Flows of Vector fields on manifolds

We have proved in class the following theorems for integral curves of vector fields on manifolds.

Theorem 1 (Existence). *If v is a C^1 vector field on a smooth manifold M, for any point $p \in M$, there exists some $\epsilon > 0$ and an integral curve of v*

$$\gamma : (-\epsilon, \epsilon) \longrightarrow M$$

so that $\gamma(0) = p$.

Theorem 2 (Uniqueness). *If v is a C^1 vector field on M, let $\gamma_i : I_i \to M$, $i = 1, 2$, be integral curves of v. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$ then $\gamma_1 \equiv \gamma_2$ on $I_1 \cap I_2$ and the curve $\gamma : I_1 \cup I_2 \to M$ defined by*

$$\gamma(t) = \begin{cases} \gamma_1(t)\,, & t \in I_1 \\ \gamma_2(t)\,, & t \in I_2 \end{cases}$$

is an integral curve.

We also have the following theorem that says the there is a unique maximal integral curve passing through the point p which either exists for all time, or runs off to infinity, or off the edge of M.

Theorem 3. *If v is a C^1 vector field on M, for any point p there exists a unique maximal integral curve*

$$\gamma : [0, b) \longrightarrow M \qquad \gamma(0) = p$$

so that at least one of the following is true

1. *either*

$$b = \infty$$

2. *or:*

   For every compact set $K \subset M$ there exists some time $T < b$ so that $\gamma(t) \notin K$ for all $t \geq T$.

Theorem 4 (Smooth dependence on initial data). *Let v be a C^k-vector field on a smooth manifold M, and $U \subset M$ an open subset. Suppose that $\Phi(u, t) : (-\epsilon, \epsilon) \times U \longrightarrow M$ has the following properties:*

(i) $\Phi(p, 0) = p$.

(ii) For all $p \in U$ the curve

$$\gamma_p : (-\epsilon, \epsilon) \to M \qquad \gamma_p(t) = \Phi(p, t)$$

is an integral curve of v.

Then Φ is C^k (If $k \geq 1$).

Definition 1. *A C^1 vector field v on M is complete if for all $p \in M$ there exists an integral curve of v,*

$$\gamma_p : \mathbb{R} \longrightarrow M \text{ so that } \gamma_p(0) = p$$

In such a case, the flow of v is defined by

$$\Phi_{tv}(p) := \gamma_p(t)$$

For example, Theorem 3 tells us that v is complete if M is compact, or if v is compactly supported.

Note that if $\gamma_p(t)$ is an integral curve, $\gamma_p(t + t_0)$ is also an integral curve. This tells us the following important identity:

$$\Phi_{t_1 v} \circ \Phi_{t_2 v} = \Phi_{(t_1+t_2)v}$$

Note that as $\Phi_{0v}(x) = x$, this in particular tells us that

$$\Phi_{tv}^{-1} = \Phi_{-tv}$$

If v is smooth, then Theorem 4 tells us that that each of these must be smooth, so Φ_{tv} is a diffeomorphism.

If v is not complete, then $\Phi_{tv}(x)$ is not defined for all t and x. We can, however still make the following definition:

Definition 2. *If v is a C^1 vector field on M define the flow Φ_{tv} as follows: If there exists an integral curve γ_p so that $\gamma_p(0) = p$, and $\gamma_p(t)$ is defined, then*

$$\Phi_{tv}(p) = \gamma_p(t)$$

If there does not exist such a vector field, then $\Phi_{tv}(p)$ is not defined.

Theorem 5. *If v is C^1, then for all $p \in M$, there exists some open set U containing p, and $\epsilon > 0$ so that for all $t < \epsilon$, $\Phi_{tv} : U \to M$ is defined, and $p \in \Phi_{tv}(U)$.*

Theorem 4 tells us $\Phi_{tv} : U \rightarrow M$ is C^k if v is. Note also that if this is the case, the inverse of Φ_{tv} is given by

$$\Phi_{tv}^{-1} : \Phi_{tv}(U) \longrightarrow U$$

$$\Phi_{tv}^{-1} = \Phi_{-tv}$$

In the following, we shall use the following notation for the pull back of a function

$$f^* g := g \circ f$$

Theorem 6. *If v is a C^1 vector field, then*

$$L_v f(p) = \frac{d}{dt}|_{t=0}(\Phi_{tv}^* f)(p)$$

Proof.

$$\frac{d}{dt}|_{t=0}(\Phi_{tv}^* f)(p) = \frac{d}{dt}|_{t=0} f(\Phi_{tv}(p)) = df(\frac{d}{dt}|_{t=0}\Phi_{tv}(p)) = df(v(p)) = L_v f(p)$$

□

This says that $L_v f$ measures how f changes in the direction of the flow of v. Recall that we defined a conserved quantity of v to be a function which is constant on all integral curves of v. The following theorem was proved in class:

Theorem 7. *If v is a C^1 vector field on M, and $f : M \longrightarrow \mathbb{R}$ is a differentiable function, f is a conserved quantity of v if and only if $L_v f = 0$.*

Now, let us define the Lie derivative of a vector field. We have defined the push forward of a vector field w by

$$f_* w := Tf \circ w \circ f^{-1}$$

Define the pull back of a vector field by

$$f^* w := (f^{-1})_* w := Tf^{-1} \circ w \circ f$$

Definition 3. *If v and w are two C^1 vector field on M, define the Lie derivative of w with respect to v as follows:*

$$L_v w(p) := \frac{d}{dt}|_{t=0}\Phi_{tv}^* w := \frac{d}{dt}|_{t=0}(\Phi_{-tv})_* w$$

Note that theorem 5 says that $\Phi^*_{tv}w(p) \in T_pM$ will be defined for t small enough, so this definition is valid even if v is not complete.

Note that if $L_v w = 0$, $\Phi^*_{tv}w = w$, in other words, the flow defined by v preserves w. For example, this means that if γ is an integral curve of w, and $L_v w = 0$, then $\Phi_{tv} \circ \gamma$ is also an integral curve.

You should think of $L_{-v}w$ as measuring how you see w change as you flow it by v. $L_v w$ measures how you see w change if you flow yourself by v. Unless you happen to be particularly diffuse and flexible, this is somewhat less intuitive than the interpretation of $L_{-v}w$, however it has the advantage of coinciding with the nice formula $L_v f = df(v)$.

Theorem 8. *If v and w are C^1 vector fields,*

$$L_v w = [v, w]$$

Proof. Recall that the Lie bracket $[v, w]$ is the unique vector field so that

$$L_{[v,w]}f = L_v L_w f - L_w L_v f \text{ for all smooth } f$$

Let us compute

$$\begin{aligned}
L_{L_v w} f &= \frac{d}{dt}|_{t=0} df(\Phi^*_{tv}w) \\
&= \frac{d}{dt}|_{t=0} df \circ T\Phi^{-1}_{tv} \circ w \circ \Phi_{tv} \\
&= \frac{d}{dt}|_{t=0} df \circ T\Phi^{-1}_{tv} \circ w + \frac{d}{dt}|_{t=0} df \circ w \circ \Phi_{tv} \\
&= \frac{d}{dt}|_{t=0} d(\Phi^*_{-tv}f)(w) + \frac{d}{dt}|_{t=0} df(w) \circ \Phi_{tv} \\
&= -L_w L_v f + L_v L_w f \\
&= [v, w]f
\end{aligned}$$

□

Note that this says that if the flow of v preserves w, the flow of w preserves v.

Strictly speaking, to know that if the $L_v w$ is 0, the flow of v preserves w, we need the following theorem:

Theorem 9. *If v and w are C^1 vector fields on M, and v is complete, then $\Phi^*_{tv}w := (\Phi_{-tv})_* w$ is the unique time dependent vector field w_t so that:*

1.
$$\frac{\partial}{\partial t} w_t = L_v w_t$$

2.

$$w_0 = w$$

Proof. First, we show that Φ^*_{tv} satisfies the above partial differential equation:

$$\frac{d}{dt}\Phi^*_{tv}w = \frac{d}{ds}|_{s=0}\Phi^*_{(s+t)v}w = \frac{d}{ds}|_{s=0}\Phi^*_{sv}(\Phi^*_{tv}w) = L_v(\Phi_{tv}w)$$

As it also obeys the initial condition $\Phi^*_{0v}w = w$, it obeys both the above conditions. We must show that this uniquely determines it.

Consider the above equation in local coordinates,

$$w_t := W_1(x,t)\frac{\partial}{\partial x_1} + \cdots + W_n(x,t)\frac{\partial}{\partial x_n}$$

$$v = \sum v_i \frac{\partial}{\partial x_i}$$

We have

$$\frac{\partial W_j}{\partial t} = L_v W_j - \sum_i W_i \frac{\partial v_j}{\partial x_i}$$

We shall solve this partial differential equation by the method of characteristics. Define the vector field

$$\tilde{v} := \frac{\partial}{\partial t} - \sum_i v_i \frac{\partial}{\partial x_i}$$

We can now rewrite our system of equations as

$$L_{\tilde{v}} W_j = -\sum_i W_i \frac{\partial v_j}{\partial x_i}$$

This gives an equation for how $W(x,t)$ changes along integral curves of $\tilde{v}$. Note that the flow defined by $\tilde{v}$ is give by

$$\Phi_{t\tilde{v}}(x,0) = (t, \Phi_{-tv}x)$$

As v is C^1, the flow it defines is C^1, and therefore the flow defined by $\tilde{v}$ is also C^1. We therefore have that along any integral curve, this gives a system of ordinary differential equations for W which is continuous, and, because it is linear at any particular time, obeys a Lipshitz condition on any finite time interval. Our uniqueness theorem for Ordinary differential equations then tells us that if two solutions agree anywhere along an integral curve of $\tilde{v}$, they agree along the entire integral curve. But any integral curve of $\tilde{v}$ passes through the slice where $t = 0$, where $w_0 = w$, so the solution to the above differential equation is unique.

□

We can also define the Lie derivative of a one form α. Recall that the pull back is defined by

$$f^*\alpha := \alpha \circ Tf$$

Definition 4. *If v is a C^1 vector field and α is a C^1 one form on M, the Lie derivative of α with respect to v is the one form $L_v\alpha$ defined by*

$$L_v\alpha := \frac{d}{dt}|_{t=0}\Phi_{tv}^*\alpha$$

1. Show that

$$L_v(\alpha(w)) = L_v(\alpha)(w) + \alpha(L_v w)$$

2. Show that

$$L_v df = dL_v f$$

3. Show that if f is a function and α a one form

$$L_v(f\alpha) = (L_v f)\alpha + fL_v\alpha$$

   Show that the same formula holds if w is a vector field

$$L_v(fw) = (L_v f)w + fL_v w$$

4. find

$$L_{x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}}(x_1 dx_1 + x_2 dx_2)$$

$$L_{x_1x_2x_3\frac{\partial}{\partial x_1}}(dx_1 + x_2 dx_3)$$

5. Suppose that v and w are complete C^1 vector fields on M Show that $L_v w = 0$ if and only if

$$\Phi_{tv} \circ \Phi_{sw} = \Phi_{sw} \circ \Phi_{tv}$$

   In such a case, we say that the flows commute.

6. Suppose that v and w are C^1 vector fields on $\mathbb{R}^N$ which are tangent to $M \subset \mathbb{R}^N$. Prove that $L_v w$ is also tangent to M. Show also that if v_1 and w_1 indicates the restriction of v and w to M, the restriction of $L_v w$ is $L_{v_1} w_1$.

7. (a) Let $U = \mathbb{R}^2$ and let v be the vector field, $x_1\partial/\partial x_2 - x_2\partial/\partial x_1$. Show that the curve

$$t \in \mathbb{R} \to (r\cos(t+\theta)\,,\, r\sin(t+\theta))$$

   is the unique integral curve of v passing through the point, $(r\cos\theta, r\sin\theta)$, at $t = 0$.

(b) Let $U = \mathbb{R}^n$ and let v be the constant vector field: $\sum c_i \partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to a + t(c_1, \ldots, c_n)$$

is the unique integral curve of v passing through $a \in \mathbb{R}^n$ at $t = 0$.

(c) Let $U = \mathbb{R}^n$ and let v be the vector field, $\sum x_i \partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to e^t(a_1, \ldots, a_n)$$

is the unique integral curve of v passing through a at $t = 0$.

8. Let U be an open subset of $\mathbb{R}^n$ and $F : U \times \mathbb{R} \to U$ a $\mathcal{C}^\infty$ mapping. The family of mappings

$$f_t : U \to U\,, \quad f_t(x) = F(x,t)$$

is said to be a *one-parameter group of diffeomorphisms* of U if f_0 is the identity map and $f_s \circ f_t = f_{s+t}$ for all s and t. (Note that $f_{-t} = f_t^{-1}$, so each of the f_t's is a diffeomorphism.) Show that the following are one-parameter groups of diffeomorphisms:

(a) $f_t : \mathbb{R} \to \mathbb{R}\,, \quad f_t(x) = x + t$

(b) $f_t : \mathbb{R} \to \mathbb{R}\,, \quad f_t(x) = e^t x$

(c) $f_t : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad f_t(x,y) = (\cos t\, x - \sin t\, y\,, \sin t\, x + \cos t\, y)$

9. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. Show that the series

$$\exp tA = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

converges and defines a one-parameter group of diffeomorphisms of $\mathbb{R}^n$.

10. (a) What are the infinitesimal generators of the one-parameter groups in exercise 8? In other words, what is the vector field v so that the flow of v gives the above group of diffeomorphisms.

(b) Show that the infinitesimal generator of the one-parameter group in exercise 9 is the vector field

$$\sum a_{i,j} x_j \frac{\partial}{\partial x_i}$$

where $[a_{i,j}]$ is the defining matrix of A.

11. Let $X \subseteq \mathbb{R}^3$ be the paraboloid, $x_3 = x_1^2 + x_2^2$ and let w be the vector field

$$w = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3}\,.$$

(a) Show that w is tangent to X and hence defines by restriction a vector field, v, on X.

(b) What are the integral curves of v?

12. Let S^2 be the unit 2-sphere, $x_1^2 + x_2^2 + x_3^2 = 1$, in $\mathbb{R}^3$ Consider the following vector fields:

$$w_1 = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$$

$$w_2 = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}$$

$$w_3 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$$

(a) Show that w_i is tangent to S^2, and hence by restriction defines a vector field, v_i, on S^2.

(b) What are the integral curves of v_i?

(c) Find $L_{v_i} v_j$ for all i, j.

(d) Give a formula for $(\Phi_{tv_1})_* v_2$.

13. Let S^2 be the unit 2-sphere in $\mathbb{R}^3$ and let w be the vector field

$$w = \frac{\partial}{\partial x_3} - x_3 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right)$$

(a) Show that w is tangent to S^2 and hence by restriction defines a vector field, v, on S^2.

(b) What do its integral curves look like?

14. Let S^1 be the unit sphere, $x_1^2 + x_2^2 = 1$, in $\mathbb{R}^2$ and let $X = S^1 \times S^1$ in $\mathbb{R}^4$ with defining equations

$$f_1 = x_1^2 + x_2^2 - 1 = 0$$

$$f_2 = x_3^2 + x_4^2 - 1 = 0\,.$$

(a) Show that the vector field

$$w = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \lambda \left(x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \right),$$

$\lambda \in \mathbb{R}$, is tangent to X and hence defines by restriction a vector field, v, on X.

(b) What are the integral curves of v?

(c) Show that $L_w f_i = 0$.

15. For the vector field, v, in problem 14a, describe the one-parameter group of diffeomorphisms it generates.

16. Let X and v be as in problem 11 and let $f : \mathbb{R}^2 \to X$ be the map, $f(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$. Show that if u is the vector field,

$$u = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2},$$

then $f_* u = v$.

17.* An elementary result in number theory asserts

Theorem. *A number, $\lambda \in \mathbb{R}$, is irrational if and only if the set*

$$\{m + \lambda n, \quad m \text{ and } n \text{ intgers}\}$$

is a dense *subset of* $\mathbb{R}$.

Let v be the vector field in problem 14a. Using the theorem above prove that if $\lambda/2\pi$ is irrational then for every integral curve, $\gamma(t)$, $-\infty < t < \infty$, of v the set of points on this curve is a dense subset of X.