df(w) \circ \Phi_{tv}$$

$$= -L_w L_v f + L_v L_w f$$

$$= [v, w] f$$

Note that this says that if the flow of $v$ preserves $w$, the flow of $w$ preserves $v$.

Strictly speaking, to know that if the $L_v w$ is 0, the flow of $v$ preserves $w$, we need the following theorem:

**Theorem 9.** If $v$ and $w$ are $C^1$ vector fields on $M$, and $v$ is complete, then $\Phi^*_{tv} w := (\Phi_{-tv})^* w$ is the unique time dependent vector field $w_t$ so that:

1. $$\frac{\partial}{\partial t} w_t = L_v w_t$$
2.

\[ w_0 = w \]

**Proof.** First, we show that \( \Phi_{tv}^* \) satisfies the above partial differential equation:

\[
\frac{d}{dt} \Phi_{tv}^* w = \frac{d}{ds}|_{s=0} \Phi_{(s+t)v}^* w = \frac{d}{ds}|_{s=0} \Phi_{sv}^* (\Phi_{tv}^* w) = L_v (\Phi_{tv}^* w)
\]

As it also obeys the initial condition \( \Phi_{0v}^* w = w \), it obeys both the above conditions. We must show that this uniquely determines it.

Consider the above equation in local coordinates,

\[
w_t := W_1(x, t) \frac{\partial}{\partial x_1} + \cdots + W_n(x, t) \frac{\partial}{\partial x_n}
\]

\[
v = \sum v_i \frac{\partial}{\partial x_i}
\]

We have

\[
\frac{\partial W_j}{\partial t} = L_v W_j - \sum_i W_i \frac{\partial v_j}{\partial x_i}
\]

We shall solve this partial differential equation by the method of characteristics. Define the vector field

\[
\bar{v} := \frac{\partial}{\partial t} - \sum v_i \frac{\partial}{\partial x_i}
\]

We can now rewrite our system of equations as

\[
L_{\bar{v}} W_j = -\sum_i W_i \frac{\partial v_j}{\partial x_i}
\]

This gives an equation for how \( W(x, t) \) changes along integral curves of \( \bar{v} \). Note that the flow defined by \( \bar{v} \) is give by

\[
\Phi_{\bar{v}}(x, 0) = (t, \Phi_{-tv} x)
\]

As \( v \) is \( C^1 \), the flow it defines is \( C^1 \), and therefore the flow defined by \( \bar{v} \) is also \( C^1 \). We therefore have that along any integral curve, this gives a system of ordinary differential equations for \( W \) which is continuous, and, because it is linear at any particular time, obeys a Lipshitz condition on any finite time interval. Our uniqueness theorem for Ordinary differential equations then tells us that if two solutions agree anywhere along an integral curve of \( \bar{v} \), they agree along the entire integral curve. But any integral curve of \( \bar{v} \) passes through the slice where \( t = 0 \), where \( w_0 = w \), so the solution to the above differential equation is unique.

\[ \square \]
We can also define the Lie derivative of a one form $\alpha$. Recall that the pull back is defined by

$$f^*\alpha := \alpha \circ T f$$

**Definition 4.** If $v$ is a $C^1$ vector field and $\alpha$ is a $C^1$ one form on $M$, the Lie derivative of $\alpha$ with respect to $v$ is the one form $L_v\alpha$ defined by

$$L_v\alpha := \frac{d}{dt}\bigg|_{t=0}\Phi^*_t \alpha$$

1. Show that

$$L_v(\alpha(w)) = L_v(\alpha)(w) + \alpha(L_vw)$$

2. Show that

$$L_v df = dL_v f$$

3. Show that if $f$ is a function and $\alpha$ a one form

$$L_v(f\alpha) = (L_v f)\alpha + fL_v\alpha$$

Show that the same formula holds if $w$ is a vector field

$$L_v(fw) = (L_v f)w + fL_vw$$

4. find

$$L_{x_1} \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} (x_1 dx_1 + x_2 dx_2)$$

$$L_{x_1 x_2 x_3} \frac{\partial}{\partial x_1} (dx_1 + x_2 dx_3)$$

5. Suppose that $v$ and $w$ are complete $C^1$ vector fields on $M$ Show that $L_vw = 0$ if and only if

$$\Phi^*_{t v} \circ \Phi^*_{s w} = \Phi^*_{s w} \circ \Phi^*_{t v}$$

In such a case, we say that the flows commute.

6. Suppose that $v$ and $w$ are $C^1$ vector fields on $\mathbb{R}^N$ which are tangent to $M \subset \mathbb{R}^N$. Prove that $L_v w$ is also tangent to $M$. Show also that if $v_1$ and $w_1$ indicates the restriction of $v$ and $w$ to $M$, the restriction of $L_v w$ is $L_{v_1} w_1$.

7. (a) Let $U = \mathbb{R}^2$ and let $v$ be the vector field, $x_1 \partial / \partial x_2 - x_2 \partial / \partial x_1$. Show that the curve

$$t \in \mathbb{R} \rightarrow (r \cos(t + \theta), r \sin(t + \theta))$$

is the unique integral curve of $v$ passing through the point, $(r \cos \theta, r \sin \theta)$, at $t = 0$. 

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(b) Let $U = \mathbb{R}^n$ and let $v$ be the constant vector field: $\sum c_i \partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to a + t(c_1, \ldots, c_n)$$

is the unique integral curve of $v$ passing through $a \in \mathbb{R}^n$ at $t = 0$.

(c) Let $U = \mathbb{R}^n$ and let $v$ be the vector field, $\sum x_i \partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to e^t(a_1, \ldots, a_n)$$

is the unique integral curve of $v$ passing through $a$ at $t = 0$.

8. Let $U$ be an open subset of $\mathbb{R}^n$ and $F : U \times \mathbb{R} \to U$ a $C^\infty$ mapping. The family of mappings

$$f_t : U \to U, \quad f_t(x) = F(x, t)$$

is said to be a one-parameter group of diffeomorphisms of $U$ if $f_0$ is the identity map and $f_s \circ f_t = f_{s+t}$ for all $s$ and $t$. (Note that $f_{-t} = f_t^{-1}$, so each of the $f_t$'s is a diffeomorphism.) Show that the following are one-parameter groups of diffeomorphisms:

(a) $f_t : \mathbb{R} \to \mathbb{R}, \quad f_t(x) = x + t$

(b) $f_t : \mathbb{R} \to \mathbb{R}, \quad f_t(x) = e^t x$

(c) $f_t : \mathbb{R}^2 \to \mathbb{R}^2, \quad f_t(x, y) = (\cos t x - \sin t y, \sin t x + \cos t y)$

9. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. Show that the series

$$\exp tA = I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \cdots$$

converges and defines a one-parameter group of diffeomorphisms of $\mathbb{R}^n$.

10. (a) What are the infinitesimal generators of the one-parameter groups in exercise 8? In other words, what is the vector field $v$ so that the flow of $v$ gives the above group of diffeomorphisms.

(b) Show that the infinitesimal generator of the one-parameter group in exercise 9 is the vector field

$$\sum a_{i,j} x_j \frac{\partial}{\partial x_i}$$

where $[a_{i,j}]$ is the defining matrix of $A$.

11. Let $X \subseteq \mathbb{R}^3$ be the paraboloid, $x_3 = x_1^2 + x_2^2$ and let $w$ be the vector field

$$w = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3}.$$
(a) Show that $w$ is tangent to $X$ and hence defines by restriction a vector field, $v$, on $X$.

(b) What are the integral curves of $v$?

12. Let $S^2$ be the unit 2-sphere, $x_1^2 + x_2^2 + x_3^2 = 1$, in $\mathbb{R}^3$ Consider the following vector fields:

$$w_1 = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$$

$$w_2 = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}$$

$$w_3 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$$

(a) Show that $w_i$ is tangent to $S^2$, and hence by restriction defines a vector field, $v_i$, on $S^2$.

(b) What do its integral curves look like?

(c) Find $L_{v_i}v_j$ for all $i, j$.

(d) Give a formula for $(\Phi_{tv_i})_*v_2$.

13. Let $S^2$ be the unit 2-sphere in $\mathbb{R}^3$ and let $w$ be the vector field

$$w = \frac{\partial}{\partial x_3} - x_3 \left( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right)$$

(a) Show that $w$ is tangent to $S^2$ and hence by restriction defines a vector field, $v$, on $S^2$.

(b) What do its integral curves look like?

14. Let $S^1$ be the unit sphere, $x_1^2 + x_2^2 = 1$, in $\mathbb{R}^2$ and let $X = S^1 \times S^1$ in $\mathbb{R}^4$ with defining equations

$$f_1 = x_1^2 + x_2^2 - 1 = 0$$

$$f_2 = x_3^2 + x_4^2 - 1 = 0$$

(a) Show that the vector field

$$w = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \lambda \left( x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \right),$$

$\lambda \in \mathbb{R}$, is tangent to $X$ and hence defines by restriction a vector field, $v$, on $X$. 

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(b) What are the integral curves of \( v \)?

(c) Show that \( L_w f_i = 0 \).

15. For the vector field, \( v \), in problem 14a, describe the one-parameter group of diffeomorphisms it generates.

16. Let \( X \) and \( v \) be as in problem 11 and let \( f : \mathbb{R}^2 \to X \) be the map, \( f(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^3) \). Show that if \( u \) is the vector field,

\[
 u = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2},
\]

then \( f_\ast u = v \).

17.* An elementary result in number theory asserts

**Theorem.** A number, \( \lambda \in \mathbb{R} \), is irrational if and only if the set

\[
\{m + \lambda n, \ m \text{ and } n \text{ integers}\}
\]

is a dense subset of \( \mathbb{R} \).

Let \( v \) be the vector field in problem 14a. Using the theorem above prove that if \( \lambda/2\pi \) is irrational then for every integral curve, \( \gamma(t), -\infty < t < \infty \), of \( v \) the set of points on this curve is a dense subset of \( X \).