18.101 Final Exam
December 18, 2006

1. (20 points)

a. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function:

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^2+y^2}\,, & (x,y) \neq 0 \\ 0\,, & (x,y) = 0\,. \end{cases}$$

For what vectors, $u \in \mathbb{R}^2$, does the directional derivative, $D_u f(0)$, exist? Is f differentiable at 0?

b. Same question for the function

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}\,, & (x,y) \neq 0 \\ 0\,, & (x,y) = 0\,. \end{cases}$$

2. (20 points)

In the exercise below Q is the rectangle, $I \times I$, $I = [0,1]$, and $f : Q \to \mathbb{R}$ is a bounded function. True or false:

a. If f is zero except on a countable set, f is integrable.

b. If f is zero except on a set of measure zero, f is integrable.

c. Let $A \subseteq Q$ be a subset of measure zero. If f is continuous at all points, $p \in Q - A$, f is integrable.

d. If f is integrable then for some point, $x \in I$, the upper and lower Riemann integrals,

$$\overline{\int}_I f(x,y)\,dy \qquad \text{and} \qquad \underline{\int}_I f(x,y)\,dy\,,$$

are equal.

e. If the upper Riemann integral, $\overline{\int}_Q f$, and the lower Riemann integral, $\underline{\int}_Q f$, are both zero, f is integrable.

3. (25 points) Let A be the subset $[1,\infty) \times [1,\infty)$, of $\mathbb{R}^2$. For which of the functions below does the improper integral of f over A exist?

a. $f(x,y) = \frac{1}{x^3 y} + \frac{1}{x y^3}$.

b. $f(x,y) = \frac{1}{x^2 y^2}$.

c. $f(x,y) = \frac{1}{x^2+y^2}$.

d. $f(x,y) = \frac{1}{(x^2+y^2)^2}$.

e. $f(x,y) = \frac{1}{x^4} + \frac{1}{y^4}$.

4. (15 points) The volume of the unit ball, $x_1^2 + \cdots + x_{2n}^2 \leq 1$, in $\mathbb{R}^{2n}$ is $\pi^n/n!$. What is the volume of the set, $4x_1^2 + x_2^2 + \cdots + x_{2n}^2 \leq 1$?

5. (20 points) Compute the integral

$$\int_{\mathbb{R}^n} e^{-(x^2+y^2)}\, dx\, dy\, .$$

Hint: Polar coordinates and the change of variables formula.

6. (20 points)

a. Let v be the vector field on $\mathbb{R}^2$:

$$v = x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2}\, .$$

What are its integral curves? Is this vector field complete?

b. Let v be the vector field

$$v = x_1 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2}\, .$$

What are its integral curves? Is this vector field complete?

7. (20 points) Let U be the open set

$$\{(x,y) \in \mathbb{R}^2\, , \quad 0 < x\, , \quad 0 < y < \frac{\pi}{2}\}\, ,$$

V the open set

$$\{(x,y) \in \mathbb{R}^2\, , \quad 0 < x\, , 0 < y\}$$

and $f : U \to V$ the diffeomorphism

$$f(x,y) = (x \cos y\, , \, x \sin y)\, .$$

Let

$$w = f_* \frac{\partial}{\partial y} = w_1(x,y)\frac{\partial}{\partial x} + w_2(x,y)\frac{\partial}{\partial y}\,.$$

What are w_1 and w_2?

8. (15 points) Let X be the n-dimensional paraboloid: $x_{n+1} = x_1^2 + \cdots + x_n^2$, in $\mathbb{R}^{n-1}$. What is the tangent space, T_pX, at $p = (1,1,\ldots,1,n)$?

9. (20 points) Prove that the subset of $\mathbb{R}^n$ defined by the pair of equations

$$4x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = 1$$

and

$$x_1 + x_2 + \cdots + x_n = 0$$

is an $(n-2)$-dimensional submanifold of $\mathbb{R}^n$.

10. (25 points)

a. (Munkres, §22, #2) Let U be a bounded subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a $\mathcal{C}^\infty$ function. Suppose the derivatives

$$\frac{\partial f}{\partial x_k}\,, \quad k = 1,\ldots,n$$

are bounded on U. Let

$$X = \{(x, f(x))\,, \quad x \in U\}$$

be the graph of f viewed as an n-dimensional submanifold of $\mathbb{R}^{n+1}$. Express $\mathrm{vol}\,(X)$ as an integral over U.

b. (Loomis–Sternberg, 10.4, #3) For $n = 2$ show that this integral reduces to standard formula of elementary calculus for the area of the surface X.