The theory of manifolds   Lecture 2

Let X be a subset of $\mathbb{R}^N$, Y a subset of $\mathbb{R}^n$ and $f : X \to Y$ a continuous map. We recall

Definition 1. *f is a $\mathcal{C}^\infty$ map if for every $p \in X$, there exists a neighborhood, U_p, of p in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $g_p : U_p \to \mathbb{R}^n$, which coincides with f on $U_p \cap X$.*

We will say that f is a *diffeomorphism* if it is one–one and onto and f and f^{-1} are both $\mathcal{C}^\infty$. In particular if Y is an open subset of $\mathbb{R}^n$, X is a simple example of what we will call a *manifold*. More generally,

Definition 2. *A subset, X, of $\mathbb{R}^N$ is an n-dimensional manifold if, for every $p \in X$, there exists a neighborhood, V, of p in $\mathbb{R}^m$, an open subset, U, in $\mathbb{R}^n$, and a diffeomorphism $\varphi : U \to X \cap V$.*

Thus X is an n-dimensional manifold if, locally near every point p, X "looks like" an open subset of $\mathbb{R}^n$.

Often, we are only interested in the manifold X itself, and not in how it is sitting inside $\mathbb{R}^N$. Properties of X that don't depend on this embedding are called intrinsic properties of X. The following is the notion of when two manifolds are the same for this purpose.

Definition 3. *Two manifolds $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ are diffeomorphic if there exists a diffeomorphism $f : X \longrightarrow Y$*

Note that the chain rule tells us that the above is an equivalence relation.

We'll now describe how manifolds come up in concrete applications. Let U be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a $\mathcal{C}^\infty$ map.

Definition 4. *A point, $a \in \mathbb{R}^k$, is a regular value of f if for every point, $p \in f^{-1}(a)$, f is a submersion at p.*

Note that for f to be a submersion at p, $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ has to be onto, and hence k has to be less than or equal to N. Therefore this notion of "regular value" is interesting only if $N \geq k$.

Theorem 1. *Let $N - k = n$. If a is a regular value of f, the set, $X = f^{-1}(a)$, is an n-dimensional manifold.*

Proof. Replacing f by $\tau_{-a} \circ f$ we can assume without loss of generality that $a = 0$. Let $p \in f^{-1}(0)$. Since f is a submersion at p, the canonical submersion theorem tells us that there exists a neighborhood, $\mathcal{O}$, of 0 in $\mathbb{R}^N$, a neighborhood, U_0, of p in U and a diffeomorphism, $g : \mathcal{O} \to U_0$ such that

$$f \circ g = \pi \tag{1}$$

where π is the projection map

$$\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k\,, \quad (x,y) \to x\,.$$

Hence $\pi^{-1}(0) = \{0\} \times \mathbb{R}^n = \mathbb{R}^n$ and by (1), g maps $\mathcal{O} \cap \pi^{-1}(0)$ diffeomorphically onto $U_0 \cap f^{-1}(0)$. However, $\mathcal{O} \cap \pi^{-1}(0)$ is a neighborhood, V, of 0 in $\mathbb{R}^n$ and $U_0 \cap f^{-1}(0)$ is a neighborhood of p in X, and, as remarked, these two neighborhoods are diffeomorphic.

□

Some examples:

1. *The n-sphere.* Let

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

be the map,

$$(x_1, \ldots, x_{n+1}) \to x_1^2 + \cdots + x_{n+1}^2 - 1\,.$$

Then

$$Df(x) = 2(x_1, \ldots, x_{n+1})$$

so, if $x \neq 0$ f is a submersion at x. In particular f is a submersion at all points, x, on the n-sphere

$$S^n = f^{-1}(0)$$

so the n-sphere is an n-dimensional submanifold of $\mathbb{R}^{n+1}$.

2. *Graphs.* Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and let

$$X = \text{ graph } g = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k\,, \quad y = g(x)\}\,.$$

We claim that X is an n-dimensional submanifold of $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$.

Proof. Let

$$f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$$

be the map, $f(x,y) = y - g(x)$. Then

$$Df(x,y) = [-Dg(x)\,,\, I_k]$$

where I_k is the identity map of $\mathbb{R}^k$ onto itself. This map is always of rank k. Hence graph $g = f^{-1}(0)$ is an n-dimensional submanifold of R^{n+k}.

□

3. Munkres, §24, #6. Let $\mathcal{M}_n$ be the set of all $n \times n$ matrices and let $\mathcal{S}_n$ be the set of all symmetric $n \times n$ matrices, i.e., the set

$$\mathcal{S}_n = \{A \in \mathcal{M}_n \, , \, A = A^t\} \, .$$

The map

$$[a_{i,j}] \rightarrow (a_{11}, a_{12}, \dots, a_{1n}, a_{2,1}, \dots, a_{2n}, \dots)$$

gives us an identification

$$\mathcal{M}_n \cong \mathbb{R}^{n^2}$$

and the map

$$[a_{i,j}] \rightarrow (a_{11}, \dots a_{1n}, a_{22}, \dots a_{2n}, a_{33}, \dots a_{3n}, \dots)$$

gives us an identification

$$\mathcal{S}_n \cong \mathbb{R}^{\frac{n(n+1)}{2}} \, .$$

(Note that if A is a symmetric matrix,

$$a_{12} = a_{21} \, , \, a_{13} = a_{13} = a_{31} \, , \, a_{32} = a_{23}, \text{ etc.}$$

so this map avoids redundancies.) Let

$$O(n) = \{A \in \mathcal{M}_n \, , \, A^t A = I\} \, .$$

This is the set of *orthogonal* $n \times n$ matrices, and the exercise in Munkres requires you to show that it's an $n(n-1)/2$-dimensional manifold.

Hint: Let $f : \mathcal{M}_n \rightarrow \mathcal{S}_n$ be the map $f(A) = A^t A - I$. Then

$$O(n) = f^{-1}(0) \, .$$

These examples show that lots of interesting manifolds arise as zero sets of submersions, $f : U \rightarrow \mathbb{R}^k$. We'll conclude this lecture by showing that locally *every* manifold arises this way. More explicitly let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold, p a point of X, U a neighborhood of 0 in $\mathbb{R}^n$, V a neighborhood of p in $\mathbb{R}^N$ and $\varphi : (U, 0) \rightarrow (V \cap X, p)$ a diffeomorphism. We will for the moment think of φ as a $\mathcal{C}^\infty$ map $\varphi : U \rightarrow \mathbb{R}^N$ whose image happens to lie in X.

Lemma 2. *The linear map*

$$D\varphi(0) : \mathbb{R}^n \rightarrow \mathbb{R}^N$$

is injective.

Proof. $\varphi^{-1} : V \cap X \to U$ is a diffeomorphism, so, shrinking V if necessary, we can assume that there exists a $\mathcal{C}^\infty$ map $\psi : V \to U$ which coincides with φ^{-1}on $V \cap X$ Since φ maps U onto $V \cap X$, $\psi \circ \varphi = \varphi^{-1} \circ \varphi$ is the identity map on U. Therefore,

$$D(\psi \circ \varphi)(0) = (D\psi)(p)D\varphi(0) = I$$

by the chain rule, and hence if $D\varphi(0)v = 0$, it follows from this identity that $v = 0$. □

Lemma 6 says that φ is an immersion at 0, so by the canonical immersion theorem there exists a neighborhood, U_0, of 0 in U a neighborhood, V_p, of p in V, a neighborhood, $\mathcal{O}$, of 0 in $\mathbb{R}^N$ and a diffeomorphism

$$g : (V_p, p) \to (\mathcal{O}, 0)$$

such that

$$\iota^{-1}(\mathcal{O}) \;=\; U_0 \tag{2}$$

and

$$g \circ \varphi \;=\; \iota\,, \tag{3}$$

ι being, as in lecture 1, the canonical immersion

$$\iota(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots 0)\,. \tag{4}$$

By (3) g maps $\varphi(U_0)$ diffeomorphically onto $\varphi(U_0)$. However, by (2) and (3) $\iota(U_0)$ is the subset of $\mathcal{O}$ defined by the equations, $x_i = 0$, $i = n+1, \ldots, N$. Hence if $g = (g_1, \ldots, g_N)$ the set, $\varphi(U_0) = V_p \cap X$ is defined by the equations

$$g_i = 0\,, \quad i = n+1, \ldots, N\,. \tag{5}$$

Let $\ell = N - n$, let

$$\pi : \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^\ell$$

be the canonical submersion,

$$\pi(x_1, \ldots, x_N) = (x_{n+1}, \ldots x_N)$$

and let $f = \pi \circ g$. Since g is a diffeomorphism, f is a submersion and (5) can be interpreted as saying that

$$V_p \cap X = f^{-1}(0)\,. \tag{6}$$

A nice way of thinking about Theorem 2 is in terms of the coordinates of the mapping, f. More specifically if $f = (f_1, \ldots, f_k)$ we can think of $f^{-1}(a)$ as being the set of solutions of the system of equations

$$f_i(x) = a_i\,, \quad i = 1, \ldots, k \tag{7}$$

and the condition that a be a regular value of f can be interpreted as saying that for every solution, p, of this system of equations the vectors

$$(df_i)_p = \sum \frac{\partial f_i}{\partial x_j}(0)\, dx_j \tag{8}$$

in $T_p^*\mathbb{R}^n$ are linearaly independent, i.e., the system (7) is an "independent system of defining equations" for X.

Problem set

1. Show that the set of solutions of the system of equations

$$x_1^2 + \cdots + x_n^2 = 1$$

and

$$x_1 + \cdots + x_n = 0$$

is an $n-2$-dimensional submanifold of $\mathbb{R}^n$.

2. Let S^{n-1} be the $(n-1)$-sphere in $\mathbb{R}^n$ and let

$$X_q = \{x \in S^{n-1}\,, \quad x_1 + \cdots + x_n = q\}\,.$$

For what values of q is X_q an $(n-2)$-dimensional submanifold of S^{n-1}?

3. Show that if X_i, $i = 1, 2$, is an n_i-dimensional submanifold of $\mathbb{R}^{N_i}$ then

$$X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

is an $(n_1 + n_2)$-dimensional submanifold of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

4. Show that the set

$$X = \{(x, \mathrm{v}) \in S^{n-1} \times \mathbb{R}^n\,, \quad x \cdot \mathrm{v} = 0\}$$

is a $2n-2$-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. (Here "$x \cdot \mathrm{v}$" is the dot product, $\sum x_i v_i$.)