Flux and the divergence theorem

We now know one way of calculating how an integral changes under the flow of a vector field, namely,

$$\frac{d}{dt}|_{t=0}\int_{\Phi_{tv}U}\sigma=\int_U L_v\sigma$$

When we flow a region U with a nice boundary, all changes to U happen close to the boundary, so we should be able to express with something which depends only on σ and v close to the boundary of U.

We shall use the following lemma which is a version of Fubini's theorem:

Lemma 1. *If M is a compact manifold, $I\subset\mathbb{R}$ is an interval, and σ is a continuous density on $M\times I$, then*

$$\int_{M\times I}\sigma=\int_I f$$

where the function $f: I\longrightarrow\mathbb{R}$ is continuous, and defined by

$$f(t):=\int_{M\times\{t\}} i_{\frac{\partial}{\partial t}}\sigma:=\int_{M\times\{t\}}\sigma\left(\frac{\partial}{\partial t},\cdot,\dots,\cdot\right)$$

We shall leave the proof of this as an exercise. It amounts to using Fubini's theorem in the correct coordinates.

To reduce our computations of how the integral changes under the flow of a vector field to this case, we shall use the following:

Theorem 2. *Suppose that v is a C^k vector field on a smooth $(n+1)$ dimensional manifold N, and M is a compact n dimensional sub manifold of N. If v is transverse to M (in other words, for all $p\in M$ $v(p)\notin T_pM\subset T_pN$), then there exists some $\epsilon>0$ and a C^k diffeomorphism ψ from $M\times(-\epsilon,\epsilon)$ onto some open neighborhood of M in N, where*

$$\psi(p,t):=\Phi_{tv}(p) \text{ where } p\in M \text{ and } t\in(-\epsilon,\epsilon)$$

Proof. First, note that as M is compact, there exists some $\epsilon'>0$ so that $\Phi_{tv}p$ is defined for all $|t|<\epsilon'$ and all $p\in M$, so it is possible to define ψ as above. We must verify that ψ is indeed a diffeomorphism onto a neighborhood of $M\subset N$.

Note that $\psi(\cdot,0)$ is a diffeomorphism $M\times\{0\}\longrightarrow M\subset N$. Also note that

$$\psi_*\frac{\partial}{\partial t}=v$$

The fact that v is transverse to M then tells us that $T_{(p,0)}\psi$ is surjective. Therefore the inverse function theorem states that, ψ is a diffeomorphism from a neighborhood of $(p, 0)$ onto a neighborhood of $p \in N$. By using the fact that $M \subset N$ is closed, we can arrange that the image of this diffeomorphism intersects $M \subset N$ only where it should, in the image of $M \times \{0\}$.

Then using this, and the fact that M is compact, we get that there exists some $\epsilon > 0$ so that $\psi : M \times (-2\epsilon, 2\epsilon) \longrightarrow N$ is a local diffeomorphism, and so that $\psi(p, t) \notin M$ if $t \neq 0$.

We claim that $\psi : M \times (-\epsilon, \epsilon) \longrightarrow N$ is injective: To see this, suppose that $\psi(p, t) = \psi(p', t')$. Then

$$\psi(p, 0) = \Phi_{-tv}\psi(p, t) = \Phi_{-tv}\psi(p', t') = \psi(p', t - t')$$

As $|t - t'| < 2\epsilon$, and $\psi(p', t - t') \in M$, we must conclude that $t' = t$, and therefore, $p' = p$. As ψ is injective, and a local diffeomorphism, it is our required diffeomorphism onto a neighborhood of $M \subset N$.

$\square$

Theorem 3. *Suppose that $U \subset N$ is an open subset so that $\bar{U}$ is compact, and the boundary of U, $bdy(U) = bdy(\bar{U}) = M \subset N$ is a manifold with one dimension less than N. Let σ be a continuous density on N, and v a C^1 vector field on N which is transverse to M, and which points out of U. Then*

$$\frac{d}{dt}|_{t=0} \int_{\Phi_{tv}(U)} \sigma = \int_M i_v \sigma := \int_M \sigma(v, \cdot, \ldots, \cdot)$$

Proof. Use theorem 2 to get a C^1 diffeomorphism ψ from $M \times (-\epsilon, \epsilon)$ onto a neighborhood of $M \subset N$ so that

$$\psi(\cdot, 0) : M \times \{0\} \longrightarrow M \subset N \text{ is a diffeomorphism}$$

and

$$\psi_* \frac{\partial}{\partial t} = v$$

Note that for $t \in (-\epsilon, \epsilon)$ we can write $\Phi_{tv}(U)$ as a disjoint union:

$$\Phi_{tv}(U) = \Phi_{-\epsilon v}(\bar{U}) \cup \psi\left(M \times (-\epsilon, t)\right)$$

Therefore,

$$\frac{d}{dt}|_{t=0} \int_{\Phi_{tv}(U)} \sigma = \frac{d}{dt}|_{t=0} \int_{\psi(M \times (-\epsilon, t))} \sigma = \frac{d}{dt}|_{t=0} \int_{M \times (-\epsilon, t)} \psi^* \sigma$$

We are now placed to use Lemma 1,

$$\int_{M \times (-\epsilon, t)} \psi^* \sigma = \int_{(-\epsilon, t)} f$$

where

$$f(t) = \int_{M\times\{t\}} i_{\frac{\partial}{\partial t}}(\psi^*\sigma)$$

Therefore, using the fundamental theorem of calculus,

$$\frac{d}{dt}|_{t=0}\int_{\Phi_{tv}(U)}\sigma = \int_{M\times\{0\}} i_{\frac{\partial}{\partial t}}(\psi^*\sigma)$$

But

$$i_{\frac{\partial}{\partial t}}(\psi^*\sigma) = (\psi^*\sigma)\left(\frac{\partial}{\partial t},\cdot\right) = \psi(\cdot,0)^*(\sigma\left(\psi_*\frac{\partial}{\partial t},\cdot\right) = \psi(\cdot,0)^*(i_v\sigma)$$

Therefore,

$$\frac{d}{dt}|_{t=0}\int_{\Phi_{tv}(U)}\sigma = \int_{M\times\{0\}} i_{\frac{\partial}{\partial t}}(\psi^*\sigma) = \int_M i_v\sigma$$

as required. □

The set $\bar{U}$ in the theorem above is what is known as a manifold with boundary. This is a subset of $\mathbb{R}^N$ so that every point has an open neighborhood diffeomorphic to either an open subset of $\mathbb{R}^{n+1}$, or an open subset of $[0,\infty)\times R^n$.

Theorem 4 (Divergence theorem)**.** *Let N be an $n+1$ dimensional manifold, and $\bar{U}$ a compact $n+1$ dimensional manifold with boundary the n dimensional manifold M, and interior the open set $U\subset N$. If v is a C^1 vector field on N, and σ is a C^1 density, then*

$$\int_U L_v\sigma = \int_M sgn(v) i_v\sigma$$

where

$$sgn(v) = \begin{cases} 1 \text{ if } v \text{ points out of } U \\ -1 \text{ if } v \text{ points into } U \\ 0 \text{ otherwise} \end{cases}$$

Proof. We may assume that v is complete by smoothly cutting it off outside $\bar{U}$ to make it compactly supported. (To prove that this is possible, use a smooth, compactly supported partition of unity $\{\phi_i\}$, and replace v with the sum of $\phi_i v$ for all ϕ_i who's support intersects $\bar{U}$.) By using a partition of unity, we can construct a smooth, complete vector field w so that w and $v+w$ both point out of U. Then

$$\frac{d}{dt}|_{t=0}\int_{\Phi_{tw}(U)}\sigma = \int_U L_w\sigma = \int_M i_w\sigma$$

and

$$\frac{d}{dt}|_{t=0}\int_{\Phi_{t(w+v)}(U)}\sigma = \int_U L_{w+v}\sigma = \int_M i_{w+v}\sigma$$

so

$$\int_U L_v\sigma = \int_U L_{v+w}\sigma - \int_U L_w\sigma = \int_M i_{v+w}\sigma - i_w\sigma = \int_M sgn(v) i_v\sigma$$

$\square$

Exercises

1. Prove Lemma 1

2. Show that if N is a manifold with a smooth metric, we have the following version of the divergence theorem:

$$\int_U L_v\sigma_{vol,N} = \int_M (v\cdot n)\sigma_{vol,M}$$

   where n is an outward pointing unit normal vector to M, and $\sigma_{vol,N}$ means the volume form on N coming from the metric, and $\sigma_{vol,M}$ means the volume form on M coming from the restriction of this metric to M.

3. Use the divergence theorem to calculate the ratio between the volume of the unit sphere in $\mathbb{R}^n$ and the unit ball in $\mathbb{R}^n$.