The theory of Densities

For those of you who have a good grounding in linear algebra (and I hope most of you do) this set of notes is intended to give a slightly more algebraic approach to the theory of densities than that in §3 of Loomis–Sternberg.

We recall that if V is an n-dimensional vector space, $u_1, \dots u_n$, a basis of V and $A : V \to V$ a linear map then from the identities

$$Au_i = \sum_{j=1}^{n} a_{i,j,} u_j$$

one gets an $n \times n$ matrix, $[a_{i,j}]$, and the *determinant* of A is defined to be the determinant of this matrix. It's easy to check that this definition of determinant doesn't depend on the choice of $u_1, \dots, u_n$ and also to check that if one is given a linear mapping, $B : V \to V$, then $\det BA = \det B \det A$. Now let

$$V^n = V \times \cdots \times V \quad (n \text{ copies}) .$$

Definition 1. *A map* $\sigma : V^n \to \mathbb{R}$ *is a density on* V *if for all* $(\mathrm{v}_1, \dots \mathrm{v}_n) \in V^n$ *and all linear mappings* $A : V \to V$

$$\sigma(A\mathrm{v}_1, \dots, A\mathrm{v}_n) = |\det A| \sigma(\mathrm{v}_1, \dots, \mathrm{v}_n) . \tag{1}$$

Check:

1. If $\sigma_i : V^n \to \mathbb{R}$ $i = 1, 2$ is a density, $\sigma_1 + \sigma_2$ is a density.
2. If $\sigma : V^n \to \mathbb{R}$ is a density and $c \in \mathbb{R}$, $c\sigma$ is a density.

Thus the set of densities on V form a vector space. We'll denote this vector space by $|V|$.

Claim:

$|V|$ is a one-dimensional vector space.

Proof. Let $u = (u_1, \dots, u_n)$ be a basis of V. Then for every $(\mathrm{v}_1 \dots, \mathrm{v}_n) \in V^n$ there exists a unique linear mapping, $A : V \to V$, with

$$Au_i = \mathrm{v}_i \qquad i = 1, \dots, n .$$

Hence if $\sigma : V^n \to \mathbb{R}$ is a density

$$\sigma(\mathrm{v}_1, \dots, \mathrm{v}_n) = \sigma(Au_1, \dots, Au_n)$$

and hence

$$\sigma(\mathrm{v}_1,\dots,\mathrm{v}_n)=|\det A|\sigma(u_1,\dots,u_n) \tag{2}$$

i.e., σ is completely determined by its value at u.

□

Note that if $\sigma(v_1,\dots,v_n)>0$ for any basis $v_1,\dots,v_n$, then given any other basis $u_1,\dots,u_n$, $\sigma(u_1,\dots,u_n)>0$ (of course, σ applied to any collection of vectors which is not a basis will be 0.) Therefore, it makes sense to call σ a positive density, and use the notation

$$\sigma>0$$

It also makes sense to say that one density is larger than another, we write $\sigma_1\geq\sigma_2$ or $\sigma_1>\sigma_2$ if these inequalities hold for σ_1 and σ_2 applied to any basis.

Some examples of densities

1. In formula 2 set $\sigma(u_1,\dots,u_n)=1$. Then the density defined by this formula will be denoted by σ_u.

   Exercise 1. Show that σ_u *is* a density, i.e., show that if $B:V\to V$ is a linear map

   $$\sigma_u(B\mathrm{v}_1,\dots,B\mathrm{v}_n)=|\det B|\sigma_u(\mathrm{v}_1,\dots,\mathrm{v}_n)\,.$$

   Hint: $\det BA=\det B\det A$.

2. In particular let $V=\mathbb{R}^n$ and let $(e_1,\dots,e_n)=e$ be the standard basis of $\mathbb{R}^n$. Then $\sigma_e\in|\mathbb{R}^n|$ is the unique density which is 1 on $(e_1,\dots,e_n)$.

3. More generally if $p\in\mathbb{R}^n$ and

   $$V=T_p\mathbb{R}^n=\{(p,\mathrm{v})\,,\ \mathrm{v}\in\mathbb{R}^n\}\,,$$

   $\sigma_{p,e}\in|T_p\mathbb{R}^n|$ is the unique density which is 1 on the basis of vectors: $(p,e_1),\dots,(p,e_n)$.

4. Let $\langle\,,\,\rangle$ be an inner product on V and for $(\mathrm{v}_1,\dots,\mathrm{v}_n)\in V$ let

   $$C=[c_{i,j}]\,,\qquad c_{i,j}=\langle\mathrm{v}_i,\mathrm{v}_j\rangle\,.$$

   Then the *volume density* on V

   $$\sigma_{\mathrm{vol}}:V^n\to\mathbb{R}$$

   is defined by setting

   $$\sigma_{\mathrm{vol}}(\mathrm{v}_1,\dots,\mathrm{v}_n)=|\det C|^{1/2}\,. \tag{3}$$

   Exercise 2. Check that σ_{vol} is a positive density.

Hint: Let $u_1, \dots, u_n$ be an orthogonal basis of V, i.e., a basis having the property, $\langle u_i, u_j \rangle = 0$ for $i \neq j$ and $\langle u_i, u_i \rangle = 1$. If $(\mathrm{v}_1, \dots, \mathrm{v}_n) \in V^n$ then $\mathrm{v}_i = \sum a_{i,j} u_j$, hence

$$c_{i,j} = \langle \mathrm{v}_i, \mathrm{v}_j \rangle = \sum_{k=1}^{n} a_{ik} a_{jk}\,,$$

so if $C = [c_{i,j}]$ and $A = [a_{i,j}]$

$$C = AA^t$$

and $\det C = (\det A)^2$.

5. In particular if $V = T_p\mathbb{R}^n$ (which we can identify with $\mathbb{R}^n$ via the map, $(p, \mathrm{v}) \to \mathrm{v}$) and $\langle \mathrm{v}, w \rangle$ is the usual dot product of v with w, the vectors (p, e_i), $i = 1, \dots, n$ are an orthonormal basis of $T_p\mathbb{R}^n$ and $\sigma_{\mathrm{vol}} = \sigma_{p,e}$.

6. Let V_1 be an $(n-1)$-dimensional subspace of V. Then for $\mathrm{v} \in V$, $\iota(\mathrm{v})\sigma$ is the density on V_1 defined by

$$\iota_{\mathrm{v}}\sigma(\mathrm{v}_1, \dots, \mathrm{v}_{n-1}) = \sigma(\mathrm{v}, \mathrm{v}_1, \dots, \mathrm{v}_{n-1})\,. \tag{4}$$

7. Let V and W be n-dimensional vector spaces and $A : V \to W$ a bijective linear mapping. Given $\sigma \in |W|$, one defines $A^*\sigma \in |V|$ by the recipe

$$A^*\sigma(\mathrm{v}_1, \dots, \mathrm{v}_n) = \sigma(A\mathrm{v}_1, \dots, A\mathrm{v}_n)\,. \tag{5}$$

We call $A^*\sigma$ the *pull-back* of σ to V by A.

Exercise 3. Check that $A^*\sigma$ is a density.

Hint: If $B : V \to V$ is a linear map then

$$\begin{aligned} A^*\sigma(B\mathrm{v}_1, \dots, B\mathrm{v}_n) &= \sigma(AB\mathrm{v}_1, \dots, AB\mathrm{v}_n) \\ &= \sigma(B'A\mathrm{v}_1, \dots, B'A\mathrm{v}_n) \end{aligned} \tag{6}$$

where $B' = ABA^{-1}$.

Exercise 4.

Let $u = (u_1, \dots, u_n)$ be a basis of V and $w = (w_1, \dots, w_n)$ a basis of W. Show that if

$$Au_i = \sum a_{ij} w_j$$

and $\mathcal{A} = [a_{i,j}]$

$$A^*\sigma_w = |\det \mathcal{A}|\sigma_u\,.$$

8. In particular let U and U' be open subsets of $\mathbb{R}^n$ and

$$f : (U, p) \to (U', q)$$

a diffeomorphism. Then if $u_i = (p, e_i)$ and $w_i = (q, e_i)$,

$$T_p f(u_i) = \sum \frac{\partial f_i}{\partial x_j}(p) w_j$$

so

$$(T_p f)^* \sigma_{q,e} = |\det \partial f_i / \partial x_j(p)| \sigma_{p,e} \,. \tag{7}$$

This terminates our discussion of "densities on vector spaces". We will next discuss the notion of densities on manifolds. These densities will be more or less identical with the "densities" in §3 of Loomis–Sternberg.

Definition 2. *Let M be a smooth manifold. A* density *on U is a map, σ, which assigns to each point, $p \in M$, an element, $\sigma(p)$ of $|T_p M|$.*

Examples.

1. If $U \subset \mathbb{R}^n$ is open the Lebesgue density, σ_{Leb}. This is the density

$$p \in U \to \sigma_{p,e} \in |T_p \mathbb{R}^n|$$

2. (a) If σ is any density on M and $\varphi : M \longrightarrow \mathbb{R}$ is any real valued function on M, $\varphi\sigma$ is the density

$$p \in U \to \varphi(p)\sigma \,.$$

(b) If σ_1 and σ_2 are densities on M, the density $\sigma_1 + \sigma_2$ is defined by

$$(\sigma_1 + \sigma_2)(p) = \sigma_1(p) + \sigma_2(p)$$

(c) $|\sigma|$ is the density defined by

$$|\sigma|(v_1, \ldots, v_n) := |\sigma(v_1, \ldots, v_n)|$$

3. Note that we can write any density σ on an open subset $U \subset \mathbb{R}^n$ as $\varphi\sigma_{\text{Leb}}$ using

$$\varphi = \sigma \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$$

We will say that a density on U is a $\mathcal{C}^\infty$ *density* if it is of the form $\varphi\sigma_{\text{Leb}}$, with φ in $\mathcal{C}^\infty(U)$, and is *compactly supported* if $\varphi \in \mathcal{C}^\infty_o(U)$.

4. *Pull-backs.* Let $f : M \longrightarrow N$ be a diffeomorphism. Then if σ is density on N, we define a density $f^*\sigma$ on U by the formula

$$f^*\sigma(p) = (T_p f)^*\sigma(q) \qquad q = f(p)\,. \tag{8}$$

Another way of writing this is

$$f^*\sigma = (T_p f)^*\sigma \circ f$$

Since $T_p f : T_p M \to T_q N$ is a bijective linear map and $\sigma(q)$ is in $|T_q N|$ the left hand side of (8) is in $|T_p M|$, so the formula (8) defines a density

$$p \in M \to f^*\sigma(p) \in |T_p M|$$

as claimed.

Note that if $v_1, \dots, v_n$ are vector fields on M,

$$f^*\sigma(v_1, \dots, v_n) = f^*(\sigma(f_* v_1, \dots, f_* v_n)) = \sigma(f_* v_1, \dots, f_* v_n) \circ f$$

In other words, if $f(p) = q$

$$f^*\sigma(p)(v_1(p), \dots, v_n(p)) = \sigma(q)(T_p f v_1(p), \dots, T_p f v_n(p))$$

Exercise 5. Show that if f is a C^1 diffeomorphism between open subsets of $\mathbb{R}^n$,

$$f^*\sigma_{\text{Leb}} = |\det(Df)|\sigma_{\text{Leb}}\,. \tag{9}$$

Hint: Formula (7).

More generally show that if $\sigma = \varphi\sigma_{\text{Leb}}$, then

$$f^*\sigma = f^*\varphi|\det(Df)|\sigma_{\text{Leb}}\,. = \varphi \circ f|\det(Df)|\sigma_{\text{Leb}} \tag{10}$$

Definition 3. *A density σ on M is C^k if for all smooth coordinate charts $\psi : U \longrightarrow M$,*

$$\psi^*\sigma = \varphi\sigma_{\text{Leb}}$$

where φ is a C^k function.

Definition 4. *A Riemannian metric $< \cdot, \cdot >$ on a manifold M is a map which assigns to every point p, an inner product $< \cdot, \cdot >_p$ on $T_p M$ for all $p \in M$. We can put any two vector fields v_1, v_2 on M into the metric $< \cdot, \cdot >$ to obtain a function*

$$< v_1, v_2 >: M \longrightarrow \mathbb{R}$$

defined by

$$< v_1, v_2 > (p) = < v_1(p), v_2(p) >_p$$

We call a Riemannian metric C^k if for any pair of smooth vector fields $< v_1, v_2 >$ is a C^k function.

5. If $< \cdot, \cdot >$ is a Riemannian metric on a manifold M, define the density σ_{vol} by defining $\sigma_{vol}(p) \in |T_pM|$ by $\sigma_{vol}(p) = \sigma_{vol}$ on $(T_pM, < \cdot, \cdot >_p)$.

   In particular, if M is n dimensional and $v_1, \dots, v_n$ are n vector fields, we can find $\sigma(v_1, \dots, v_n)$ as follows: Define the matrix C to have entries $c_{i,j} =< v_i, v_j >$, then

$$\sigma_{vol}(v_1, \dots, v_n) := \sqrt{\det(C)}$$

   Note that if $< \cdot, \cdot >$ is a C^k metric, σ_{vol} is a C^k density.

6. If $M \subset \mathbb{R}^N$ is a smooth submanifold, we can put a metric on M which is the restriction of the Euclidean metric on $\mathbb{R}^N$. In particular, for $v, w \in T_pM$ define $< v, w >:= v \cdot w$, the usual dot product. We therefore get a corresponding volume density σ_{vol} defined on M.

Problem set.

Exercises 1, 2, 3, 4 and 5.

6. Verify that if V_1 is an $(n-1)$-dimensional subspace of V the density, $\iota_{\mathrm{v}}\sigma$, defined by formula (4) is, in fact, a density on V_1.

7. Let V_i, $i = 1, 2, 3$ be n-dimensional vector spaces and $A_i : V_i \to V_{i+1}$, $i = 1, 2$, bijective linear mapping. Show that if $\sigma \in |V_3|$, $A_1^*(A_2^*\sigma) = (A_2A_1)^*\sigma$.