Integrating Densities on Manifolds

Suppose that we have a density σ on an an open set $U \subset \mathbb{R}^n$. Recall that we can write σ as

$$\sigma = h\sigma_{\text{Leb}}$$

where $h : U \longrightarrow \mathbb{R}$ is give by

$$h = \sigma\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$$

Definition 1. *The integral of $\sigma = h\sigma_{\text{Leb}}$ over U is defined to be*

$$\int_U \sigma = \int_U h\sigma_{\text{Leb}} := \int_U h$$

where the the right hand side indicates the extended Riemann integral of h over U. The integral of σ over U is defined if and only if $\int_U h$ is.

In other words,

$$\int_U \sigma := \int_U \sigma\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$$

The change of variable theorem for the extended Riemann integral tells us the following:

Theorem 1. *if $f : U' \longrightarrow U$ is a C^1 diffeomorphism,*

$$\int_U \sigma = \int_{U'} f^*\sigma$$

Proof. Recall that $f^*\sigma_{\text{Leb}} = |\det Df|$. If $\sigma = h\sigma_{\text{Leb}}$,

$$\int_U \sigma = \int_U h = \int_{U'} h \circ f |\det Df| = \int_{U'} f^*(h\sigma_{\text{Leb}}) = \int_{U'} f^*\sigma$$

□

We shall now define the integral of a density on a manifold M.

Definition 2. *Suppose that σ is a density on a manifold M, and $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha \subset M$ is a collection of diffeomorphisms so that $\{V_\alpha\}$ is an open cover of M. Choose a partition of unity $\{\phi_i\}$ subordinate to V_α, so the support of ϕ_i is contained inside V_i.*

Note that $\sigma = \sum_i \phi_i\sigma$.

Then if

1.

$$\int_{U_i} \varphi_i^*(\phi_i \sigma) \text{ exists for all } i$$

2.

$$\sum_{i=1}^{\infty} \int_{U_i} \varphi_i^*(\phi_i |\sigma|) < \infty$$

Then we say that σ is integrable. In that case, we define

$$\int_M \sigma = \sum_{i=1}^{\infty} = \int_{U_i} \varphi^*(\phi_i \sigma)$$

We must check that this definition is well defined. Suppose that σ is integrable, and that we have another partition of unity, ϕ'_j so that the support of each ϕ'_j is contained in the open set $V'_j \subset M$ which is the image of the diffeomorphism $\varphi' : U'_j \longrightarrow V'_j$.

Then

$$\begin{aligned}
\sum_i \int_{U_i} \varphi_i^*(\phi_i |\sigma|) &= \sum_i \sum_j \int_{U_i} \varphi_i^*(\phi_i \phi'_j |\sigma|) \\
&= \sum_{i,j} \int_{\varphi_i^{-1}(V_i \cap V'_j)} \varphi_i^*(\phi_i \phi'_j |\sigma|) \\
&= \sum_j \sum_i \int_{(\varphi'_j)^{-1}(V_i \cap V'_j)} (\varphi_i^{-1} \varphi'_j)^* \varphi_i^*(\phi_i \phi'_j |\sigma|) \\
&= \sum_j \int_{U'_j} (\varphi'_j)^*(\phi_j |\sigma|)
\end{aligned}$$

We can repeat the same argument with σ in place of $|\sigma|$, noting that we can rearrange the sums involved because they are absolutely convergent to get that

$$\sum_i \int_{U_i} \varphi_i^* \phi_i = \sum_j \int_{U_j} (\varphi'_j)^*(\phi'_j \sigma)$$

(and in particular all the integrals on the right hand side are defined), so the integral is well defined. Note also that this agrees with our definition of the integral on open sets.

We shall leave the proof of the following four important theorems as exercises:

Theorem 2. *If σ is a continuous density on a compact manifold M, $\int_M \sigma$ exists.*

Theorem 3. *If $f : M \longrightarrow N$ is a C^1 diffeomorphism, and σ is a density on N,*

$$\int_M f^* \sigma = \int_N \sigma$$

Recall that diffeomorphisms preserve the property of a set having measure 0. We can therefore say that a subset X of a manifold M has measure 0 if it has measure 0 in any coordinate chart.

Theorem 4. *If $X \subset M$ is a closed subset of M with measure 0, and σ is integrable on M, then*

$$\int_{M-X} \sigma = \int_M \sigma$$

Theorem 5. *Let M be a smooth manifold.*

1. *If σ_1 and σ_2 are integrable, so is $a\sigma_1 + b\sigma_2$, and*

$$\int_M (a\sigma_1 + b\sigma_2) = a \int_M \sigma_1 + b \int_M \sigma_2$$

2. *If σ is nonnegative (ie $\sigma(v_1, \dots, v_n) \geq 0$ for any collection of vector fields),*

$$\int_M \sigma \geq 0$$

3. *If U is an open subset of M and σ is integrable on M, σ is integrable on U. If σ is a positive density, then*

$$\int_U \sigma \leq \int_M \sigma$$

4. *If σ is integrable on open subsets U and V of M, σ is integrable on $U \cap V$ and $U \cup V$, and*

$$\int_{U \cup V} \sigma = \int_U \sigma + \int_V \sigma - \int_{U \cap V} \sigma$$

Examples

1. Suppose that $M^n \subset \mathbb{R}^N$ is an n dimensional submanifold of $\mathbb{R}^N$. The n-dimensional volume of M is defined to be

$$vol(M) := \int_M \sigma_{vol}$$

Where σ_{vol} is the volume density on M given by the Riemannian metric $< \cdot, \cdot >$ given by restricting the Euclidean metric on $\mathbb{R}^N$ to M. This agrees with our earlier definition of volume in the case that M is an open subset of $\mathbb{R}^N$.

2. For example, consider a one dimensional submanifold $M \subset \mathbb{R}^n$. To calculate $vol(M)$ (which we'd usually call the length of M), suppose that there exists a diffeomorphism $\gamma : (0,1) \longrightarrow M$. Then

$$vol(M) = \int_{(0,1)} \gamma^* \sigma_{vol} = \int_{(0,1)} \left| \frac{\partial \gamma}{\partial x} \right| \sigma_{\text{Leb}}$$

If M was diffeomorphic to a circle instead of an interval, we could either use a partition of unity and two coordinate patches, or we could note that as σ_{vol} is continuous, and M is compact the integral of σ_{vol} must exist, so we can simply integrate σ_{vol} over M minus 1 point, because the point will have measure 0. This will be diffeomorphic to an inverval, so we can integrate it in one step as above.

3. Suppose that M^n is an n dimensional submanifold of $\mathbb{R}^N$, and $\varphi : U \longrightarrow M$ is a diffeomorphism. To calculate $vol(M)$, we need to calculate $\varphi^* \sigma_{vol}$.

$$\begin{aligned} \varphi^* \sigma_{vol}(p)(e_1, \ldots, e_n) &= \sigma_{vol}(D\varphi(p)e_1, \ldots, D\varphi(p)e_n) \\ &= \sqrt{\det A} \end{aligned}$$

where A is the matrix with entries $a_{i,j} = D\varphi(p)e_i \cdot D\varphi(p)e_j$. In other words, $A = (D\varphi(p))^T D\varphi(p)$, where $(D\varphi(p))^T$ is the transpose, or adjoint of $D\varphi(p)$. Therefore

$$\varphi^* \sigma_{vol} = \sqrt{\det(D\varphi^T D\varphi)} \sigma_{\text{Leb}}$$

so

$$\int_M \sigma = \int_U \sqrt{\det(D\varphi^T D\varphi)} \sigma_{\text{Leb}}$$

This formula also works if $\varphi : U \longrightarrow M$ is a diffeomorphism onto M minus a set of measure 0.

Exercises

1. Prove theorem 2
2. Prove theorem 3
3. Prove theorem 4
4. Prove theorem 5
5. State and prove the formula for the area of a two dimensional submanifold of $\mathbb{R}^3$ which you remember from multivariable calculus.
6. Let

$$M = \{x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1\} \subset \mathbb{R}^4$$

Calculate $vol(M)$.

7. Calculate the volume of the unit sphere in $\mathbb{R}^4$.