HOMEWORK FOR 18.101, FALL 2007
ASSIGNMENT 1
DUE 11AM FRIDAY SEPTEMBER 21 IN ROOM 108

Each part of each problem is worth 10 points.

(1) Given a linear map

$$T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

Define the operator norm of T as follows:

$$\|T\| := \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}$$

Similarly, if A is a matrix, define the operator norm of A by

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

(a) Show that $\|T\|$ is finite.
(b) Show that $\|\cdot\|$ is a norm.
(c) Show that if AB is defined,

$$\|A\| \, \|B\| \geq \|AB\|$$

(2) Pove that the two norms $|\cdot|_s$ and $\|\cdot\|$ on $\mathbb{R}^n$ give the same topology in the sense that if U is an open set using the metric from one norm, it is open using the metric from the other norm.
(Recall that we defined $|x|_s := \max |x_i|$, and $\|x\| := \sqrt{\sum x_i^2}$.)

(3) (a) Show that given any $m \times n$ matrix A, the transpose of the matrix A^T is the unique $n \times m$ matrix with the property that

$$(Ax) \cdot y = x \cdot (A^T)y \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m$$

(b) Show that

$$\|A\| = \left\|A^T\right\|$$

(4) (a) Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable, and suppose that for all x, $\|Df(x)\| \leq 1$. Prove that

$$\|f(x) - f(y)\| \leq \|x - y\|$$

(Hint: Try taking the dot product with $f(x) - f(y)$, and use the chain rule to convert this into a single variable problem. Then you can use the mean value theorem.)

(b) Find a counterexample to the following naive generalization of the mean value theorem: Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ differentiable and points $x, y \in \mathbb{R}^n$, there exists some point c on the line segement between x and y so that

$$Df(c)(x - y) = f(x) - f(y)$$

(5) (a) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be of class C^1. Prove that the set $S \subset \mathbb{R}^n$ consisting of points $x \in \mathbb{R}^n$ where $Df(x)$ has rank n is open.
(Hint: The determinent has a formula which is a polynomial in the coefficients of the matrix. This tells you that the determinent in a continuous function of the coefficients of a matrix. Use this.)
(b) Use the inverse function theorem to prove that $f(S) \subset \mathbb{R}^n$ is also open.