6 Existence and Uniqueness Theorems

1 Introduction

We now resume the study of general normal systems of first-order DE's, of the form

$$\begin{aligned} dx_1/dt &= X_1(x_1, \ldots, x_n; t), \\ &\cdots\cdots \\ dx_n/dt &= X_n(x_1, \ldots, x_n; t). \end{aligned} \tag{1}$$

Our main concern in this chapter is to prove general theorems about the existence, uniqueness, continuity, and differentiability of solution of such systems for given fixed and variable initial conditions. These theorems will generalize results proved in Chapter 1 for the special case $n = 1$. In Chapters 7 and 8, we shall describe limiting processes for constructing and even computing such solutions.

We continue to make the assumptions of Chapter 5, § 1: that the X_i are continuous, real-valued functions of the independent variables $x_1, x_2, \ldots, x_n, t$ in some region $\mathscr{R}$ of interest in $(x_1, \ldots, x_n, t)$-space. We shall also use the vector notation introduced there, rewriting (1) as

$$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t) \qquad \text{or} \qquad \mathbf{x}'(t) = \mathbf{X}(\mathbf{x}, t). \tag{2}$$

The curve $\mathbf{x}(t)$ in $\mathscr{R}$ defined by any solution of (1) will be called a *solution curve* of the system (1).

Note that we can trivially inflate *any* normal system (1) of n first-order DE's to a normal *autonomous* system of $n + 1$ DE's by the simple device of writing $t = x_{n+1}$. This gives the equivalent system

$$dx_j/dt = X_j(x_1, \ldots, x_n; x_{n+1}), \qquad j = 1, \ldots, n + 1,$$

where X_{n+1} is the function 1. However, this does not help to prove the theorems of major interest.

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A nonconstant function $U(x_1, \ldots, x_n; t)$ which is defined in $\mathscr{R}$ and of class $\mathscr{C}^1$ there is called an *integral* of (1) when $U(x_1(t), \ldots, x_n(t); t)$ is constant for every solution of (1). This means that each solution curve of (1) stays on a single level surface $U(x_1, \ldots, x_n; t) = C$ of the function U in $(x_1, \ldots, x_n; t)$-space ("space-time") and, thus, generalizes the definition of "integral" in Chapter 1, § 3.

EXAMPLE 1. Consider the system

$$dx/dt = tx/(x^2 - y^2), \qquad dy/dt = ty/(y^2 - x^2).$$

It is easy to verify that, when $x = x(t)$ and $y = y(t)$ are solutions, the two functions

$$V(x, y, t) = xy \qquad \text{and} \qquad W(x, y, t) = x^2 + y^2 - t^2$$

satisfy $dV/dt = x\, dy/dt + y\, dx/dt = 0$ and $dW/dt = 0$. Therefore, V and W are integrals of the system. The intersection of two surfaces $V = c_1$ and $W = c_2$ is a solution curve of the system. Thus, every solution curve lies on the intersection of a hyperbolic cylinder $xy = c_1$ and a hyperboloid (or cone) $x^2 + y^2 - t^2 = c_2$.

First-order systems of DE's (1) provide a standard form to which *all* normal ordinary DE's and systems of DE's can be reduced. Using (1), we can thus give a unified theory for the existence and uniqueness of solutions of DE's and systems of DE's of all orders, as we shall see in this chapter.

For example, one can reduce the solution of a normal nth order DE to the solution of a system of n first-order DE's as follows. Let $u(t)$ be any solution of the given nth order DE,

$$d^n u/dt^n = F(u, du/dt, d^2u/dt^2, \ldots, d^{n-1}u/dt^{n-1}; t).$$

Then the n functions $x_1(t) = u,\ x_2(t) = du/dt, \ldots, x_n(t) = d^{n-1}u/dt^{n-1}$ satisfy the normal first-order system

$$dx_k/dt = x_{k+1}, \qquad k = 1, \ldots, n-1; \qquad dx_n/dt = F(x_1, x_2, \ldots, x_n; t).$$

Conversely, given any solution of the preceding first-order system, the first component $x_1(t)$ will have the other components $x_2, \ldots, x_n$ as its derivatives of orders $1, \ldots, n-1$. Hence, substituting back, $x_1(t)$ will satisfy the given nth order equation.

2 Lipschitz condition

In order to make use of vector notation for systems of DE's, we recall a few facts about vectors in n-dimensional Euclidean spaces. Addition of two vectors and multiplication of vectors by scalars are defined component-wise, as in the plane and in space.† The *length* of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is defined as

$$|\mathbf{x}| = (x_1^{\,2} + \cdots + x_n^{\,2})^{1/2}.$$

Length satisfies the triangle inequality

$$|\mathbf{x} + \mathbf{y}| \leqq |\mathbf{x}| + |\mathbf{y}|.$$

The dot product or *inner product* of two vectors is defined as

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$$

and satisfies the Schwarz inequality $|\mathbf{x} \cdot \mathbf{y}| \leqq |\mathbf{x}| \cdot |\mathbf{y}|$.

We shall integrate, differentiate, and take limits of vector functions $\mathbf{x}(t)$ of a scalar (real) variable t. All these operations can be carried out component by component, as in vector addition.

For example, the derivative of a vector function

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

is the vector function $\mathbf{x}'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t))$. The integral $\int_a^b \mathbf{x}(t)\,dt$ is the vector with components $\int_a^b x_1(t)\,dt$, $\int_a^b x_2(t)\,dt, \ldots,$ $\int_a^b x_n(t)\,dt$. We shall often make use of the fundamental inequality‡

$$\left|\int_a^b \mathbf{x}(t)\,dt\right| \leqq \int_a^b |\mathbf{x}(t)|\,dt. \tag{3}$$

A vector function $\mathbf{X}(\mathbf{x})$ of a vector variable is said to be *continuous* when each component X_i of $\mathbf{X}$ is a continuous function of the n variables $x_1, \ldots, x_n$, the components of the vector independent variable $\mathbf{x}$. This is equivalent to the following statement: The function $\mathbf{X}(\mathbf{x})$ is continuous at the point (vector) $\mathbf{c}$ whenever, given $\epsilon > 0$, there exists $\delta > 0$ such that, if $|\mathbf{x} - \mathbf{c}| < \delta$, then $|\mathbf{X}(\mathbf{x}) - \mathbf{X}(\mathbf{c})| < \epsilon$. We leave it as an exercise to verify that these definitions are equivalent.

Note that there is no such thing as "the" derivative of a function $\mathbf{X}(\mathbf{x})$ of a *vector* independent variable, but only a matrix of *partial* derivatives $\partial X_i/\partial x_j$ relative to the different components $x_1, \ldots, x_n$.

The reader who is not accustomed to working with functions of vectors should familiarize himself with the differences between the following types of functions: vector-valued functions of a scalar variable

† Birkhoff and Mac Lane, Chapter 7. The three-dimensional case is treated in Courant, Vol. 2, Chapter 2; Widder, Chapter 2, and in most other texts on the advanced calculus.
‡ This inequality is the continuous analog of the triangle inequality

$$|\mathbf{x}^{(1)} + \mathbf{x}^{(2)} + \cdots + \mathbf{x}^{(n)}| \leqq |\mathbf{x}^{(1)}| + |\mathbf{x}^{(2)}| + \cdots + |\mathbf{x}^{(n)}|.$$

It can be obtained from this inequality by recalling the definition of the integral $\int_a^b \mathbf{x}(t)\,dt$ as a limit of Riemann sums, using the triangle inequality for each of the Riemann sums, and passing to the limit on both sides.

(such as $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots,$
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$\mathbf{X}(\mathbf{x}) = (X_1(x_1, x_2, \ldots, x_n),$

vector-valued function of a vect
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A vector-valued function $\mathbf{X}($
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■ DEFINITION. *A vector-value*
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$|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)| \leqq L|\mathbf{x}$ -

Note that both terms on the l

■ LEMMA. *If* $\mathbf{X}(\mathbf{x}, t)$ *is of cla*
convex† domain D, then it satisfi

Proof — Let M be the maxim
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and for variable s,

$\frac{d}{ds}[X_i(\mathbf{x} + s\mathbf{y}, t)$

Hence, by the mean-value theo
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$X_i(\mathbf{x} + \mathbf{y}, t) - X_i(\mathbf{x},$

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(such as $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$), scalar-valued functions of a vector variable (such as $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$), vector-valued functions of a vector variable such as

$$\mathbf{X}(\mathbf{x}) = (X_1(x_1, x_2, \ldots, x_n), X_2(x_1 \ldots, x_n), \ldots, X_m(x_1, \ldots, x_n)),$$

vector-valued function of a vector variable $\mathbf{x}$ and a parameter t (such as $\mathbf{X}(\mathbf{x}, t)$).

A vector-valued function $\mathbf{X}(\mathbf{x})$ of a vector variable is said to be of class $\mathscr{C}^n$ in a given region when each of the component functions $X_i(x_1, \ldots, x_n)$ is of class $\mathscr{C}^n$ there.

One can also easily extend the definition of a Lipschitz condition to vector-valued functions; as we shall see, this provides a simple sufficient condition for the uniqueness and existence of solutions for normal systems.

■ Definition. *A vector-valued function* $\mathbf{X}(\mathbf{x}, t)$ *satisfies a* Lipschitz condition *in a region* $\mathscr{R}$ *of* $\mathbf{x}t$*-space if and only if, for some Lipschitz constant* L,

$$|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)| \leqq L|\mathbf{x} - \mathbf{y}| \qquad \textit{if} \quad (\mathbf{x}, t) \in \mathscr{R}, \quad (\mathbf{y}, t) \in \mathscr{R}. \tag{4}$$

Note that both terms on the left side of (4) involve the same value of t.

■ Lemma. *If* $\mathbf{X}(\mathbf{x}, t)$ *is of class* $\mathscr{C}^1$ *in a bounded closed ("compact") convex*† *domain* D, *then it satisfies a Lipschitz condition there.*

Proof — Let M be the maximum of all partial derivatives $|\partial X_i/\partial x_j|$ in the closed domain D. For each component X_i we have, for fixed $\mathbf{x}, \mathbf{y}, t$ and for variable s,

$$\frac{d}{ds}[X_i(\mathbf{x} + s\mathbf{y}, t)] = \sum_{k=1}^{n} \frac{\partial X_i}{\partial x_k}(\mathbf{x} + s\mathbf{y}, t)y_k.$$

Hence, by the mean-value theorem applied to the function $X_i(\mathbf{x} + s\mathbf{y}, t)$ of the variable s on the interval $0 \leqq s \leqq 1$, we have

$$X_i(\mathbf{x} + \mathbf{y}, t) - X_i(\mathbf{x}, t) = \sum_{k=1}^{n} \frac{\partial X_i}{\partial x_k}(\mathbf{x} + \sigma_i \mathbf{y}, t)y_k,$$

for some σ_i between 0 and 1. Squaring, and applying the Schwarz

† A set S in n-space is *convex* when the segment joining any two points of the set S lies entirely within S. This definition applies both to closed domains D and open regions $\mathscr{R}$.

inequality to the right side, we get

$$|X_i(\mathbf{x}+\mathbf{y},t)-X_i(\mathbf{x},t)|^2 \leqq \left(\sum_{k=1}^{n}\left|\frac{\partial X_i}{\partial x_k}\right|^2\right)\left(\sum_{k=1}^{n}|y_k|^2\right) \leqq nM^2|\mathbf{y}|^2.$$

Hence, summing over all components i, we obtain

$$|\mathbf{X}(\mathbf{x}+\mathbf{y},t)-\mathbf{X}(\mathbf{x},t)|^2 \leqq n^2M^2|\mathbf{y}|^2.$$

Taking square roots, the Lipschitz condition follows with Lipschitz constant nM.

3 Well-set problems

For DE's to be useful in predicting the future behavior of a physical system from its present state, their solutions must exist, be unique, and depend continuously on their initial values. As stated in Chapter 1, § 9, an initial-value problem is said to be *well-set* when these conditions are satisfied. We now show that, if $\mathbf{X}$ satisfies a Lipschitz condition, the vector DE (2) defines a well-set initial-value problem.

We begin by proving uniqueness. As in the special case $n = 1$ of Chapter 1, §10, uniqueness does not follow from the continuity of $\mathbf{X}(\mathbf{x}, t)$ alone.

■ THEOREM 1 (Uniqueness Theorem). *If the function* $\mathbf{X}(\mathbf{x}, t)$ *satisfies a Lipschitz condition* (4) *in a domain* $\mathcal{R}$, *there is at most one solution* $\mathbf{x}(t)$ *of the vector* DE (2) *which satisfies a given initial condition* $\mathbf{x}(a) = \mathbf{c}$ *in* $\mathcal{R}$.

The proof of this theorem parallels that of Theorem 5 of Chapter 1. We show that, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are both solutions of (2) and if they are equal for one value of t, say $t = a$, it follows that $\mathbf{x}(t) \equiv \mathbf{y}(t)$ in any domain in which a Lipschitz condition is satisfied.

Consider the square of the n-dimensional distance between the two vectors $\mathbf{x}(t)$ and $\mathbf{y}(t)$. By definition, this is

$$\sigma(t) = \sum [x_k(t) - y_k(t)]^2 = |\mathbf{x}(t) - \mathbf{y}(t)|^2 \geqq 0.$$

Differentiating $\sigma(t)$, and using the fact that $\mathbf{x}$ and $\mathbf{y}$ are solutions of the normal system (2), we get

$$\begin{aligned}\sigma'(t) &= 2\sum [x_k(t) - y_k(t)][X_k(\mathbf{x}(t), t) - X_k(\mathbf{y}(t), t)]\\ &= 2(\mathbf{x}(t) - \mathbf{y}(t)) \cdot [\mathbf{X}(\mathbf{x}(t), t) - \mathbf{X}(\mathbf{y}(t), t)].\end{aligned}$$

By the Schwarz inequality, therefore, we have

$$\begin{aligned}\sigma'(t) &\leqq |\sigma'(t)| = 2|(\mathbf{x}-\mathbf{y}) \cdot (\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t))|\\ &\leqq 2|\mathbf{x}-\mathbf{y}| \cdot |\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)| \leqq 2L|\mathbf{x}-\mathbf{y}|^2 = 2L\sigma(t).\end{aligned}$$

By the Corollary of Lemma
$\mathbf{x}(a) = \mathbf{y}(a)$, that is, if $\sigma(a) = 0$,
for all $t \geqq a$.
A similar argument works fo
$d\sigma/d(-t)$
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(2) depend continuously on the

■ THEOREM 2 (Continuity T
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Lipschitz condition (4). *Then*
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Proof — Replacing $a + t$ b
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Theorem 1,
$\sigma'(t) = 2[\mathbf{x}(t) - \mathbf{y}(t)] \cdot [\mathbf{X}(\mathbf{x}$
Applying Lemma 2 of Chapter
Taking the square root of both
From Theorem 2 we can easil
of the solutions of the DE (2).

■ COROLLARY. *Let* $\mathbf{x}(t, \mathbf{c})$ *be the*
condition $\mathbf{x}(a, \mathbf{c}) = \mathbf{c}$. *Let the hy*
the functions $\mathbf{x}(t, \mathbf{c})$ *be defined fo*
(*a*) $\mathbf{x}(t, \mathbf{c})$ *is a continuous fu*
(*b*) *if* $\mathbf{c} \to \mathbf{c}^0$, *then* $\mathbf{x}(t, \mathbf{c}) \to \mathbf{x}($

Both properties follow from
In view of the preceding resu
theorem, in order to show that
normal first-order systems (1).

EXERCISES A

1. Show that $u = x + y + z$ and
system $dx/dt = y - z$, $dy/dt = z - x$
2. Reduce each of the following D
determine in which domain or do
a half-plane, etc.) the resulting syst

(a) $d^3x/dt^3 + x^2 = 1$, (b) $d^2x/$

$\left.\frac{X_i}{x_k}\right|^2\Bigg)\left(\sum_{k=1}^{n}|y_k|^2\right) \leqq nM^2|\mathbf{y}|^2.$

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$(t), t) - X_k(\mathbf{y}(t), t)]$

$t) - \mathbf{X}(\mathbf{y}(t), t)].$

have

$- \mathbf{X}(\mathbf{y}, t))|$

$\leqq 2L|\mathbf{x} - \mathbf{y}|^2 = 2L\sigma(t).$

By the Corollary of Lemma 2 of Chapter 1, §10, it follows that if $\mathbf{x}(a) = \mathbf{y}(a)$, that is, if $\sigma(a) = 0$, then $\sigma(t) \equiv 0$ (that is, $|\mathbf{x}(t) - \mathbf{y}(t)|^2 \equiv 0$) for all $t \geqq a$.

A similar argument works for $t < a$: replacing t by $-t$, we obtain

$$d\sigma/d(-t) \leqq |\sigma'(t)| \leqq 2L\sigma(t),$$

again using the preceding inequality.

We shall prove next that the solutions of a normal first-order system (2) depend continuously on their initial values.

■ THEOREM 2 (Continuity Theorem). *Let* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *be any two solutions of the vector* DE (2), *where* $\mathbf{X}(\mathbf{x}, t)$ *is continuous and satisfies the Lipschitz condition* (4). *Then*

$$|\mathbf{x}(a+h) - \mathbf{y}(a+h)| \leqq e^{L|h|}\,|\mathbf{x}(a) - \mathbf{y}(a)|. \tag{5}$$

Proof — Replacing $a + t$ by $a - t$, we can always reduce to the case $h \geqq 0$. Consider again $\sigma(t) = |\mathbf{x}(t) - \mathbf{y}(t)|^2$. As in the proof of Theorem 1,

$$\sigma'(t) = 2[\mathbf{x}(t) - \mathbf{y}(t)] \cdot [\mathbf{X}(\mathbf{x}(t) - \mathbf{X}(\mathbf{y}(t)] \leqq 2L|\mathbf{x} - \mathbf{y}|^2 = 2L\sigma(t).$$

Applying Lemma 2 of Chapter 1, § 10 to $\sigma(t)$, we get $\sigma(a+h) \leqq \sigma(a)e^{2Lh}$. Taking the square root of both sides, we get the desired result.

From Theorem 2 we can easily infer the following important property of the solutions of the DE (2).

■ COROLLARY. *Let* $\mathbf{x}(t, \mathbf{c})$ *be the solution of the* DE (2) *satisfying the initial condition* $\mathbf{x}(a, \mathbf{c}) = \mathbf{c}$. *Let the hypotheses of Theorem 2 be satisfied, and let the functions* $\mathbf{x}(t, \mathbf{c})$ *be defined for* $|\mathbf{c} - \mathbf{c}^0| \leqq K$ *and* $|t - a| \leqq T$. *Then*:

(*a*) $\mathbf{x}(t, \mathbf{c})$ *is a continuous function of both variables*;

(*b*) *if* $\mathbf{c} \to \mathbf{c}^0$, *then* $\mathbf{x}(t, \mathbf{c}) \to \mathbf{x}(t, \mathbf{c}^0)$ uniformly *for* $|t - a| \leqq T$.

Both properties follow from the inequality (5).

In view of the preceding results, it remains only to prove an *existence* theorem, in order to show that the initial value problem is well-set for normal first-order systems (1). This will be done in Theorems 6–8 below.

EXERCISES A

1. Show that $u = x + y + z$ and $v = x^2 + y^2 + z^2$ are integrals of the linear system $dx/dt = y - z$, $dy/dt = z - x$, $dz/dt = x - y$. Interpret geometrically.
2. Reduce each of the following DE's to an equivalent first-order system, and determine in which domain or domains (e.g., entire plane, any bounded region, a half-plane, etc.) the resulting system satisfies a Lipschitz condition:

(a) $d^3x/dt^3 + x^2 = 1$, (b) $d^2x/dt^2 = x^{-1/2}$, (c) $d^3x/dt^3 = (1 + (d^2x/dt^2)^2)^{1/2}$.

3. Reduce the following system to normal form, and determine in which domains a Lipschitz condition is satisfied:

$$du/dt + dv/dt = u^2 + v^2, \qquad 2du/dt + 3dv/dt = 2uv.$$

4. Show that the vector-valued function $(t + be^{at}, -e^{-at}/ab)$ satisfies the DE (2) with $\mathbf{X} = (1 - (1/x_2), 1/(x_1 - t))$, for any nonzero constants a, b.
5. State and prove a uniqueness theorem for the DE $y'' = F(x, y, y')$, with $F \in \mathscr{C}^1$. [HINT: Reduce to a first-order system, and use Theorem 1.]
6. (a) Show that any solution of the linear system $dx/dt = y$, $dy/dt = z$, $dz/dt = x$ satisfies the vector DE $d^3\mathbf{x}/dt^3 = \mathbf{x}$, where $\mathbf{x} = (x, y, z)$.
(b) Show that every solution of the preceding system can be written $\mathbf{x} = e^t\mathbf{a} + e^{-t/2}[\mathbf{b}\cos\sqrt{3}\,t/2 + \mathbf{c}\sin\sqrt{3}\,t/2]$, for suitable constant vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
(c) Express $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in terms of $\mathbf{x}(0)$, $\mathbf{x}'(0)$ and $\mathbf{x}''(0)$.
7. Show that the general solution of the system $dx/dt = x^2/y$, $dy/dt = x/2$ is $x = 1/(at + b)^2$, $y = -1/[2a(at + b)]$.
8. Show that the curves defined parametrically as solutions $dx/dt = \partial F/\partial x$, $dy/dt = \partial F/\partial y$, $dz/dt = \partial F/\partial z$ are orthogonal to the surfaces $F(x, y, z) =$ constant. What differentiability condition on F must be assumed to make the above system satisfy a Lipschitz condition?
9. (a) Find a system of first-order DE's satisfied by all curves orthogonal to the spheres $x^2 + y^2 + z^2 = 2ax - a^2$.
(b) By integrating the preceding system, find the orthogonal trajectories in question. Describe the solution curves geometrically.
10. (a) In what sense is the following statement inexact? "The general solution of the DE $cy'' = (1 + y'^2)^{3/2}$ is the circle $(x - a)^2 + (y - b)^2 = c^2$, where a and b are arbitrary constants."
(b) Correct the preceding statement, distinguishing carefully between explicit, implicit, and multiple-valued functions.
11. (a) Given $\dot{x} = a(t)x + b(t)y$ and $\dot{y} = c(t)x + d(t)y$, prove that, if $b \neq 0$,

$$\ddot{x} - [(a + d) + \dot{b}/b]\dot{x} + [(ad - bc) - \dot{a} + (a\dot{b}/b)]x = 0.$$

(b) Given that $\ddot{x} + p(t)\dot{x} + q(t)x = r(t)$, prove that, if $q \neq 0$, $v = \dot{x}$ satisfies

$$\ddot{v} + [p - (\dot{q}/q)]\dot{v} + [\dot{p} + q - (\dot{q}p/q)]v = \dot{r} - r\dot{q}/q.$$

12. For which values of α, β does the function $x^\alpha y^\beta$ satisfy a Lipschitz condition: (a) in the open square $0 < x, y < 1$, (b) in the quadrant $0 < x, y < +\infty$, (c) in the part of the quadrant of (b) exterior to the square of (a)?
13. For each of the following scalar-valued functions of a vector $\mathbf{x}$ and each of the following domains, state whether a Lipschitz condition is satisfied or not:
(a) $x_1 + x_2 + \cdots + x_n$, (b) $x_1x_2 \cdots x_n$, (c) $y/(x^2 + y^2)$, (d) $|\mathbf{x}|$,
in (i) $|x| < 1$, (ii) $-\infty < x_k < \infty$, (iii) $-\infty < x_1 < \infty$, $|x_k| < 1$, $k \geq 2$.
14. Let $\mathbf{X}(\mathbf{x}, t) = (X_1(\mathbf{x}, t), \ldots, X_n(\mathbf{x}, t))$ be a vector-valued function. Show that $\mathbf{X}$ satisfies a Lipschitz condition if and only if each scalar-valued component X_i satisfies a Lipschitz condition, and relate the Lipschitz constant of $\mathbf{X}$ to those of the X_i.

4 Continuity

We shall now prove a much stronger continuity property of the solutions of systems of DE's, namely that the solutions of (2) vary

continuously when the functio
ing, the solution of a DE depe
initial values.

■ THEOREM 3. *Let* $\mathbf{x}(t)$ *and* $\mathbf{y}($

$$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t)$$

respectively, on $a \leq t \leq b$. *Furt*
continuous in a common domai

$$|\mathbf{X}(\mathbf{z}, t) - \mathbf{Y}(\mathbf{z}, t)|$$

Finally, let $\mathbf{X}(\mathbf{x}, t)$ *satisfy the L*

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leq |\mathbf{x}(a) -$$

The function $\mathbf{Y}$ is not requir

Proof — Consider the real-v
by

$$\sigma(t) = |\mathbf{x}(t) - \mathbf{y}$$

From the last expression we se
can be written in the form

$$\sigma'(t) = 2[\mathbf{X}(\mathbf{x}(t), t)$$
$$= 2\{[\mathbf{X}(\mathbf{x}(t), t)$$
$$+ 2\{[\mathbf{X}(\mathbf{y}(t)$$

We now apply the triangle in
Schwarz inequality to each of t
gives the inequality

$$|\sigma'(t)| \leq 2|\mathbf{X}(\mathbf{x}(t), t$$
$$+ 2|\mathbf{X}(\mathbf{y}(t$$

To the first term on the right si
which $\mathbf{X}$ satisfies; to the seco
following differential inequalit

$$\sigma'(t) \leq$$

The theorem is now an immed

continuously when the function $\mathbf{X}$ varies continuously. Loosely speaking, the solution of a DE depends continuously upon the DE for given initial values.

Theorem 3. *Let* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *satisfy the* DE's

$$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t) \qquad \text{and} \qquad d\mathbf{y}/dt = \mathbf{Y}(\mathbf{y}, t),$$

respectively, on $a \leqq t \leqq b$. *Further, let the functions* $\mathbf{X}$ *and* $\mathbf{Y}$ *be defined and continuous in a common domain* D, *and let*

$$|\mathbf{X}(\mathbf{z}, t) - \mathbf{Y}(\mathbf{z}, t)| \leqq \epsilon, \qquad a \leqq t \leqq b, \quad \mathbf{z} \in D. \tag{6}$$

Finally, let $\mathbf{X}(\mathbf{x}, t)$ *satisfy the Lipschitz condition* (4). *Then*

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leqq |\mathbf{x}(a) - \mathbf{y}(a)|\, e^{L|t-a|} + \frac{\epsilon}{L}\left[e^{L|t-a|} - 1\right]. \tag{7}$$

The function $\mathbf{Y}$ is not required to satisfy a Lipschitz condition.

Proof — Consider the real-valued function $\sigma(t)$, defined for $a \leqq t \leqq b$ by

$$\sigma(t) = |\mathbf{x}(t) - \mathbf{y}(t)|^2 = \sum_{k=1}^{n} [x_k(t) - y_k(t)]^2.$$

From the last expression we see that σ is differentiable. Its derivative can be written in the form

$$\begin{aligned} \sigma'(t) &= 2[\mathbf{X}(\mathbf{x}(t), t) - \mathbf{Y}(\mathbf{y}(t), t)] \cdot [\mathbf{x}(t) - \mathbf{y}(t)] \\ &= 2\{[\mathbf{X}(\mathbf{x}(t), t) - \mathbf{X}(\mathbf{y}(t), t)] \cdot [\mathbf{x}(t) - \mathbf{y}(t)]\} \\ &\quad + 2\{[\mathbf{X}(\mathbf{y}(t), t) - \mathbf{Y}(\mathbf{y}(t), t)] \cdot [\mathbf{x}(t) - \mathbf{y}(t)]\}. \end{aligned}$$

We now apply the triangle inequality to the right side, and then the Schwarz inequality to each of the two terms of the last expression. This gives the inequality

$$\begin{aligned} |\sigma'(t)| &\leqq 2|\mathbf{X}(\mathbf{x}(t), t) - \mathbf{X}(\mathbf{y}(t), t)|\, |\mathbf{x}(t) - \mathbf{y}(t)| \\ &\quad + 2|\mathbf{X}(\mathbf{y}(t), t) - \mathbf{Y}(\mathbf{y}(t), t)|\, |\mathbf{x}(t) - \mathbf{y}(t)|. \end{aligned}$$

To the first term on the right side we now apply the Lipschitz condition which $\mathbf{X}$ satisfies; to the second term, we apply (6). This gives the following differential inequality for σ:

$$\sigma'(t) \leqq 2L\sigma(t) + 2\epsilon\sqrt{\sigma(t)}. \tag{8}$$

The theorem is now an immediate consequence of the following

■ LEMMA. *Let $\sigma(t) \geqq 0$, $a \leqq t \leqq b$ be a differentiable function satisfying the differential inequality* (8). *Then*

$$\sigma(t) \leqq [\sqrt{\sigma(a)}\, e^{L(t-a)} + \frac{\epsilon}{L}\,(e^{L(t-a)} - 1)]^2, \qquad a \leqq t \leqq b. \tag{9}$$

Proof — We shall apply Theorem 7 of Chapter 1 on differential inequalities to (8). The right side of (8), the function $F(\sigma, t) = 2L\sigma + 2\epsilon\sqrt{\sigma}$, satisfies a Lipschitz condition in any half plane $\sigma \geqq \sigma_0$ which does not include the line $\sigma = 0$. Therefore, Theorem 7 of Chapter 1 applies when $\sigma(a) > 0$. For, if $\sigma(a) > 0$, then the solution of the DE

$$du/dt = 2\epsilon\sqrt{u} + 2Lu, \qquad u \geqq 0, \tag{9'}$$

which satisfies the initial condition $u(a) = \sigma(a)$, will have a nonnegative derivative, and therefore will remain, for $t > a$, within the half-plane $u \geqq \sigma(a)$.

The DE (9′) is a Bernoulli DE (Chapter 1, Exercise B2). To find the solution satisfying $u(a) = \sigma(a)$, make the substitution $v(t) = \sqrt{u(t)}$. (The square root is well-defined because $u(t) \geqq \sigma(a) > 0$.) This gives the equivalent DE

$$2vv' = 2\epsilon v + 2Lv^2.$$

If $u(a) > 0$, it follows that $u(t) > 0$ for all later t, since the derivative of u is positive. This gives $v(t) > 0$, and so we can divide both sides of this DE by v. The resulting DE is $v' - Lv = \epsilon$, an inhomogeneous linear DE (Chapter 1, § 5, Example 2), whose solution satisfying the initial condition $v(a) = \sqrt{u(a)}$ is the function

$$\sqrt{u(t)} = v(t) = \sqrt{u(a)}\, e^{L(t-a)} + (\epsilon/L)(e^{L(t-a)} - 1).$$

On applying Theorem 7 of Chapter 1, we obtain the inequality (9).

We must now consider the case $\sigma(a) = 0$ when Theorem 7 of Chapter 1 does not apply directly. In this case, we consider the solution $u_n(t)$ of the differential equation (9′) which satisfies the initial condition $u_n(a) = 1/n$. Since the right side of (9′) is positive, $u_n(t)$ is an increasing function of t. We shall prove that $u_n(t) \geqq \sigma(t)$. Suppose that at some point $t_1 > a$ we had $u_n(t_1) < \sigma(t_1)$. Then among all numbers t with $a < t < t_1$ such that $u_n(t) \geqq \sigma(t)$ there would be a largest, say t_0. Hence, we would have $u_n(t_0) = \sigma(t_0) > 0$ and $u_n(t) < \sigma(t)$ for $t_0 < t \leqq t_1$. But this is impossible by what we have already proved, since in the interval $t_0 \leqq t \leqq t_1$ the functions $u(t)$ and $\sigma(t)$ stay away from 0, and therefore a Lipschitz condition is satisfied for (9′). We infer that

$$\sigma(t) \leqq [\, n^{-1/2} e^{L|t-a|} + (\epsilon/L)(e^{L|t-a|} - 1)]^2$$

for all $n > 0$. Letting $n \to \infty$,
case.

The following corollary foll

■ COROLLARY. *Let* $\mathbf{X}(\mathbf{x}, t; \epsilon)$
defined in the domain D: $|t - a$
values of a parameter ϵ. Sup
uniformly in D to a function $\mathbf{X}$
For each $\epsilon > 0$, let $\mathbf{x}(t; \epsilon)$ be
the initial condition $\mathbf{x}(a; \epsilon) =$
of $d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, y)$ satisfying $\mathbf{x}($
$|t - a| \leq T_1 \leq T$ where all fun

EXERCISES B

1. Let $\mathbf{X}$ and $\mathbf{Y}$ be as in Theorem
$|t - a|$ remains bounded as $t \to a$.
2. Let $f(x)$ and $g(x)$ be respective
$f(0) = g(0)$. Show that $|f(x) - g(x$

Bound the differences betwee
$y(0) = c$, for the pairs of DE's in E

3. $y' = \sin xy$ and $y' = x$
4. $y' = e^y$ and $y' = 1 + y + \cdots +$
5. $d^2\theta/dt^2 = -\theta$ and $d^2\theta/dt^2 = -$si
6. To what explicit formulas does
$\mathbf{X}(t)$? For the DE $dx/dt = ax + b$
7*. Show that the conclusion of
condition $(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t$
8. Let $\mathbf{X}(\mathbf{x}, t, s)$ be continuous for
let it satisfy $|\mathbf{X}(\mathbf{x}, t, s) - \mathbf{X}(\mathbf{y}, t, s))$
$\mathbf{x}' = \mathbf{X}(\mathbf{x}, t, s)$ satisfying $\mathbf{x}(a, s) = \mathbf{c}$

5* Normal systems

Many important mathema
DE's of order $m > 1$ as their
examples, and show how to
DE's to a first-order system.

■ DEFINITION. *A* normal sy
functions $\xi_1(t), \xi_2(t), \ldots, \xi_m(t)$

$$\frac{d^{n(k)}\xi_k}{dt^{n(k)}} = F_k\Big(\xi_1, \frac{d\xi_1}{dt},$$

differentiable function satisfying

$^{a)} - 1)]^2, \qquad a \leqq t \leqq b.$ (9)

7 of Chapter 1 on differential
), the function $F(\sigma, t) = 2L\sigma +$
ny half plane $\sigma \geqq \sigma_0$ which does
Theorem 7 of Chapter 1 applies
e solution of the DE

$u, \qquad u \geqq 0,$ (9′)

$= \sigma(a)$, will have a nonnegative
for $t > a$, within the half-plane

ter 1, Exercise B2). To find the
the substitution $v(t) = \sqrt{u(t)}$.
$u(t) \geqq \sigma(a) > 0$.) This gives the

$2Lv^2$:

all later t, since the derivative
d so we can divide both sides of
$v' - Lv = \epsilon$, an inhomogeneous
), whose solution satisfying the
ction

$+ (\epsilon/L)(e^{L(t-a)} - 1).$

e obtain the inequality (9).
0 when Theorem 7 of Chapter 1
ve consider the solution $u_n(t)$ of
satisfies the initial condition
is positive, $u_n(t)$ is an increasing
$) \geqq \sigma(t)$. Suppose that at some
en among all numbers t with
uld be a largest, say t_0. Hence,
$u_n(t) < \sigma(t)$ for $t_0 < t \leqq t_1$. But
ady proved, since in the inter-
stay away from 0, and therefore
. We infer that

$L)(e^{L|t-a|} - 1)]^2$

for all $n > 0$. Letting $n \to \infty$, we obtain the inequality (9) also in this case.

The following corollary follows immediately from Theorem 3.

■ COROLLARY. *Let* $\mathbf{X}(\mathbf{x}, t; \epsilon)$ *be a set of continuous functions of* $\mathbf{x}$ *and* t, *defined in the domain* D: $|t - a| \leqq T$, $|\mathbf{x} - \mathbf{c}| \leqq K$ *for all sufficiently small values of a parameter* ϵ. *Suppose that, as* $\epsilon \to 0$, *the functions converge uniformly in* D *to a function* $\mathbf{X}(\mathbf{x}, t)$ *which satisfies a Lipschitz condition. For each* $\epsilon > 0$, *let* $\mathbf{x}(t; \epsilon)$ *be a solution of* $d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t; \epsilon)$ *satisfying the initial condition* $\mathbf{x}(a; \epsilon) = \mathbf{c}$. *Then the* $\mathbf{x}(t; \epsilon)$ *converge to the solution of* $d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, y)$ *satisfying* $\mathbf{x}(a) = \mathbf{c}$, *uniformly in any closed subinterval* $|t - a| \leqq T_1 \leqq T$ *where all functions are defined.*

EXERCISES B

1. Let $\mathbf{X}$ and $\mathbf{Y}$ be as in Theorem 3, and let $\mathbf{x}(a) = \mathbf{y}(a)$. Show that $|\mathbf{x}(t) - \mathbf{y}(t)|/|t - a|$ remains bounded as $t \to a$.

2. Let $f(x)$ and $g(x)$ be respective solutions of $y' = xy$ and $y' = y \sin x$ such that $f(0) = g(0)$. Show that $|f(x) - g(x)| < \frac{1}{640}$ for $|x| < \frac{1}{2}$.

Bound the differences between solutions having the same initial value $y(0) = c$, for the pairs of DE's in Exercises 3 through 5.

3. $y' = \sin xy$ and $y' = xy - \dfrac{x^3y^3}{3!} + \dfrac{x^5y^5}{5!} + \cdots + (-1)^n \dfrac{(xy)^{2n+1}}{(2n+1)!}$.

4. $y' = e^y$ and $y' = 1 + y + \cdots + \dfrac{y^n}{n!}$.

5. $d^2\theta/dt^2 = -\theta$ and $d^2\theta/dt^2 = -\sin\theta$.

6. To what explicit formulas does formula (7) specialize for the system $d\mathbf{x}/dt = \mathbf{X}(t)$? For the DE $dx/dt = ax + b$?

7*. Show that the conclusion of Theorem 3 holds if only a one-sided Lipschitz condition $(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)) \leqq L|\mathbf{x} - \mathbf{y}|^2$ is assumed for $\mathbf{X}$.

8. Let $\mathbf{X}(\mathbf{x}, t, s)$ be continuous for $|\mathbf{x} - \mathbf{c}| \leqq K$, $|t - a| \leqq T$, and $|s - s_0| \leqq S$, and let it satisfy $|\mathbf{X}(\mathbf{x}, t, s) - \mathbf{X}(\mathbf{y}, t, s)) \leqq L|\mathbf{x} - \mathbf{y}|$. Show that the solution $\mathbf{x}(t, s)$ of $\mathbf{x}' = \mathbf{X}(\mathbf{x}, t, s)$ satisfying $\mathbf{x}(a, s) = \mathbf{c}$ is a continuous function of s.

5* Normal systems

Many important mathematical problems have normal systems of DE's of order $m > 1$ as their natural formulation. We now give two examples, and show how to reduce every normal system of ordinary DE's to a first-order system.

■ DEFINITION. *A* normal system *of ordinary* DE's *for the unknown functions* $\xi_1(t), \xi_2(t), \ldots, \xi_m(t)$ *is any system of the form*

$$\frac{d^{n(k)}\xi_k}{dt^{n(k)}} = F_k\left(\xi_1, \frac{d\xi_1}{dt}, \ldots; \xi_2, \frac{d\xi_2}{dt}, \ldots; \xi_m, \frac{d\xi_m}{dt}, \ldots; t\right), \qquad (10)$$

$k = 1, \ldots, m$, in which for each k only derivatives $d^p\xi_j/dt^p$ of any ξ_j of orders $p < n(j)$ occur on the right side.

In other words, the requirement is that the derivative $d^{n(k)}\xi_k/dt^{n(k)}$ of highest order of each ξ_k constitutes the left-hand side of one equation and occurs nowhere else.

■ THEOREM 4. *Each normal system* (10) *of ordinary* DE's *is equivalent to a first-order normal system* (1) *(with $n \geqq m$).*

Proof — Each function F_k appearing on the right side of (10) is a function of several real variables. Denote these variables by $x_1, x_2, \ldots, x_n, t$, where $n = n_1 + \cdots + n_m$. We now construct a system for $x_1, \ldots, x_n$ by replacing each derivative $d^p\xi_j/dt^p$ of ξ_j of order $p < n(j)$ by a new variable, that is, by setting

$$x_1 = \xi_1,\ x_2 = d\xi_1/dt,\ x_3 = d^2\xi_1/dt^2, \ldots, x_{n_1} = d^{n_1-1}\xi_1/dt^{n_1-1},$$

$$x_{n_1+1} = \xi_2,\quad x_{n_1+2} = d\xi_2/dt, \ \ldots,\ \ x_{n_1+n_2} = d^{n_2-1}\xi_2/dt^{n_2-1}, \text{ etc.}$$

The equivalent system for the x_k is then the system

$$dx_1/dt = x_2,\ dx_2/dt = x_3, \ldots, dx_{n_1-1}/dt = x_{n_1},\ dx_{n_1}/dt = F_1(x_1, \ldots, x_n),$$

$$dx_{n_1+1}/dt = x_{n_1+2}, \ldots, dx_{n_1+n_2}/dt = F_2(x_1, \ldots, x_n, t), \text{ etc.}$$

It is clear that this system satisfies the requirements of the theorem.

The *initial value problem* for the normal system (10) is the problem of finding a solution for which the variables

$$\xi_1,\ d\xi_1/dt, \ldots, d^{n_1-1}\xi_1/dt^{n_1-1}, \ldots, d^{n_m-1}\xi_m/dt^{n_m-1}$$

assume given values at $t = a$.

It is easily seen from the proof of Theorem 4 that, if the functions F_k, considered as functions of the vector variable $\mathbf{x} = (x_1, \ldots, x_n)$ and t, satisfy Lipschitz conditions, then so do the functions $X_k(\mathbf{x}, t)$ in the associated first-order systems (1). This gives the following.

■ COROLLARY. *If the functions F_k of the normal system* (10) *satisfy Lipschitz conditions in a domain D, then the system has at most one solution in D satisfying given initial conditions.*

EXAMPLE 2 (the n-body problem). Let n mass points with masses m_j attract each other according to an inverse αth power law of attraction. Then, in suitable units, their position coordinates satisfy a normal system of $3n$ second-order differential equations of the form

$$d^2x_i/dt^2 = \sum_{j \neq i} m_j(x_j - x_i)/r_{ij}^{\alpha+1}$$

and the same is true for d^2y_i/d

$$r_{ij} = [(x_i - x_j)^2 + ($$

Then Theorem 1 asserts that t
velocities $(x_i'(0), y_i'(0), z_i'(0))$ of
subsequent motion (if any such
ness theorem asserts the de
theorem, taken with the conti
asserts that the n-body proble

To see this, let $\boldsymbol{\xi} = (\xi_1, \ldots$
defined as follows, for $k = 1, \ldots$

$$\xi_k = x_k, \quad \xi$$

$$\xi_{3n+k} = x_k', \quad$$

In this notation, the system (
system of the form (1):

$$\frac{d\xi_h}{dt} = F_h(\xi) = \begin{cases} \xi_{h+3n}, \\ \sum_j m_j(\xi_{j-3n} - \end{cases}$$

where $k(j)$ is the remainder of
extended to those $n - 1$ values
distinct and have the same in
is, so long as there are *no colli*
satisfied by the functions F_h.
some of the functions F_h bec
Theorem 1 is inapplicable.

EXAMPLE 3. The *Frenet-S*
normal system of first-order D

$$\frac{d\boldsymbol{\alpha}}{ds} = \frac{\boldsymbol{\beta}}{R(s)}, \qquad \frac{d\boldsymbol{\beta}}{ds} = -$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$ a
tangent, normal, and binorma
ture $\kappa(s) = 1/R(s)$ and torsion
length s; $\boldsymbol{\alpha} = d\mathbf{x}/ds$ is the deriva
length.

† Widder, p. 101.

‡ $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ denotes the cross product of

y derivatives $d^p\xi_j/dt^p$ *of any* ξ_j *of*

hat the derivative $d^{n(k)}\xi_k/dt^{n(k)}$ of
he left-hand side of one equation

0) *of ordinary* DE's *is equivalent to*
$\le m$).

g on the right side of (10) is a func-
hese variables by $x_1, x_2, \ldots, x_n, t$,
onstruct a system for $x_1, \ldots, x_n$
of ξ_j of order $p < n(j)$ by a new

$^2, \ldots, x_{n_1} = d^{n_1-1}\xi_1/dt^{n_1-1}$,

$x_{n_1+n_2} = d^{n_2-1}\xi_2/dt^{n_2-1}$, etc.

en the system
$t = x_{n_1},\ dx_{n_1}/dt = F_1(x_1, \ldots, x_n)$,

$t = F_2(x_1, \ldots, x_n, t)$, etc.

requirements of the theorem.
nal system (10) is the problem of
les

$^1, \ldots, d^{n_m-1}\xi_m/dt^{n_m-1}$

orem 4 that, if the functions F_k,
variable $\mathbf{x} = (x_1, \ldots, x_n)$ and t,
to the functions $X_k(\mathbf{x}, t)$ in the
gives the following.

the normal system (10) *satisfy*
hen the system has at most one
litions.

Let n mass points with masses
nverse αth power law of attrac-
ion coordinates satisfy a normal
quations of the form

$-x_i)/r_{ij}^{\alpha+1}$

and the same is true for d^2y_i/dt^2 and d^2z_i/dt^2, where

$$r_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2} = r_{ji}.$$

Then Theorem 1 asserts that the initial positions $(x_i(0), y_i(0), z_i(0))$ and velocities $(x_i'(0), y_i'(0), z_i'(0))$ of the mass points *uniquely* determine their subsequent motion (if any such motion is possible). That is, the uniqueness theorem asserts the *determinacy of the n-body problem.* This theorem, taken with the continuity theorem, and Theorem 8 to follow, asserts that the n-body problem is well-set.

To see this, let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{6n})$ be the vector with components defined as follows, for $k = 1, \ldots, n$.

$$\xi_k = x_k, \quad \xi_{n+k} = y_k, \quad \xi_{2n+k} = z_k,$$

$$\xi_{3n+k} = x_k', \quad \xi_{4n+k} = y_k', \quad \xi_{5n+k} = z_k'.$$

In this notation, the system (10) is equivalent to a first-order normal system of the form (1):

$$\frac{d\xi_h}{dt} = F_h(\xi) = \begin{cases} \xi_{h+3n}, & h = 1, \ldots, 3n, \\ \sum_j m_j(\xi_{j-3n} - \xi_{h-3n})/r_{hk(j)}^{\alpha+1}, & h = 3n+1, \ldots, 6n, \end{cases}$$

where $k(j)$ is the remainder of j when divided by n, and summation is extended to those $n - 1$ values of j such that $(h - 1)/n$ and $(j - 1)/n$ are distinct and have the same integral part. So long as no $r_{hj} = 0$, that is, so long as there are *no collisions*, a Lipschitz condition is evidently satisfied by the functions F_h. When one or more r_{hj} vanish, however, some of the functions F_h become *singular* (they are undefined) and Theorem 1 is inapplicable.

Example 3. The *Frenet-Serret formulas*† comprise the following normal system of first-order DE's:

$$\frac{d\boldsymbol{\alpha}}{ds} = \frac{\boldsymbol{\beta}}{R(s)}, \qquad \frac{d\boldsymbol{\beta}}{ds} = -\frac{\boldsymbol{\alpha}}{R(s)} + \frac{\boldsymbol{\gamma}}{T(s)}, \qquad \frac{d\boldsymbol{\gamma}}{ds} = -\frac{\boldsymbol{\beta}}{T(s)},$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$ are three-dimensional‡ vectors: the unit tangent, normal, and binormal vectors to a space curve. The curvature $\kappa(s) = 1/R(s)$ and torsion $\tau(s) = 1/T(s)$ are functions of the arc length s; $\boldsymbol{\alpha} = d\mathbf{x}/ds$ is the derivative of vector position with respect to arc length.

† Widder, p. 101.
‡ $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ denotes the cross product of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

If we let $\eta(s)$ be the nine-dimensional vector $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3)$, the system can be written as the first-order vector DE

$$d\boldsymbol{\eta}/ds = \mathbf{Y}(\boldsymbol{\eta}; s).$$

Here $\mathbf{Y}(\boldsymbol{\eta}; s)$ is obtained by setting

$$Y_h(\eta; s) = \begin{cases} \kappa(s)\eta_{h+3}, & h = 1, 2, 3, \\ -\kappa(s)\eta_{h-3} + \tau(s)\eta_{h+3}, & h = 4, 5, 6, \\ -\tau(s)\eta_{h-3}, & h = 7, 8, 9. \end{cases}$$

If $\kappa(s)$ and $\tau(s)$ are bounded, the function $\mathbf{Y}(\boldsymbol{\eta}; s)$satisfies a Lipschitz condition (4) with $L = \sup\{|\kappa(s)| + |\tau(s)|\}$; hence, for given initial tangent direction $\boldsymbol{\alpha}(0)$, normal direction $\boldsymbol{\beta}(0)$ perpendicular to $\boldsymbol{\alpha}(0)$, and binormal direction $\boldsymbol{\gamma}(0) = \boldsymbol{\alpha}(0) \times \boldsymbol{\beta}(0)$, there is only one set of directions satisfying the Frenet-Serret formulas. This proves that *a curve is determined up to a rigid motion by its curvature and torsion.*†

EXERCISES C

1. Find all solutions of the system

$$x\frac{d^2x}{dt^2} - y\frac{d^2y}{dt^2} = 0, \qquad \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + x + y = 0.$$

2. Show that, if $a_{ihk} = -a_{khi}$, then $\sum x_i^2$ is an integral of the system

$$dx_i/dt = \sum a_{ihk} x_h x_k.$$

3. The *one-body* problem is defined in space by the system

$$\ddot{x} = -xf(r), \qquad \ddot{y} = -yf(r), \qquad \ddot{z} = -zf(r), \qquad r^2 = x^2 + y^2 + z^2.$$

(a) Show that the components $L = y\dot{z} - z\dot{y}$, $M = z\dot{x} - x\dot{z}$, and $N = x\dot{y} - y\dot{x}$ of the angular momentum vector (L, M, N) are integrals of this system.
(b) Show that any solution of the system lies in a plane $Ax + By + Cz = 0$.
(c) Construct an energy integral for the system.

4. Let $\alpha = 2$ in the n-body problem (Newton's law of gravitation), and define the potential energy as $V = -\sum_{i<j} m_i m_j / r_{ij}$.
(a) Show that the n-body problem is defined by the system

$$m_i d^2x_i/dt^2 = -\partial V/\partial x_i.$$

(b) Show that the total energy $\sum m_i \dot{x}_i^2/2 + V(\mathbf{x})$ is an integral of the system.
(c) Show that the components $\sum m_i \dot{x}_i$, etc., of linear momentum are integrals.
(d) Do the same as in (c) for the components $\sum m_i(y_i \dot{z} - z_i \dot{y}_i)$, etc., of angular momentum.

5. Show that the general solution of the vector DE $d^3\mathbf{x}/dt^3 = d\mathbf{x}/dt$ is $\mathbf{a} + \mathbf{b}e^t + \mathbf{c}e^{-t}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are arbitrary vectors.

† This theorem of differential geometry can fail when $\kappa(s)$ is zero, because $\boldsymbol{\beta} = x''(s)/|x''(s)|$ is then geometrically undefined, so that the Frenet-Serret formulas do not necessarily hold.

Exercises 6–9 refer to the Frene

6. Show that, if $\boldsymbol{\alpha}(s)$, $\boldsymbol{\beta}(s)$, $\boldsymbol{\gamma}(s)$ are
this is true for all s, provided they
7. Show that if $1/T(s) \equiv 0$, and d
Consider the dot product $\boldsymbol{\gamma} \cdot \mathbf{x}$.]
8*. Show that, if $T = kR$ (k const
9*. Show that, if $R/T + (TR')' = ($

6 Equivalent integral equation

We now establish the *existe*
system of DE's for arbitrary in
to reduce the problem to an
place of derivatives. One reas
makes it easier to treat is tha
able functions directly, but o
integrals. Every continuous
continuous functions are not d

■ THEOREM 5. *Let* $\mathbf{X}(\mathbf{x}; t)$ *be*
$\mathbf{x}$ *and* t. *Then any solution* $\mathbf{x}(t)$

$$\mathbf{x}(t) = \mathbf{c}$$

is a solution of the vector DE (2)
and conversely.

The vector integral equation
r unknown scalar functions $x_1(t$
function $\mathbf{x}(t)$. That is,

$$x_k(t) = c_k + \int_a^t X_k(x_1(s), x$$

(In §§ 6-7, we will deal with

Proof — If $\mathbf{x}(t)$ satisfies the
by the Fundamental Theoren
$k = 1, \ldots, r$, so that $\mathbf{x}(t)$ also
Fundamental Theorem of th
$\int_a^t x_k'(s)\, ds$ for all continuousl
satisfies the normal system of D
if, in addition, $\mathbf{x}(a) = \mathbf{c}$, the int

EXAMPLE 4. Consider the
$x(0) = 0$. Separating variables

Exercises 6–9 refer to the Frenet-Serret formulas.

6. Show that, if $\boldsymbol{\alpha}(s)$, $\boldsymbol{\beta}(s)$, $\boldsymbol{\gamma}(s)$ are orthogonal vectors of length one when $s = 0$, this is true for all s, provided they satisfy the Frenet-Serret formulas.

7. Show that if $1/T(s) \equiv 0$, and $d\mathbf{x}/ds = \boldsymbol{\alpha}$, the curve $\mathbf{x}(s)$ lies in a plane. [HINT: Consider the dot product $\boldsymbol{\gamma} \cdot \mathbf{x}$.]

8*. Show that, if $T = kR$ (k constant), the curve $\mathbf{x}(s)$ lies on a cylinder.

9*. Show that, if $R/T + (TR')' = 0$, the curve $\mathbf{x}(s)$ lies on a sphere.

6 Equivalent integral equation

We now establish the *existence* of a solution of a normal first-order system of DE's for arbitrary initial values. To this end, it is convenient to reduce the problem to an equivalent one where integrals take the place of derivatives. One reason why this restatement of the problem makes it easier to treat is that we do not have to deal with differentiable functions directly, but only with continuous functions and their integrals. Every continuous function has an integral, while many continuous functions are not differentiable.

■ THEOREM 5. *Let $\mathbf{X}(\mathbf{x}; t)$ be a continuous vector function of the variables $\mathbf{x}$ and t. Then any solution $\mathbf{x}(t)$ of the vector integral equation*

$$\mathbf{x}(t) = \mathbf{c} + \int_a^t \mathbf{X}(\mathbf{x}(s), s)\, ds \tag{11}$$

is a solution of the vector DE (2) *which satisfies the initial condition* $\mathbf{x}(a) = \mathbf{c}$, *and conversely.*

The vector integral equation (11) is a system of integral equations for r unknown scalar functions $x_1(t), \ldots, x_r(t)$, the components of the vector function $\mathbf{x}(t)$. That is,

$$x_k(t) = c_k + \int_a^t X_k(x_1(s), x_2(s), \ldots, x_r(s), s)\, ds, \qquad 1 \leq k \leq r.$$

(In §§ 6-7, we will deal with r-dimensional vectors.)

Proof — If $\mathbf{x}(t)$ satisfies the integral equation (11), then $\mathbf{x}(a) = \mathbf{c}$ and, by the Fundamental Theorem of the Calculus, $\mathbf{x}_k'(t) = X_k(\mathbf{x}(t); t)$ for $k = 1, \ldots, r$, so that $\mathbf{x}(t)$ also satisfies the system (2). Conversely, the Fundamental Theorem of the Calculus shows that $x_k(t) = x_k(a) + \int_a^t x_k'(s)\, ds$ for all continuously differentiable functions $\mathbf{x}(t)$. If $\mathbf{x}(t)$ satisfies the normal system of DE's (2), then $\mathbf{x}(t) = \mathbf{x}(a) + \int_a^t \mathbf{X}(\mathbf{x}(s); s)\, ds$; if, in addition, $\mathbf{x}(a) = \mathbf{c}$, the integral equation (11) is obtained, q.e.d.

EXAMPLE 4. Consider the DE $dx/dt = e^x$ for the initial condition $x(0) = 0$. Separating variables, we see that this initial-value problem has the (unique) solution $1 - e^{-x} = t$, $x = -\ln(1-t)$. Theorem 5 shows that it is equivalent to the integral equation $x(t) = \int_0^t e^{x(s)}\,ds$, which therefore has the same (unique) solution. Since the solution is defined only in the interval $-\infty < t < 1$, we see again that only a *local* existence theorem can be proved.

Operator interpretation. The problem of finding a solution to the integral equation (11) can be rephrased in terms of operators on vector functions as follows. We define an operator $\mathbf{y} = U[\mathbf{x}] = U\mathbf{x}$, transforming vector functions $\mathbf{x}$ into vector functions $\mathbf{y}$ by the identity

$$\mathbf{y}(t) = U[\mathbf{x}(t)] = \mathbf{c} + \int_a^t \mathbf{X}(\mathbf{x}(s), s)\,ds. \tag{12}$$

If $\mathbf{X}(\mathbf{x}, t)$ is defined for all $\mathbf{x}$ in the slab $|t-a| \leqq T$ and is continuous, the domain of this operator can be taken to be the family of continuous vector functions defined in the interval $|t-a| \leqq T$; its range consists of all continuously differentiable vector functions defined in this interval, satisfying $\mathbf{y}(a) = \mathbf{c}$. In this case, Theorem 5 has the following.

■ COROLLARY. *The* DE (2) *has a solution satisfying* $\mathbf{x}(a) = \mathbf{c}$ *if and only if the mapping* U *of* (12) *has a fixpoint in* $\mathscr{C}[a, b]$.

However, if $\mathbf{X}(\mathbf{x}, t)$ is not defined for all $\mathbf{x}$, the domain of the operator U has to be determined with care. This is done in Theorem 8 below.

7 Successive approximation

Picard had the idea of iterating the integral operator U defined by (12), and proving that, for any initial trial function $\mathbf{x}^0$, the successive integral transforms (Picard approximations)

$$\mathbf{x}^0,\ \mathbf{x}^1 = U\mathbf{x}_0, \qquad \mathbf{x}^2 = U^2[\mathbf{x}^0] = U[\mathbf{x}^1], \qquad \mathbf{x}^3 = U[\mathbf{x}^2], \ldots$$

converge to a solution. This idea works under various sets of hypotheses; one such set is the following.

■ THEOREM 6. *Let the vector function* $\mathbf{X}(\mathbf{x}; t)$ *be continuous and satisfy the Lipschitz condition* (4) *on the interval* $|t-a| \leqq T$ *for all* $\mathbf{x}, \mathbf{y}$. *Then, for any constant vector* $\mathbf{c}$, *the vector* DE $\mathbf{x}'(t) = \mathbf{X}(\mathbf{x}; t)$ *has a solution defined on the interval* $|t-a| \leqq T$, *which satisfies the initial condition* $\mathbf{x}(a) = \mathbf{c}$.

Proof — As remarked at the end of the preceding section, the operator U is defined by (12) for all functions $\mathbf{x}(t)$ continuous for $|t-a| \leqq T$. In particular, since $U\mathbf{x}$ is again a continuous function on $|t-a| \leqq T$, the function $\mathbf{x}^2 = U^2[\mathbf{x}] = U^2\mathbf{x}$ is well-defined. Similarly,

the *iterates* $U^3\mathbf{x}$, $U^4\mathbf{x}$, etc., are
converge; a typical case is de

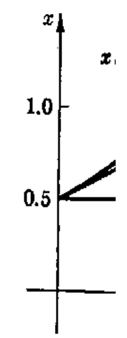

FIGURE 6.1 Picard A

■ LEMMA. *If* $\mathbf{x}^0(t) \equiv \mathbf{c}$, *the se*
$\mathbf{x}^1 = U[\mathbf{x}^0]$, $\mathbf{x}^2 = U[\mathbf{x}^1] = U$
converges uniformly for $|t-a|$

Proof — Let $M = \sup_{|t-a|\leqq}$
continuous functions are boun
generality we can assume that
is $0 \leqq t \leqq T$; the proof for gen
this case by the substitutions
By the basic inequality (3)
$\mathbf{x}^1(t)$ satisfies the inequality

$$|\mathbf{x}^1(t) - \mathbf{x}^0(t)| = \left|\int_0^t \mathbf{X}(\mathbf{x}^0(s), s)\,ds\right.$$

Again, by (3), the function $\mathbf{x}^2$

$$|\mathbf{x}^2(t) - \mathbf{x}^1(t)| = \left|\int_0^t\right.$$

$$\leqq \int_0^t$$

We now use the assumption
condition with Lipschitz const

$$|\mathbf{x}^2(t) - \mathbf{x}^1(t)| \leqq \int_0^t |\mathbf{X}(\mathbf{x}^1(s)$$

$$\leqq \int_0^t L|\mathbf{x}^1(s)$$

$= -\ln(1-t)$. Theorem 5 shows
equation $x(t) = \int_0^t e^{x(s)}\, ds$, which
on. Since the solution is defined
again that only a *local* existence

m of finding a solution to the
l in terms of operators on vector
operator $\mathbf{y} = U[\mathbf{x}] = U\mathbf{x}$, trans-
unctions $\mathbf{y}$ by the identity

$$\int_a^t \mathbf{X}(\mathbf{x}(s), s)\, ds. \qquad (12)$$

$-a| \leq T$ and is continuous, the
to be the family of continuous
l $|t-a| \leq T$; its range consists
ctor functions defined in this
se, Theorem 5 has the following.

ution satisfying $\mathbf{x}(a) = \mathbf{c}$ *if and*
nt in $\mathscr{C}[a, b]$.

all $\mathbf{x}$, the domain of the operator
s is done in Theorem 8 below.

integral operator U defined by
trial function $\mathbf{x}^0$, the successive
ions)

$[\mathbf{x}^1], \quad \mathbf{x}^3 = U[\mathbf{x}^2], \ldots$

under various sets of hypotheses;

$\mathbf{X}(\mathbf{x}; t)$ *be continuous and satisfy*
$t - a| \leq T$ *for all* $\mathbf{x}, \mathbf{y}$. *Then, for*
$= \mathbf{X}(\mathbf{x}; t)$ *has a solution defined on*
initial condition $\mathbf{x}(a) = \mathbf{c}$.

of the preceding section, the
functions $\mathbf{x}(t)$ continuous for
again a continuous function on
$U^2\mathbf{x}$ is well-defined. Similarly,

the *iterates* $U^3\mathbf{x}$, $U^4\mathbf{x}$, etc., are well-defined. These iterates will always converge; a typical case is depicted in Figure 6.1.

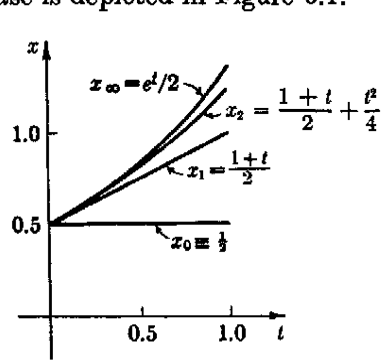

FIGURE 6.1 Picard Approximation for $dx/dt = x$, $x(0) = 1/2$.

■ LEMMA. *If* $\mathbf{x}^0(t) \equiv \mathbf{c}$, *the sequence of functions defined recursively by* $\mathbf{x}^1 = U[\mathbf{x}^0], \quad \mathbf{x}^2 = U[\mathbf{x}^1] = U^2[\mathbf{x}^0], \quad \ldots, \quad \mathbf{x}^n = U[\mathbf{x}^{n-1}] = U^n[\mathbf{x}^0], \ldots$ *converges uniformly for* $|t-a| \leq T$.

Proof — Let $M = \sup_{|t-a|\leq T} |\mathbf{X}(\mathbf{c}; t)|$; the number M is finite because continuous functions are bounded on a closed interval. Without loss of generality we can assume that $a = 0$ and $t \geq a$, that is, that the interval is $0 \leq t \leq T$; the proof for general a and for $t < a$ can be deduced from this case by the substitutions $t \to t + a$ and $t \to a - t$.

By the basic inequality (3) for vector-valued functions, the function $\mathbf{x}^1(t)$ satisfies the inequality

$$|\mathbf{x}^1(t) - \mathbf{x}^0(t)| = \left|\int_0^t \mathbf{X}(\mathbf{x}^0(s), s)\, ds\right| \leq \int_0^t \left|\mathbf{X}(\mathbf{x}^0, s)\right| ds \leq M \int_0^t ds = Mt. \qquad (13)$$

Again, by (3), the function $\mathbf{x}^2 = U[\mathbf{x}^1]$ satisfies the inequality

$$|\mathbf{x}^2(t) - \mathbf{x}^1(t)| = \left|\int_0^t [\mathbf{X}(\mathbf{x}^1(s), s) - \mathbf{X}(\mathbf{x}^0(s), s)]\, ds\right|$$
$$\leq \int_0^t |\mathbf{X}(\mathbf{x}^1(s), s) - \mathbf{X}(\mathbf{x}^0(s), s)|\, ds.$$

We now use the assumption that the function $\mathbf{X}$ satisfies a Lipschitz condition with Lipschitz constant L. This gives, by (13), the inequality

$$|\mathbf{x}^2(t) - \mathbf{x}^1(t)| \leq \int_0^t |\mathbf{X}(\mathbf{x}^1(s), s) - \mathbf{X}(\mathbf{x}^0(s), s)|\, ds$$
$$\leq \int_0^t L|\mathbf{x}^1(s) - \mathbf{x}^0(s)|\, ds \leq L \int_0^t Ms\, ds = LMt^2/2.$$

Similarly, for any $n = 1, 2, 3, \ldots,$

$$|\mathbf{x}^{n+1}(t) - \mathbf{x}^n(t)| \leqq \int_0^t |\mathbf{X}(\mathbf{x}^n(s), s) - \mathbf{X}(\mathbf{x}^{n-1}(s), s)|\, ds$$
$$\leqq L \int_0^t |\mathbf{x}^n(s) - \mathbf{x}^{n-1}(s)|\, ds.$$

We now proceed by induction. Assuming that

$$|\mathbf{x}^n(t) - \mathbf{x}^{n-1}(t)| \leqq (M/L)(Lt)^n/n!,$$

we infer that

$$|\mathbf{x}^{n+1}(t) - \mathbf{x}^n(t)| \leqq L\left(\frac{M}{L}\right)\int_0^t \frac{(Ls)^n}{n!}\, ds = M(Lt)^{n+1}/L(n+1)! \qquad (14)$$

Next, we show that the sequence of functions $\mathbf{x}^n(t)$ $(n = 0, 1, 2, \ldots)$ is uniformly convergent for $0 \leqq t \leqq T$. Indeed, the infinite series

$$(M/L)\sum_{n=0}^{\infty} (Lt)^{n+1}/(n+1)!$$

of positive terms is convergent to $(M/L)(e^{Lt} - 1)$, and uniformly convergent for $0 \leqq t \leqq T$. Hence, by the Comparison Test,† the series $x^0(t) + \sum_{k=0}^{\infty} [\mathbf{x}^{k+1}(t) - \mathbf{x}^k(t)]$ is uniformly convergent for $0 \leqq t \leqq T$. The nth partial sum of this series is the function $\mathbf{x}^n(t)$. It follows that the sequence of functions $\mathbf{x}^n(t)$ is uniformly convergent. This completes the proof of the lemma.

To complete the proof of Theorem 6, let $\mathbf{x}^\infty(t)$ denote the limit function of the sequence $\mathbf{x}^n(t)$; it suffices by Theorem 5 to show that $\mathbf{x}^\infty(t)$ is a solution of the integral equation (11). To this end, we consider the limit of the equations $\mathbf{x}^{n+1} = U[\mathbf{x}^n]$, namely the equations

$$\mathbf{x}^{n+1}(t) = \mathbf{c} + \int_0^t \mathbf{X}(\mathbf{x}^n(s), s)\, ds.$$

The left side converges uniformly, by the preceding lemma. By the Lipschitz condition, $|\mathbf{X}(\mathbf{x}^m(s), s) - \mathbf{X}(\mathbf{x}^n(s), s)| \leqq L|\mathbf{x}^m(s) - \mathbf{x}^n(s)|$, and so the integrand on the right side also converges uniformly. Since the integrands on the right side form a uniformly convergent sequence, their indefinite integrals are also uniformly convergent‡; hence, passing to the limit, we have

$$\mathbf{x}^\infty(t) = \mathbf{c} + \int_0^t \mathbf{X}(\mathbf{x}^\infty(s), s)\, ds.$$

This demonstrates (11) and completes the proof of Theorem 6.

† Courant and John, p. 535; see also Widder, p. 285.
‡ Courant and John, p. 537; Widder, p. 304.

EXERCISES D

In Exercises 1 through 5, solve

1. $u(t) = 1 + \int_0^t su(s)\, ds.$
3. $u(t) + e^t = \int_0^t su(s)\, ds.$
5. $u(t) = \int_0^t [u(s) + v(s)]\, ds, \quad v(t)$

6. Show that the nth iterate for th
sum of the first $n + 1$ terms of the

For the initial value problems in
for the nth function of the sequenc
exact solutions.

7. $dx/dt = x, \quad x(0) = 1.$
8. $dx/dt = y, \quad dy/dt = -4x.$
$x(0) = 0, \quad y(0) = 1.$

For the initial value problems
x^1, x^2, x^3 of the sequence of Picard

11. $dx/dt = x^2 + t^2, \quad x(0) = 0.$
12. $dx/dt = y^2 + t^2, \quad dy/dt = x^2 +$
13. $dx/dt = x(1 - 2t), \quad x(0) = 1.$
14*. Show that, in Ex. 13, the sequ
all t, but that this is not so in Ex. 1
15. Let $\mathbf{X}(\mathbf{x}, t) = A\mathbf{x}$, where A is a
of the nth Picard approximation
degree at most n.
16. Establish the following inequa
tions.

$$|\mathbf{x}^n(t) - \mathbf{x}^{n-1}(t)| \leqq \frac{M}{L}\left(\frac{(L|t-a|)^n}{n!}\right.$$

8 Linear systems

A first-order system of DE's
form

$$dx_i/dt = \sum_{j=1}^n a_{ij}(t)$$

In this case, we have $X_i(\mathbf{x}, t) =$
the linear system (15) is writte

$$d\mathbf{x}/dt$$

where $A(t)\,\mathbf{x}$ stands† for the m
$\mathbf{b}$ stands for the vector $(b_1, \ldots$

† For the definition and elementary p
Chapters 8, 10.

$), s) - \mathbf{X}(\mathbf{x}^{n-1}(s), s)|\, ds$

$) - \mathbf{x}^{n-1}(s)|\, ds.$

uming that

$(M/L)(Lt)^n/n!,$

$ds = M(Lt)^{n+1}/L(n+1)! \qquad (14)$

functions $\mathbf{x}^n(t)\ (n = 0, 1, 2, \ldots)$ is
Indeed, the infinite series

$/(n+1)!$

$/L)(e^{Lt} - 1)$, and uniformly con-
e Comparison Test,† the series
mly convergent for $0 \leq t \leq T$.
e function $\mathbf{x}^n(t)$. It follows that
rmly convergent. This completes

n 6, let $\mathbf{x}^\infty(t)$ denote the limit
ices by Theorem 5 to show that
ion (11). To this end, we consider
], namely the equations

$\mathbf{x}^n(s), s)\, ds.$

the preceding lemma. By the
$\mathbf{x}^n(s), s)| \leq L|\mathbf{x}^m(s) - \mathbf{x}^n(s)|$, and
converges uniformly. Since the
uniformly convergent sequence,
mly convergent‡; hence, passing

$\mathbf{x}^\infty(s), s)\, ds.$

the proof of Theorem 6.

. 285.

EXERCISES D

In Exercises 1 through 5, solve the integral equations specified.

1. $u(t) = 1 + \int_0^t su(s)\, ds.$
2. $u(t) = 1 + \int_0^t su^2(s)\, ds.$
3. $u(t) + e^t = \int_0^t su(s)\, ds.$
4. $u(t) = 1 - \int_0^t u(s) \tan s\, ds.$
5. $u(t) = \int_0^t [u(s) + v(s)]\, ds, \qquad v(t) = 1 - \int_0^t u(s)\, ds.$

6. Show that the nth iterate for the solution of $y' = yx$ such that $y(0) = 1$ is the sum of the first $n + 1$ terms of the power series expansion of $e^{x^2/2}$.

For the initial value problems in Exercises 7 through 10, obtain an expression for the nth function of the sequence of Picard approximations $\mathbf{x}^n = U^n[\mathbf{x}^0]$ to the exact solutions.

7. $dx/dt = x, \quad x(0) = 1.$
8. $dx/dt = y, \quad dy/dt = -4x.$ $\quad x(0) = 0, \quad y(0) = 1.$
9. $dx/dt = tx, \quad x(0) = 1.$
10. $dx/dt = ty, \quad dy/dt = -tx.$ $\quad x(0) = 0, \quad y(0) = 1.$

For the initial value problems of Exercises 11–13, compute the functions x^1, x^2, x^3 of the sequence of Picard approximations.

11. $dx/dt = x^2 + t^2, \quad x(0) = 0.$
12. $dx/dt = y^2 + t^2, \quad dy/dt = x^2 + t^2, \quad x(0) = y(0) = 0.$
13. $dx/dt = x(1 - 2t), \quad x(0) = 1.$

14*. Show that, in Ex. 13, the sequence of Picard approximations converges for all t, but that this is not so in Ex. 11. In Ex. 13, is the convergence uniform?

15. Let $\mathbf{X}(\mathbf{x}, t) = A\mathbf{x}$, where A is a constant matrix. Show that each component of the nth Picard approximation to any solution is a polynomial function of degree at most n.

16. Establish the following inequalities for the sequence of Picard approximations.

$$|\mathbf{x}^n(t) - \mathbf{x}^{n-1}(t)| \leq \frac{M}{L}\left(\frac{(L|t-a|)^n}{n!}\right), \qquad |\mathbf{x}^n(t) - \mathbf{x}^\infty(t)| \leq \frac{M}{L}\sum_{k=n+1}^{\infty}\frac{(L|t-a|)^k}{k!}.$$

8 Linear systems

A first-order system of DE's (1) is said to be *linear* when it is of the form

$$dx_i/dt = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + b_i(t), \qquad 1 \leq i \leq n. \tag{15}$$

In this case, we have $X_i(\mathbf{x}, t) = \sum_{j=1}^n a_{ij}(t)x_j + b_i(t)$. In vector notation, the linear system (15) is written in the form

$$d\mathbf{x}/dt = A(t)\,\mathbf{x} + \mathbf{b}(t), \tag{16}$$

where $A(t)\,\mathbf{x}$ stands† for the matrix $\|a_{ij}(t)\|$ applied to the vector $\mathbf{x}$, and $\mathbf{b}$ stands for the vector $(b_1, \ldots, b_n)$.

† For the definition and elementary properties of matrices, see Birkhoff and Mac Lane, Chapters 8, 10.

When $\mathbf{b}(t) \equiv 0$, the system (16) is said to be *homogeneous*. Otherwise, it is called *inhomogeneous*. The homogeneous system obtained from a given inhomogeneous system (15) by setting the b_j equal to zero is called the *reduced* system associated with (15).

A basic property of a linear system of DE's (16) is that the difference $\mathbf{x} - \mathbf{y}$ of any two solutions of (16) is a solution of the reduced system. It can be immediately verified that any linear combination $a\mathbf{x}(t) + b\mathbf{y}(t)$ of solutions $\mathbf{x}(t)$ and $\mathbf{y}(t)$ of a homogeneous linear system is again a solution.

We shall now establish the existence of solutions of linear systems and describe the set of all solutions.

■ Lemma. *Any linear system* (15) *with continuous coefficient functions on a closed interval I satisfies a Lipschitz condition* (4) *with*

$$L = \sum_{i,j} \sup_{t \in I} |a_{ij}(t)|. \tag{17}$$

Proof — Since $\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)$ is the vector sum of n^2 vectors $\mathbf{z}_{ij}$, with ith component $a_{ij}(x_j - y_j)$ and other components zero, repeated use of the triangle inequality gives

$$|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)| \leqq \sum_{i,j} |\mathbf{z}_{ij}| \leqq \sum_{i,j} |a_{ij}(t)| \cdot |x_j - y_j|$$
$$\leqq \sum_{i,j} \sup_{t \in I} |a_{ij}(t)| \cdot |\mathbf{x} - \mathbf{y}|.$$

The functions $a_{ij}(t)$, being continuous on a closed interval, are bounded.† Hence, the Lipschitz constant L of (17) is finite. This completes the proof of the lemma.

We can now state the existence theorem for linear systems.

■ Theorem 7. *A linear system* (15), *with the $a_{ij}(t)$ and $b_i(t)$ defined and continuous for $|t - a| \leqq T$, has a unique solution on $|t - a| \leqq T$ satisfying any given initial condition $\mathbf{x}(a) = \mathbf{c}$.*

Proof — The preceding lemma shows that such a system satisfies the hypotheses of Theorem 6. This gives the existence of the solution. The uniqueness follows from Theorem 1, again by the preceding lemma.

For *homogeneous* systems, we can construct a basis of solutions, as follows.

■ Corollary 1. *Let $\mathbf{x}^i(t)$ be the solution of a homogeneous linear system $d\mathbf{x}/dt = A(t)\mathbf{x}$ which satisfies the initial condition $x_k^i(a) = 0$, $i \neq k$, $x_i^i(a) = 1$.*

† Courant and John, p. 101.

Then the solution satisfying th
is equal to the linear combinatic

Proof — The vector-valuec
solution of the linear system, si
This function satisfies the init
of the way in which the initial
chosen. Since the identically
linear system, it follows from tl
q.e.d.†

The reduction of an nth o
sketched in § 1, when applied t

$$\frac{d^n u}{dt^n} = p_1(t) \frac{d^{n-1}u}{dt^{n-1}} + p_2(t)$$

transforms the DE into a ho
where the matrix $\|a_{ij}(t)\| =$
$1 \leqq i \leqq n - 1$ and $j \neq i + 1$;
$p_{n-j+1}(t)$.

We therefore obtain the foll

■ Corollary 2. *An nth order*
continuous for $|t - a| \leqq T$, ha
satisfying the initial‡ conditions

More results about solutions
in the Appendix.

9 Local existence theorem

In Theorem 6, it was assum
that a Lipschitz condition hel
case. For instance, this assump
of Example 4. The ratio

$$|X(x, t) - X(0$$

is unbounded if the domain of

† In algebraic terms, Corollary 1 states
of dimension n form an n-dimensional
solutions of such a system are always
‡ δ_j^i is the Kronecker delta function;
basis of solutions of nth order DE'S, s
†† The same complications arise with
Chapter 3, § 9.

id to be *homogeneous*. Otherwise
geneous system obtained from
etting the b_j equal to zero is called
5).
of DE's (16) is that the difference
a solution of the reduced system
y linear combination $a\mathbf{x}(t) + b\mathbf{y}(t)$
geneous linear system is again a

ce of solutions of linear systems

ith continuous coefficient functions
tz condition (4) *with*

$_{I}|a_{ij}(t)|.$ (17)

he vector sum of n^2 vectors $\mathbf{z}_{ij}$,
other components zero, repeated

$| \leq \sum_{i,j} |a_{ij}(t)| \cdot |x_j - y_j|$

$_{\in I}|a_{ij}(t)| \cdot |\mathbf{x}-\mathbf{y}|.$

ous on a closed interval, are
stant L of (17) is finite. This

orem for linear systems.

vith the $a_{ij}(t)$ and $b_i(t)$ defined and
solution on $|t-a| \leq T$ satisfying

s that such a system satisfies the
he existence of the solution. The
gain by the preceding lemma.
onstruct a basis of solutions, as

m of a homogeneous linear system
ndition $x_k^i(a) = 0, i \neq k, x_i^i(a) = 1$.

Then the solution satisfying the initial condition $\mathbf{x}(a) = \mathbf{c} = (c_1, \ldots, c_n)$ *is equal to the linear combination* $\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\,\mathbf{x}^2(t) + \cdots + c_n\mathbf{x}^n(t)$.

Proof — The vector-valued function $\mathbf{y}(t) = \mathbf{x}(t) - \sum_{j=1}^{n} c_j\mathbf{x}^j(t)$ is a solution of the linear system, since it is a linear combination of solutions. This function satisfies the initial condition $\mathbf{y}(a) = (0, 0, \ldots, 0)$ because of the way in which the initial conditions for the solutions $\mathbf{x}^i$ have been chosen. Since the identically zero function is also a solution of the linear system, it follows from the uniqueness in Theorem 7 that $\mathbf{y}(t) \equiv 0$, q.e.d.†

The reduction of an nth order normal DE to a first-order system sketched in § 1, when applied to a linear nth order DE in normal form

$$\frac{d^n u}{dt^n} = p_1(t)\frac{d^{n-1}u}{dt^{n-1}} + p_2(t)\frac{d^{n-2}u}{dt^{n-2}} + \cdots + p_{n-1}(t)\frac{du}{dt} + p_n(t)u,$$

transforms the DE into a homogeneous linear system $d\mathbf{x}/dt = A(t)\mathbf{x}$, where the matrix $\|a_{ij}(t)\| = A(t)$ is defined as follows: $a_{ij}(t) = 0$ if $1 \leqq i \leqq n-1$ and $j \neq i+1$; $a_{i,\,i+1}(t) = 1$ if $1 \leqq i \leqq n-1$; $a_{nj}(t) = p_{n-j+1}(t)$.

We therefore obtain the following result.

■ Corollary 2. *An nth order* DE *in normal form, with coefficients $p_j(t)$ continuous for $|t-a| \leq T$, has a basis of solutions $u_j(t)$ $(1 \leqq j \leqq n)$ satisfying the initial‡ conditions $u_j^{(i)}(0) = \delta_j^{i+1}$, $0 \leqq i \leqq n-1$.*

More results about solutions of linear systems of DE's will be derived in the Appendix.

9 Local existence theorem

In Theorem 6, it was assumed that $\mathbf{X}(\mathbf{x}, t)$ was defined for all $\mathbf{x}$ and that a Lipschitz condition held for all $\mathbf{x}, \mathbf{y}$. But often this is not the case. For instance, this assumption does not hold for the DE $dx/dt = e^x$ of Example 4. The ratio

$$|X(x, t) - X(0, t)|/|x - 0| = (e^x - 1)/x$$

is unbounded if the domain of e^x is unrestricted††.

† In algebraic terms, Corollary 1 states that the solutions of a homogeneous linear system of dimension n form an n-dimensional vector space of functions. Therefore, any $n+1$ solutions of such a system are always linearly dependent.

‡ δ_j^i is the Kronecker delta function: $\delta_j^i = 0$ if $i \neq j$ and $\delta_i^i = 1$. For the concept of a basis of solutions of nth order DE'S, see Chapter 4, § 4.

†† The same complications arise with the DE $y' = 1 + y^2$, as has already been shown in Chapter 3, § 9.

Correspondingly, the conclusion of Theorem 6 fails for this DE: the solution which takes the value c at $t = 0$ is the function $x(t) = -\ln(e^{-c} - t)$, and this function is defined only in the interval $-\infty < t < e^{-c}$. Hence, there is *no* $\epsilon > 0$ such that the DE $dx/dt = e^x$ has a solution defined on all of $|t| < \epsilon$ for every initial value: the interval of definition of a solution changes with the initial value.

To take care of this situation, and also of cases where the function $\mathbf{X}$ is defined only in a small region of $(x_1, \ldots, x_n)$-space, we now prove a *local* existence theorem, whose assumptions and conclusions refer only to neighborhoods of a given point.

■ THEOREM 8. *Suppose that the function* $\mathbf{X}(\mathbf{x}, t)$ *in* (2) *is defined and continuous in the closed domain* $|\mathbf{x} - \mathbf{c}| \leq K$, $|t - a| \leq T$ *and satisfies a Lipschitz condition* (4) *there. Let* $M = \sup|\mathbf{X}(\mathbf{x}, t)|$ *in this domain. Then the* DE (2) *has a unique solution satisfying* $\mathbf{x}(a) = \mathbf{c}$ *and defined on the interval* $|t - a| \leq \min(T, K/M)$.

Proof — All steps in the proof of Theorem 6 can be carried out, provided we know that the functions $\mathbf{x}^n(t)$ referred to there take their values within the domain D_1: $|\mathbf{x} - \mathbf{c}| \leq K$, $|t - a| \leq \min(T, K/M)$, in which $\mathbf{x}(t)$ is surely defined. In particular, note that since $D_1 \subset D$, the bound M and Lipschitz constant L of Theorem 6 can be used in D_1. Therefore, the proof is a corollary of the following

■ LEMMA. *Under the hypotheses of Theorem* 8, *the operator* U *defined by* (12) *carries functions* $\mathbf{x}(t)$ *satisfying the conditions* (i) $\mathbf{x}(t)$ *is defined and continuous on* $|t - a| \leq \min(T, K/M)$, (ii) $\mathbf{x}(a) = \mathbf{c}$, (iii) $|\mathbf{x}(t) - \mathbf{c}| \leq K$ *on the interval* $|t - a| \leq \min(T, K/M)$, *into functions satisfying the same conditions.*

Proof — In (12), suppose that $\mathbf{x}(s)$ satisfies conditions (i), (ii), (iii). We must show that $\mathbf{y}(t)$ satisfies the same conditions. Clearly (i) and (ii) are satisfied by $\mathbf{y}(t)$. By the inequality (3) we have (taking again $t \geq a$ for simplicity)

$$|\mathbf{y}(t) - \mathbf{c}| = \left|\int_a^t \mathbf{X}(\mathbf{x}(s), s)\, ds\right| \leq \int_a^t |\mathbf{X}(\mathbf{x}(s), s)|\, ds.$$

If M is the maximum of $\mathbf{X}$ and if $|t - a| \leq K/M$, this gives

$$|\mathbf{y}(t) - \mathbf{c}| \leq MK/M = K.$$

Therefore, (iii) is satisfied and $\mathbf{y}(t)$ is defined for $|t - a| \leq \min(T, K/M)$, completing the proof.

Using the reduction of § 1, t

$$u^{(n)} = F(u$$

into an equivalent first-order r

■ COROLLARY. *Let the functi the cylinder* $|t - a| \leq T$, $|\mathbf{x} - \mathbf{c}|$ *M, and let F satisfy a Lipsch* $|t - a| \leq \min(T, K/M)$, *the* DI *satisfies the initial conditions* $u^{(i}$

10* Analytic equations

We shall now consider the that $\mathbf{X}(\mathbf{x}, t)$ is an analytic f The essential principle to be es DE's *are analytic functions.*†

The result is true whether th first consider the complex case. complex variables, we rewrite

$$d\mathbf{z}/dt = \mathbf{z}'(t) =$$

where $z_j = x_j + iy_j$ and $Z_j = X$

We assume that the $Z_j(z_1,$ variables $z_1, z_2, \ldots, z_n$ and t $|t - a| \leq T$, $|\mathbf{z} - \mathbf{c}| \leq K$, with n this implies that a Lipschitz co

Vector notation can be adapt changes. The length (or norn complex components z_k is defin

$$|\mathbf{z}| = (z_1 z_1^* +$$

The *Hermitian inner product* of

$$\mathbf{w} = ($$

is defined as

$$\mathbf{z} \cdot \mathbf{w} = (z_1 w_1^*$$

note that $\mathbf{z} \cdot \mathbf{w} = (\mathbf{w} \cdot \mathbf{z})^*$. (T above inner product is called a

† This section requires a knowledge o found in the books by Hille (Vol. 1) an

Theorem 6 fails for this DE: at $t = 0$ is the function $x(t) =$ efined only in the interval $-\infty$ ch that the DE $dx/dt = e^x$ has a ery initial value: the interval of e initial value.
also of cases where the function $_1, \ldots, x_n$)-space, we now prove a tions and conclusions refer only

tion $\mathbf{X}(\mathbf{x}, t)$ *in* (2) *is defined and* $\leqq K$, $|t - a| \leqq T$ *and satisfies a* $\sup|\mathbf{X}(\mathbf{x}, t)|$ *in this domain. Then fying* $\mathbf{x}(a) = \mathbf{c}$ *and defined on the*

Theorem 6 can be carried out, $\mathbf{x}^n(t)$ referred to there take their $\leqq K$, $|t - a| \leqq \min(T, K/M)$, in icular, note that since $D_1 \subset D$, L of Theorem 6 can be used in of the following

eorem 8, *the operator U defined by conditions* (i) $\mathbf{x}(t)$ *is defined and* (ii) $\mathbf{x}(a) = \mathbf{c}$, (iii) $|\mathbf{x}(t) - \mathbf{c}| \leqq K$ *into functions satisfying the same*

satisfies conditions (i), (ii), (iii). ame conditions. Clearly (i) and ality (3) we have (taking again

$$\Big| \leqq \int_a^t |\mathbf{X}(\mathbf{x}(s), s)|\, ds.$$

$a| \leqq K/M$, this gives

$$/M = K.$$

fined for $|t - a| \leqq \min(T, K/M)$,

Using the reduction of § 1, taking an nth-order normal DE

$$u^{(n)} = F(u, u', u'', \ldots, u^{(n-1)}, t) \tag{18}$$

into an equivalent first-order normal system (1), we obtain the

■ Corollary. *Let the function* $F(x_1, x_2, \ldots, x_n, t)$ *be continuous in the cylinder* $|t - a| \leqq T$, $|\mathbf{x} - \mathbf{c}| \leqq K$. *Let* $(x_2^2 + x_3^2 + \cdots + x_n^2 + F^2)^{1/2} \leqq M$, *and let F satisfy a Lipschitz condition there. Then, on the interval* $|t - a| \leqq \min(T, K/M)$, *the* DE (18) *has one and only one solution which satisfies the initial conditions* $u^{(i)}(a) = c_{i+1}$, $0 \leqq i \leqq n - 1$.

10* Analytic equations

We shall now consider the vector DE (2) under the assumption that $\mathbf{X}(\mathbf{x}, t)$ is an analytic function of all variables $x_1, \ldots, x_n, t$. The essential principle to be established is that *all solutions of analytic* DE's *are analytic functions.*†

The result is true whether the variables are real or complex; we shall first consider the complex case. To emphasize that we are dealing with complex variables, we rewrite the vector DE (2) as

$$d\mathbf{z}/dt = \mathbf{z}'(t) = \mathbf{Z}(\mathbf{z}, t), \qquad t = r + is, \tag{19}$$

where $z_j = x_j + iy_j$ and $Z_j = X_j + iY_j$ are complex-valued functions.

We assume that the $Z_j(z_1, \ldots, z_n, t)$ are analytic functions of the variables $z_1, z_2, \ldots, z_n$ and t in the closed cylindrical domain C: $|t - a| \leqq T$, $|\mathbf{z} - \mathbf{c}| \leqq K$, with maximum M there. By the lemma of § 2, this implies that a Lipschitz condition holds in C, for some constant L.

Vector notation can be adapted to complex vectors with the following changes. The length (or norm) of a vector $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ with complex components z_k is defined as

$$|\mathbf{z}| = (z_1 z_1^* + z_2 z_2^* + \cdots + z_n z_n^*)^{1/2}.$$

The *Hermitian inner product* of two complex vectors $\mathbf{z}$ and

$$\mathbf{w} = (w_1, w_2, \ldots, w_n)$$

is defined as

$$\mathbf{z} \cdot \mathbf{w} = (z_1 w_1^* + z_2 w_2^* + \cdots + z_n w_n^*).$$

note that $\mathbf{z} \cdot \mathbf{w} = (\mathbf{w} \cdot \mathbf{z})^*$. (The set of complex n-vectors with the above inner product is called a unitary space.)

† This section requires a knowledge of elementary complex function theory such as is found in the books by Hille (Vol. 1) and Ahlfors.

Now let γ be any path in the complex t-plane, defined parametrically by the equation $t = t(\sigma) = r(\sigma) + is(\sigma)$, where $r, s \in \mathscr{C}^1$ and σ is a *real* parameter. On the path γ, (19) is equivalent to the system of *real* DE's

$$\begin{aligned} x'(\sigma) &= \mathbf{X}(\mathbf{x}, \mathbf{y}, \sigma)r'(\sigma) - \mathbf{Y}(\mathbf{x}, \mathbf{y}, \sigma)s'(\sigma), \\ y'(\sigma) &= \mathbf{X}(\mathbf{x}, \mathbf{y}, \sigma)s'(\sigma) + \mathbf{Y}(\mathbf{x}, \mathbf{y}, \sigma)r'(\sigma). \end{aligned} \tag{19'}$$

Theorems 1 through 8 apply to this system, which satisfies a Lipschitz condition.

Using the complex vector notation described above, we can also prove analogs of these theorems directly, for the DE $\mathbf{z}'(\sigma) = \mathbf{Z}(\mathbf{z}, t(\sigma))t'(\sigma)$ equivalent to (19′), and hence to (19), on the path γ.

The analog of the operator U of formula (12) is the operator W, defined by the line integral

$$\mathbf{w}(t) = W[\mathbf{z}(t)] = \mathbf{c} + \int_a^t \mathbf{Z}(\mathbf{z}(\zeta).\ \zeta)\, d\zeta. \tag{20}$$

Since each component function Z_j is analytic, the line integral defining the operator W is independent of the path from 0 to t in the complex t-plane, provided that this path stays within the domain C where the function $\mathbf{Z}$ is defined.† By Morera's theorem, the function $\mathbf{w}$ is therefore also analytic, in the sense that each component $w_j(z)$ is. Thus, the operator W transforms analytic functions into analytic functions. Moreover, the lemma of § 9 still holds, because the integrals in (20) can be taken along straight line segments in the complex ζ-plane. This gives the following lemma.

■ LEMMA. *For $|t - a| \leqq \min(T, K/M)$, the operator W defined by* (20) *takes analytic complex-valued vector functions $\mathbf{z}(t)$ with $|\mathbf{z}(t) - \mathbf{c}| \leqq K$ into analytic vector functions $\mathbf{w}(t)$ with $|\mathbf{w}(t) - \mathbf{c}| \leqq K$.*

By repeated applications of this lemma, it follows that the functions $W^n[\mathbf{w}^0] = \mathbf{w}^n(t)$ defined by the Picard process of iterated quadrature, in the domain $|t - a| \leqq \min(T, K/M)$ of the complex t-plane, are all analytic.

We now apply the following result‡ from function theory.

Weierstrass Convergence Theorem. If a sequence $\{f_n(t)\}$ of complex analytic functions converges uniformly to $f(t)$ in a domain D, then $f(t)$ is analytic in D.

† This is true because the disc $|t - a| \leqq T$ where $\mathbf{Z}$ is defined is *simply connected*.

‡ Ahlfors p. 173. The result contrasts sharply with the case of functions of a real variable. By the Weierstrass approximation theorem, every continuous function on a real interval $a \leqq x \leqq b$ is a uniform limit of polynomial (hence analytic) functions.

By this theorem, the sequenc
for $|t - a| \leqq \min(T, K/M)$ to
equation

$$\mathbf{z}(t) = \mathbf{c} + \int_a^t \mathbf{Z}$$

and hence of the complex DE
of § 8 for *real* DE's to the syst

■ THEOREM 9. *In Theorem 8,*
complex variables t, z_j, Z_j. Un
complex analytic functions, the
analytic solution $\mathbf{z}(t)$ for given i

From this result and the uni
obtain

■ COROLLARY 1. *Let $\mathbf{Z}(\mathbf{z}, t)$ be*
of the complex t-plane, and let $\mathbf{z}$
$\mathbf{z}(t)$ is analytic.†

Real analytic DE's. A *real* fu
and t is said to be analytic at
power series with real coefficier
convergent in the cylinder $|\mathbf{x} -$
positive η and ϵ. When $X(\mathbf{x}, t)$ i
also in the *complex* cylinder |
defines a complex-valued analy

Now, let a normal system of
analytic. From Theorem 9, it
$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t)$ has a unique c
initial values. On the other
solution by Theorems 1 and 8.
proving

■ COROLLARY 2. *If $\mathbf{X}(\mathbf{x}, t)$ is*
ables $x_1, \ldots, x_n$ and t, then ever

EXERCISES E

1. (a) Obtain an equivalent first-
term of the Picard sequen
$x'(0) = 0$.

† An alternative proof can be based
continuously differentiable. Hence,
Vol. 1, pp. 72, 88).

x t-plane, defined parametrically
, where $r, s \in \mathscr{C}^1$ and σ is a *real*
valent to the system of *real* DE's

$$- \mathbf{Y}(\mathbf{x}, \mathbf{y}, \sigma)s'(\sigma), \qquad (19')$$
$$+ \mathbf{Y}(\mathbf{x}, \mathbf{y}, \sigma)r'(\sigma).$$

stem, which satisfies a Lipschitz

ı described above, we can also
, for the DE $\mathbf{z}'(\sigma) = \mathbf{Z}(\mathbf{z}, t(\sigma))t'(\sigma)$
on the path γ.
ormula (12) is the operator W,

$$\int_a^t \mathbf{Z}(\mathbf{z}(\zeta). \zeta)\, d\zeta. \qquad (20)$$

is analytic, the line integral de-
of the path from 0 to t in the
path stays within the domain C
Morera's theorem, the function $\mathbf{w}$
that each component $w_j(z)$ is
nalytic functions into analytic
still holds, because the integrals
segments in the complex ζ-plane.

t), the operator W defined by (20)
ctions $\mathbf{z}(t)$ with $|\mathbf{z}(t) - \mathbf{c}| \leq K$ into
$- \mathbf{c}| \leq K$.

ma, it follows that the functions
process of iterated quadrature, in
of the complex t-plane, are all

from function theory.

If a sequence $\{f_n(t)\}$ of complex
to $f(t)$ in a domain D, then $f(t)$ is

e $\mathbf{Z}$ is defined is *simply connected*.
th the case of functions of a real variable.
every continuous function on a real in-
d (hence analytic) functions.

By this theorem, the sequence of functions $\mathbf{w}^n(t)$ converges uniformly for $|t - a| \leq \min(T, K/M)$ to an *analytic* solution $\mathbf{w}^\infty(t)$ of the integral equation

$$\mathbf{z}(t) = \mathbf{c} + \int_a^t \mathbf{Z}(\mathbf{z}(\zeta), \zeta)\, d\zeta = W[\mathbf{z}(t)], \qquad (21)$$

and hence of the complex DE (19). Applying the Existence Theorem of § 8 for *real* DE's to the system (21), we infer the next theorem.

■ THEOREM 9. *In Theorem* 8, *replace the real variables t, x_j, X_j with complex variables t, z_j, Z_j. Under the same hypotheses, if the $Z_j(\mathbf{z}, t)$ are complex analytic functions, the vector* DE (19) *has a unique complex analytic solution $\mathbf{z}(t)$ for given initial conditions.*

From this result and the uniqueness theorem, again for real DE's, we obtain

■ COROLLARY 1. *Let $\mathbf{Z}(\mathbf{z}, t)$ be analytic in any simply-connected domain of the complex t-plane, and let $\mathbf{z}(t)$ be any solution of the* DE (19). *Then $\mathbf{z}(t)$ is analytic.*†

Real analytic DE's. A *real* function $X(\mathbf{x}, t)$ of real variables $x_1, \ldots, x_n$ and t is said to be analytic at $(\mathbf{c}, a)$ when it can be expanded into a power series with real coefficients in the variables $(x_k - c_k)$ and $(t - a)$, convergent in the cylinder $|\mathbf{x} - \mathbf{c}| < \eta$, $|t - a| < \epsilon$, for sufficiently small positive η and ϵ. When $X(\mathbf{x}, t)$ is analytic, its power series is convergent also in the *complex* cylinder $|\mathbf{z} - \mathbf{c}| < \eta$, $|t - a| < \epsilon$ (t complex), and defines a complex-valued analytic function there.

Now, let a normal system of real DE's (1) be given, the $X_j(x, t)$ being analytic. From Theorem 9, it follows that the resulting complex DE $d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t)$ has a unique complex analytic solution for given real initial values. On the other hand, it also has a unique (local) real solution by Theorems 1 and 8. Hence the two solutions must coincide, proving

■ COROLLARY 2. *If $\mathbf{X}(\mathbf{x}, t)$ is an analytic real function of the real variables $x_1, \ldots, x_n$ and t, then every solution of* (1) *is analytic.*

EXERCISES E

1. (a) Obtain an equivalent first-order system for $d^2x/dt^2 = t^2x$. Find the nth term of the Picard sequence of iterates for the initial values $x(0) = 1$, $x'(0) = 0$.

† An alternative proof can be based directly on (19). If the $z_k(t)$ satisfy (19), they are continuously differentiable. Hence, they are analytic (Ahlfors, pp. 24, 105; Hille, Vol. 1, pp. 72, 88).

(b) Prove that this initial-value problem has one and only one solution on $(-\infty, \infty)$.

2. (a) Obtain an equivalent first-order system for the DE $d^2x/dt^2 = x^2 + t^2$, and find the Lipschitz constant for the resulting system in the domain $|t| \leqq A, |x| \leqq B, |x'| \leqq C$.

(b) State and prove a local existence theorem for solutions of this DE, for the initial conditions $x(0) = b, x'(0) = c$. Estimate the largest T, U such that a solution is defined on $-U \leqq t \leqq T$.

3. Show that, if $F(y)$ is continuous for $|y| \leqq K$, and $|F(y)| \leqq M$, every solution of $y' = F(y)$ can be uniformly approximated arbitrarily closely for $|x| \leqq K/M$ by a solution of a DE $y' = P(y)$, where P is a polynomial.

4. Compute the nth Picard approximation to the solution of the complex system $dw/dt = iz$, $dz/dt = w$, which satisfies the initial conditions $w(0) = 1, z(0) = i$.

5. In the complex t-plane, determine a domain in which the system $dw/dt = tz^2$, $dz/dt = tw^2$ has an analytic solution satisfying given initial conditions $w(0) = w_0$, $z(0) = z_0$.

6. Show that the solution of the complex analytic DE

$$w'(z) = M\left[\frac{1}{2}\left(1 + \frac{w}{K}\right)\right]^{1/n}, \qquad (|z| < K)$$

which satisfies the initial condition $w(0) = 0$, is the function

$$w(z) = K\left[\left(1 + \frac{z}{c}\right)^{n/(n-1)} - 1\right], \qquad c = \frac{2^{1/n}Kn}{(n-1)M}.$$

7*. Using the result of Exercise 6, show that the bound given by Theorem 8 for the domain of existence of a solution is "best possible" for analytic functions of a complex variable.

11 Continuation of solutions

Even when the function† $\mathbf{X}(\mathbf{x}, t)$ is of class $\mathscr{C}^1$ and is defined for all $\mathbf{x}$ and t, Theorem 8 establishes the existence of solutions only in the neighborhood of a given initial value. In other words, it establishes only the *local* existence of solutions. We shall now study how such local solutions can be joined together to give a *global* solution defined up to the boundary of the domain of definition of the function $\mathbf{X}$.

■ THEOREM 10. *Let $\mathbf{X}(\mathbf{x}, t)$ be defined and of class $\mathscr{C}^1$ in an open region $\mathscr{R}$ of $(\mathbf{x}, t)$-space. For any point $(\mathbf{c}, a)$ in the region $\mathscr{R}$, the* DE (2) *has a unique solution $\mathbf{x}(t)$ satisfying the initial condition $\mathbf{x}(a) = \mathbf{c}$ and defined for an interval $a \leqq t < b$ $(b \leqq \infty)$ such that, if $b < \infty$, either $\mathbf{x}(t)$ approaches the boundary of the region or $\mathbf{x}(t)$ is unbounded as $t \to b$.*

† In this Section we consider only *real* vectors and functions. The results can, however, be extended to complex-valued and analytic functions, in much the same way as in § 10. When (2) is analytic, the continuation of solutions described here coincides with *analytic continuation* in the sense of complex function theory, by Theorem 9 (cf. Chapter 9, § 1).

Proof — Consider the set $\mathcal{S}$
which satisfy the given initial
intervals of varying lengths of
and y in this set, defined on int
$\mathbf{z}$, defined to be equal to $\mathbf{x}$ or t
also where both are defined, i
$I \cup I'$.

We now construct a single
defined on the union of *all* the
defined, by letting $\mathbf{x}(t)$ be equa
S defined at the point t. Thi
function of class $\mathscr{C}^1$, by the
interval of definition of this so
definition and, therefore, is its

Consider the limiting beha
Weierstrass Theorem,† any in
$\mathbf{x}t$-space must contain a lin
$\lim_{t \uparrow b} |\mathbf{x}(t)| = +\infty$, or at least
least one sequence of points (x
the first case, $\mathbf{x}(t)$ is unbound
solution may be said to "reced

It remains to consider the t
by choosing the region $\mathscr{R}$ as
$\cos t^{-1}$. with general solution x

We shall now prove that, in
$(\mathbf{d}, b)$ on $t = b$ of the maximal
of $\mathscr{R}$. Indeed, suppose that it
exist a closed neighborhood D:
let $M = \max_D |\mathbf{X}|$. Take $\delta <$ m
rectangle $|\mathbf{x} - \mathbf{d}| < \epsilon$, $|t - b| <$
G. Then, applying Theorem 8
we see that it stays in G until
$\lim_{t \to b} \mathbf{x}(t) = \mathbf{d}$. Hence $\mathbf{x}(t)$ wo
(by Theorem 1) local solution
would not be maximal, a contr

The maximum length $b - a$ o
escape time of the solution for
escape time for $t < a$.

A solution with a *finite* escape
bounded as $t \to b < \infty$; on the

† Cf. Courant, Vol. 2, pp. 95 ff, whe
dimensions is proved.

Proof — Consider the set S of all local solutions of the system (2) which satisfy the given initial condition $\mathbf{x}(a) = \mathbf{c}$. These are defined on intervals of varying lengths of the form $[a, T)$. Given two solutions $\mathbf{x}$ and $\mathbf{y}$ in this set, defined on intervals I and I' respectively, the function $\mathbf{z}$, defined to be equal to $\mathbf{x}$ or to $\mathbf{y}$ wherever either is defined, and hence also where both are defined, is also a solution defined on their union $I \cup I'$.

We now construct a single solution $\mathbf{x}$, called the *maximal* solution, defined on the union of *all* the intervals in which some local solution is defined, by letting $\mathbf{x}(t)$ be equal to the value of any of the solutions of S defined at the point t. This maximal solution $\mathbf{x}(t)$ is a well-defined function of class $\mathscr{C}^1$, by the Uniqueness Theorem. Furthermore, the interval of definition of this solution is the union of all the intervals of definition and, therefore, is itself an interval of the form $a \leqq t < b$.

Consider the limiting behavior of $\mathbf{x}(t)$, as $t \uparrow b$. By the Bolzano-Weierstrass Theorem,† any infinite bounded set of points $(\mathbf{x}(t_n), t_n)$ in $\mathbf{x}t$-space must contain a limit point; hence, either $b = +\infty$, or $\lim_{t\uparrow b} |\mathbf{x}(t)| = +\infty$, or at least one finite point $(\mathbf{d}, b)$ is approached by at least one sequence of points $(\mathbf{x}(t_n), t_n)$ on the above solution curve. In the first case, $\mathbf{x}(t)$ is unbounded. In the second case, the maximal solution may be said to "recede to infinity."

It remains to consider the third case; a typical example is provided by choosing the region $\mathscr{R}$ as the left-half-plane $t < 0$ and $x'(t) = t^{-2} \cos t^{-1}$. with general solution $x = C - \sin t^{-1}$.

We shall now prove that, in the third case above, *every* limit point $(\mathbf{d}, b)$ on $t = b$ of the maximal solution curve must lie on the boundary of $\mathscr{R}$. Indeed, suppose that it were in the interior; there would then exist a closed neighborhood D: $|\mathbf{x} - \mathbf{d}| \leqq \epsilon$, $|t - b| \leqq \epsilon$ of $(\mathbf{d}, b)$ also in $\mathscr{R}$; let $M = \max_D |\mathbf{X}|$. Take $\delta < \min(\epsilon, \epsilon/2M)$, and let $G \subset D$ be the open rectangle $|\mathbf{x} - \mathbf{d}| < \epsilon$, $|t - b| < \delta$. Finally, choose k so that $(\mathbf{x}(t_k), t_k) \in G$. Then, applying Theorem 8 (in G) to the solution through $(\mathbf{x}(t_k), t_k)$, we see that it stays in G until $t = b$. Since this is true for any $\epsilon > 0$, $\lim_{t\to b} \mathbf{x}(t) = \mathbf{d}$. Hence $\mathbf{x}(t)$ would have to coincide with the unique (by Theorem 1) local solution of (2) through $(\mathbf{d}, b)$. Therefore $\mathbf{x}(t)$ would not be maximal, a contradiction.

The maximum length $b - a$ of definition of the solution $\mathbf{x}$ is called the *escape time* of the solution for $t > a$. There is a similar notion of the escape time for $t < a$.

A solution with a *finite* escape time is one for which $|\mathbf{x}(t)|$ becomes unbounded as $t \to b < \infty$; on the other hand, a solution with an infinite

† Cf. Courant, Vol. 2, pp. 95 ff, where the Bolzano-Weierstrass Theorem in several dimensions is proved.

escape time is one that remains within the domain of definition of X as $t \to \infty$. For example, every solution of the DE $dx/dt = x$ has infinite escape time, whereas every nonzero solution of the DE $dx/dt = x^2$, namely every function $x = 1/(c - t)$, has finite escape time.

12* The perturbation equation

It is easy to derive a formula for the dependence on $\mathbf{c}$ of the solution $\mathbf{x} = \mathbf{f}(t, \mathbf{c})$ of the initial-value problem defined by the system $\mathbf{x}'(t) = \mathbf{X}(\mathbf{x}, t)$ and the initial condition $\mathbf{x}(a) = \mathbf{c}$. For simplicity, consider first the case $n = 1$ of a single first-order DE. Assuming that $f(t, c)$ is analytic, that is, that f has a convergent Taylor series expansion, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial c}\right) = \frac{\partial}{\partial c}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial c}[X(f(t, c), t)] = \left[\frac{\partial X}{\partial x}(f(t, c), t)\right] \cdot \left[\frac{\partial f}{\partial c}(t, c)\right]. \tag{22}$$

When we expand around $c = 0$, this gives formally

$$f(t, c) = f_0(t) + cf_1(t) + (c^2/2!)f_2(t) + \cdots, \tag{23}$$

where, by (23), $f_1(t) = \partial f/\partial c(t, 0)$ satisfies the linear perturbation equation

$$f_1'(t) = \left[\frac{\partial X}{\partial x}(f_0(t), t)\right] f_1(t), \qquad f_1(0) = 0. \tag{24}$$

Hence, if we know $f_0(t)$, we can compute $f_1(t)$ in closed form by quadrature (Chapter 1). Illustrations of this "perturbation method" are given in Exercises F5 and F6 below.

We now drop the assumption that X is a one-dimensional vector as well as the assumption that the solution has a convergent Taylor expansion, and we derive analogous results. We show that the solutions of a normal first-order system (1) depend differentiably on the initial values.

■ **Theorem 11.** *Let the vector function* $\mathbf{X}$ *be of class* $\mathscr{C}^1$, *and let* $\mathbf{x}(t, \mathbf{c})$ *be the solution of the normal system* (2), *taking the initial value* $\mathbf{c}$ *at* $t = a$. *Then* $\mathbf{x}(t, \mathbf{c})$ *is a continuously differentiable function of each of the components* c_j *of* $\mathbf{c}$.

The proof is subdivided into three steps.

A. Consider the system of DE's for the unknown functions h_i, the components of the vector $\mathbf{h} = (h_1, \ldots, h_n)$:

$$\frac{dh_i}{dt} = \sum_{k=1}^{n} \frac{\partial X_i(\mathbf{x}(t, \mathbf{c}), t)}{\partial x_k} h_k + H_i(\mathbf{h}, t, \mathbf{c}, \eta_j) \tag{25}$$

when $\mathbf{x}(t, \mathbf{c})$ is the solution of th
We assume that the functior
$|\mathbf{h} - \boldsymbol{\delta}^j| \leq K$, where $\boldsymbol{\delta}^j$ is the vec
deltas δ_i^j. We also assume that
$|t - a| \leq T$ and $|\mathbf{h} - \boldsymbol{\delta}^j| \leq K$.
3, with $\epsilon = \eta_j$, we find that the
the initial condition $h_i^j(a) = \delta_i^j$
linear system

$$\frac{df_i}{dt} = \sum$$

satisfying the same initial conc
In addition, we infer from th
tions $\mathbf{h}^j$ remain *bounded* as η_j –
B. Set

$$g_i^j(t, \mathbf{c}, \eta_j) = \frac{x_i(t, c_1, c_2, \ldots}{}$$

We next find a differential e
difference $\mathbf{g}^j = (g_1^j, g_2^j, \ldots, g_n^j)$.
By definition,

$$\frac{dg_i^j(t, \mathbf{c}, \eta_j)}{dt} = \eta_j^{-1}[X_i(\mathbf{x}($$

We now use the assumption t
functions of several variables,

$$\sum_{k=1}^{n} \frac{\partial X_i(\mathbf{x}(t, \mathbf{c}),}{\partial x_k}$$

where ϵ_i is a function of t, $\mathbf{c}$ ar
formly as the variables t and
$H_i(\mathbf{h}, t, \mathbf{c}, \eta_j) = \epsilon_i |\mathbf{h}(t, \mathbf{c})|$, we fi
tion $\mathbf{g}^j = \mathbf{h}$ of a system (25).
stated under Step A.
C. The initial conditions sa

$$g_i^j(a, \mathbf{c}, \eta_j) = \frac{x_i(a, c_1, c_2, \ldots,}{}$$

$$= \begin{cases} \dfrac{c_i - c_i}{\eta_j} = 0 \\ \dfrac{c_j + \eta_j - c_j}{\eta_j} = \end{cases}$$

ı the domain of definition of X as of the DE $dx/dt = x$ has infinite solution of the DE $dx/dt = x^2$, ıas finite escape time.

e dependence on $\mathbf{c}$ of the solution n defined by the system $\mathbf{x}'(t) =$ = $\mathbf{c}$. For simplicity, consider first er DE. Assuming that $f(t, c)$ is t Taylor series expansion, we have

$$[X(f(t, c), t)]$$

$$, t)\Big] \cdot \Big[\frac{\partial f}{\partial c}(t, c)\Big]. \qquad (22)$$

ives formally

$$(c^2/2!)f_2(t) + \cdots, \qquad (23)$$

fies the linear perturbation equa-

$$_1(t), \qquad f_1(0) = 0. \qquad (24)$$

ute $f_1(t)$ in closed form by quad- his "perturbation method" are

X is a one-dimensional vector ɔlution has a convergent Taylor ults. We show that the solutions ɪend differentiably on the initial

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ɔeps.

r the unknown functions h_i, the h_n):

$$_k + H_i(\mathbf{h}, t, \mathbf{c}, \eta_j) \qquad (25)$$

when $\mathbf{x}(t, \mathbf{c})$ is the solution of the normal system (2) for which $x(t, \mathbf{a}) = \mathbf{c}$. We assume that the functions H_i are bounded for $|t - a| \leq T$ and $|\mathbf{h} - \boldsymbol{\delta}^j| \leq K$, where $\boldsymbol{\delta}^j$ is the vector whose components are the Kronecker deltas δ_i^j. We also assume that H_i tends to zero as $\eta_j \to 0$, uniformly for $|t - a| \leq T$ and $|\mathbf{h} - \boldsymbol{\delta}^j| \leq K$. Then, applying the Corollary of Theorem 3, with $\epsilon = \eta_j$, we find that the solution $\mathbf{h}^j = \mathbf{h}(t, \mathbf{c}, \eta_j)$ of (25) satisfying the initial condition $h_i^j(a) = \delta_i^j$ tends, as $\eta_j \to 0$, to the solution $\mathbf{f}^j$ of the linear system

$$\frac{df_i}{dt} = \sum_{k=1}^{n} \frac{\partial X_i(x(t, \mathbf{c}), t)}{\partial x_k} \cdot f_i \qquad (25')$$

satisfying the same initial conditions, namely, $f_i^j(a) = \delta_i^j$.

In addition, we infer from the same Corollary that the vector functions $\mathbf{h}^j$ remain *bounded* as $\eta_j \to 0$.

B. Set

$$g_i^j(t, \mathbf{c}, \eta_j) = \frac{x_i(t, c_1, c_2, \ldots, c_{j-1}, c_j + \eta_j, c_{j+1}, \ldots, c_n) - x_i(t, \mathbf{c})}{\eta_j}.$$

We next find a differential equation satisfied by the vector partial difference $\mathbf{g}^j = (g_1^j, g_2^j, \ldots, g_n^j)$.

By definition,

$$\frac{dg_i^j(t, \mathbf{c}, \eta_j)}{dt} = \eta_j^{-1}[X_i(\mathbf{x}(t, \mathbf{c}) + \eta_j \mathbf{g}^j(t, \mathbf{c}), t) - X_i(t, \mathbf{x}(t, \mathbf{c})].$$

We now use the assumption that X_i is $\mathscr{C}^1$. By Taylor's theorem for functions of several variables, we infer that the right side equals

$$\sum_{k=1}^{n} \frac{\partial X_i(\mathbf{x}(t, \mathbf{c}), t)}{\partial x_k} \cdot g_k^j(t, \mathbf{c}) + \epsilon_i |\mathbf{g}_k(t, \mathbf{c})|,$$

where ϵ_i is a function of t, $\mathbf{c}$ and η_j which tends to zero as $\eta_j \to 0$, uniformly as the variables t and $\mathbf{c}$ range over closed intervals. Setting $H_i(\mathbf{h}, t, \mathbf{c}, \eta_j) = \epsilon_i |\mathbf{h}(t, \mathbf{c})|$, we find that the vector function $\mathbf{g}^j$ is a solution $\mathbf{g}^j = \mathbf{h}$ of a system (25). The function $\mathbf{H}$ satisfies the conditions stated under Step A.

C. The initial conditions satisfied by the g_i^j are, by definition,

$$g_i^j(a, \mathbf{c}, \eta_j) = \frac{x_i(a, c_1, c_2, \ldots, c_{j-1}, c_j + \eta_j, c_{j+1}, \ldots, c_n) - x_i(t, \mathbf{c})}{\eta_j} =$$

$$= \begin{cases} \dfrac{c_i - c_i}{\eta_j} = 0 & \text{if } \; i \neq j, \\[2ex] \dfrac{c_j + \eta_j - c_j}{\eta_j} = 1 & \text{if } \; i = j. \end{cases}$$

Combining with the results of steps A and B, we conclude that, as $\eta_j \to 0$, the function $\mathbf{g}^j$ tends to the solution $\mathbf{h}^j$ of (25′) satisfying the same initial condition. But we know that

$$\lim_{\eta_j \to 0} \mathbf{g}^j(t, \mathbf{c}, \eta_j) = \partial \mathbf{x}(t, \mathbf{c})/\partial c_j .$$

We have, therefore, shown that the derivative $\partial \mathbf{x}/\partial c_j$ exists and is indeed a solution of (25′), q.e.d.

The linear DE (24) is called the *perturbation equation* or *variational equation* of the normal system (2), because it describes approximately the perturbation of the solution caused by a small perturbation of the initial conditions.

In the course of the preceding argument we have also proved the

■ COROLLARY. *If* $\mathbf{x}(t, \mathbf{c})$ *is a solution of the normal system* (2) *satisfying the initial condition* $\mathbf{x}(a) = \mathbf{c}$ *for each* $\mathbf{c}$, *and if each component of the function* $\mathbf{X}$ *is of class* $\mathscr{C}^1$, *then for each j the partial derivative* $\partial \mathbf{x}(t, \mathbf{c})/\partial \mathbf{c}_j$ *is a solution of the perturbation equation* (25) *of the system.*

In the case of *linear* systems $d\mathbf{x}/dt = A(t)\mathbf{x} + \mathbf{b}(t)$, the perturbation equation is the reduced equation $d\mathbf{h}/dt = A(t)\mathbf{h}$ of the given system and is the same for all solutions. But in *nonlinear* systems, the perturbation equation (25) depends on the particular solution $\mathbf{x}(t, \mathbf{c})$ whose initial value is being varied.

13 Plane autonomous systems

We now apply the results of the preceding section to the trajectories of autonomous systems. The main result is that, near any noncritical point, the trajectories of an autonomous system look like a regular family of parallel straight lines. We give the proof for the case $n = 2$. Recall that a plane autonomous system is

$$dx/dt = X(x, y), \qquad dy/dt = Y(x, y). \tag{26}$$

■ THEOREM 12. *Any plane autonomous system where X and Y are of class* $\mathscr{C}^1$ *is equivalent, under a diffeomorphism in a neighborhood of any point which is not a critical point, to the system* $du/dt = 1$, $dv/dt = 0$.

Proof — Let the point be (a, b); without loss of generality, we may assume that $X(a, b) \neq 0$. Let the solution of the system for the initial values $x(0) = a$, $y(0) = c$ be $x = \xi(t, c)$, $y = \eta(t, c)$, so that $\partial\xi/\partial t = X$, $\partial\eta/\partial t = Y$. Then, by Theorem 11, the transformation $(t, c) \to (\xi(t, c),$

$\eta(t, c))$ is of class $\mathscr{C}^1$; moreove
Jacobian

$$\frac{\partial(\xi, \eta)}{\partial(t, c)} = \frac{\partial \xi}{\partial t} \cdot \frac{\partial \eta}{\partial c} - \frac{\partial \xi}{\partial c}$$

is nonvanishing at (a, b). Hen
the inverse transformation $u =$
(u, v)-coordinates, the solutio
hence, the DE assumes the forı

■ COROLLARY 1. *Any two plaı*
lent under a diffeomorphism exc

The system $\dot{u} = 1$, $\dot{v} = 0$ is,
plane autonomous systems nea
the velocity field associated wi

■ COROLLARY 2. *If the functi*
system (26) *satisfy local Lipschi*
a regular curve family in any d

Proof—By Theorem 1, there
through each point c not a crit
integral curve goes all the way t
conditions imply continuity,
$Y(x, y))$ vary continuously wi
which completes the proof.

14* The Peano existence theorem

The existence theorems for
assumed that the functions X
now derive an existence theore
in Chapter 1, solutions of suc
mined by their initial values.

■ THEOREM 13 (Peano existeı
continuous for $|\mathbf{x} - \mathbf{c}| \leq K$, $|t -$
the vector DE (2) *has at least on*

$|t - a|$

satisfying the initial condition $\mathbf{x}$

s A and B, we conclude that, as
solution $\mathbf{h}^j$ of (25′) satisfying the
v that

$= \partial\mathbf{x}(t, \mathbf{c})/\partial c_j$.

e derivative $\partial\mathbf{x}/\partial c_j$ exists and is

perturbation equation or *variational*
ause it describes approximately the
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of the normal system (2) *satisfying*
, and if each component of the func-
e partial derivative $\partial\mathbf{x}(t, \mathbf{c})/\partial c_j$ *is a*
25) of the system.

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$dy/dt = Y(x, y)$. (26)

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e system $du/dt = 1$, $dv/dt = 0$.

vithout loss of generality, we may
ution of the system for the initial
$c)$, $y = \eta(t, c)$, so that $\partial\xi/\partial t = X$,
he transformation $(t, c) \to (\xi(t, c),$

$\eta(t, c))$ is of class $\mathscr{C}^1$; moreover, since $x(0)$ does not vary with c, the Jacobian

$$\frac{\partial(\xi, \eta)}{\partial(t, c)} = \frac{\partial \xi}{\partial t} \cdot \frac{\partial \eta}{\partial c} - \frac{\partial \xi}{\partial c} \cdot \frac{\partial \eta}{\partial t} = X(a, b) \cdot 1 - 0 \cdot Y(a, b)$$

is nonvanishing at (a, b). Hence, by the Implicit Function Theorem, the inverse transformation $u = t(x, y)$, $v = c(x, y)$ is of class $\mathscr{C}^1$. In the (u, v)-coordinates, the solutions reduce to $u = t$, $v = c =$ constant; hence, the DE assumes the form stated, q.e.d.

■ COROLLARY 1. *Any two plane autonomous systems are locally equivalent under a diffeomorphism except near critical points.*

The system $\dot{u} = 1$, $\dot{v} = 0$ is, therefore, locally a canonical form for plane autonomous systems near noncritical points. In hydrodynamics, the velocity field associated with this system is called a uniform flow.

■ COROLLARY 2. *If the functions X and Y of the plane autonomous system* (26) *satisfy local Lipschitz conditions, then its integral curves form a regular curve family in any domain which contains no critical points.*

Proof—By Theorem 1, there is a unique integral curve of (26) passing through each point c not a critical point. As shown in § 12, each such integral curve goes all the way to the boundary. Finally, since Lipschitz conditions imply continuity, the directions of the vectors $(X(x, y), Y(x, y))$ vary continuously with position except near a critical point, which completes the proof.

14* The Peano existence theorem

The existence theorems for normal systems (1) proved so far have assumed that the functions X_i satisfy Lipschitz conditions. We shall now derive an existence theorem, assuming only continuity. As shown in Chapter 1, solutions of such systems need not be uniquely determined by their initial values.

■ THEOREM 13 (Peano existence theorem). *If the function* $\mathbf{X}(\mathbf{x}, t)$ *is continuous for* $|\mathbf{x} - \mathbf{c}| \leqq K$, $|t - a| \leqq T$, *and if* $|\mathbf{X}(\mathbf{x}, t)| \leqq M$ *there, then the vector* DE (2) *has at least one solution* $\mathbf{x}(t)$, *defined for*

$$|t - a| \leqq \min(T, K/M),$$

satisfying the initial condition $\mathbf{x}(a) = \mathbf{c}$.

Proof — Using an elegant method due to Tonelli, we shall consider the equivalent integral equation (11) of Theorem 5,

$$\mathbf{x}(t) = \mathbf{c} + \int_a^t \mathbf{X}(\mathbf{x}(s), s)\, ds \tag{27}$$

and prove that this has a solution. Let $T_1 = \min(T, K/M)$. We may assume that $a = 0$ and that the interval is $0 \leq t \leq T_1$. In this interval we construct a sequence of functions $\mathbf{x}^n(t)$ as follows. For $0 \leq t \leq T_1/n$, set $\mathbf{x}^n(t) \equiv \mathbf{c}$. For $T_1/n < t \leq T_1$ define $\mathbf{x}^n(t)$ by the formula

$$\mathbf{x}^n(t) = \mathbf{c} + \int_0^{t - T_1/n} \mathbf{X}(\mathbf{x}^n(s), s)\, ds. \tag{28}$$

This formula defines the value of $\mathbf{x}^n(t)$ in terms of the previous values of $\mathbf{x}^n(s)$ for $0 \leq s \leq t - T_1/n$.

It follows, as in the lemma of § 9, that the functions $\mathbf{x}^n(t)$ are defined for $0 \leq t \leq T_1$. Also, we have

$$|\mathbf{x}^n(t)| \leq |\mathbf{c}| + \int_0 M\, ds \leq |\mathbf{c}| + T_1 M.$$

Hence, the sequence of functions $|\mathbf{x}^n(t)|$ $(n = 1, 2, \ldots)$ is uniformly bounded.

Next, we prove that the sequence $\mathbf{x}^n$ is *equicontinuous* in the following sense.

■ Definition. *A family $\mathcal{F}$ of vector-valued functions $\mathbf{x}(t)$, defined on an interval I: $|t - a| \leq T$, is said to be* equicontinuous *when, given $\epsilon > 0$, a number $\delta > 0$ exists such that*

$$|t - s| < \delta \qquad \textit{implies} \qquad |\mathbf{x}(t) - \mathbf{x}(s)| < \epsilon$$

for all *functions $\mathbf{x} \in \mathcal{F}$, provided that $s, t \in I$.*

Indeed, using the inequality (3), we have

$$|\mathbf{x}^n(t_1) - \mathbf{x}^n(t_2)| \leq \int_{t_1 - T_1/n}^{t_2 - T_1/n} |\mathbf{X}(\mathbf{x}^n(s), s)|\, ds \leq M\,|t_2 - t_1|,$$

from which it is evident that the $\mathbf{x}^n(t)$ are equicontinuous.

We now apply to the sequence $\mathbf{x}^n$ the Theorem of Arzelà-Ascoli, which is stated below without proof.†

■ Arzelà-Ascoli Theorem. *Let $\mathbf{x}^n(t)$ $(n = 1, 2, 3, \ldots)$ be a bounded equicontinuous sequence of scalar or vector functions, defined for $a \leq t \leq b$.*

† Rudin, pp. 144 ff. The proof given here is for real-valued functions, but the same proof applies to vector-valued functions.

Then there exists a subsequenc
convergent in the interval.

Applying this result to the s
a uniformly convergent subse
function $\mathbf{x}^\infty(t)$ as $n_i \to \infty$.

It is now easy to verify th
integral equation (27). Indee

$$\mathbf{x}^{n_i}(t) = \mathbf{c} + \int_0^t \mathbf{X}(\mathbf{x}^{n_i}($$

As $n_i \to \infty$, $\int_0^t \mathbf{X}(\mathbf{x}^{n_i}(s), s)\, ds -$
formly continuous; and the la
the inequality (3),

$$\left|\int_{t - T_1/n_i}^t \mathbf{X}\, ds\right| \leq$$

Therefore, taking limits on bo
satisfies the integral equation

EXERCISES F

1. Let $F(x, y)$ be continuous for |
all solutions of $y' = F(x, y)$, sati
equicontinuous.
2. Show that, if $\mathbf{X}(\mathbf{x}, t)$ is conti
$a \leq t \leq b$, every solution of DE (
Show that the corresponding resu
3. Let $xX(x, y) + y^3 Y(x, y) = 0$,
system $x' = X(x, y)$, $y' = Y(x, y)$
$2x^2 + y^4$ is an integral of the syst
4. Let the function $\mathbf{X}(\mathbf{x}, t)$ be d
$|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)| \leq L(t)|\mathbf{x} - \mathbf{y}|$,
$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t)$ has a solution on 0
Show that, if one solution is boun
5. Let $\mathbf{X}(\mathbf{x}, t, s)$ be of class $\mathcal{C}^1$
$\mathbf{x}(t, s)$ be the solution of $\mathbf{x}' = \mathbf{X}$
differentiable function of s.
6*. Under the assumptions of Ex
Show that $\mathbf{x}(t, s)$ has n continuou
7*. Show that if there are two di
ing the same initial condition
$|y - c| \leq K$), there are infinitely
8*. Show that there is a *maximal*
DE in Exercise 7, such that $f_m($
that $f(a) = f_M(a) = f_m(a)$. [Hint:
9*. Let $F(x, y)$ and $G(x, y)$ be cont

due to Tonelli, we shall consider
f Theorem 5,

$\mathbf{x}(s), s)\, ds$ (27)

et $T_1 = \min(T, K/M)$. We may
al is $0 \leq t \leq T_1$. In this interval
$^n(t)$ as follows. For $0 \leq t \leq T_1/n$,
$\mathbf{x}^n(t)$ by the formula

$\mathbf{X}(\mathbf{x}^n(s), s)\, ds.$ (28)

in terms of the previous values of

at the functions $\mathbf{x}^n(t)$ are defined

$\leq |\mathbf{c}| + T_1 M.$

$^n(t)|$ $(n = 1, 2, \ldots)$ is uniformly

is *equicontinuous* in the following

alued functions $\mathbf{x}(t)$, *defined on an*
quicontinuous *when, given* $\epsilon > 0$,

$|\mathbf{x}(t) - \mathbf{x}(s)| < \epsilon$

$t \in I.$

have

$^n(s), s)|\, ds \leq M\,|t_2 - t_1|,$

are equicontinuous.
the Theorem of Arzelà-Ascoli,

$t)$ $(n = 1, 2, 3, \ldots)$ *be a bounded*
r functions, defined for $a \leq t \leq b$.

eal-valued functions, but the same proof

Then there exists a subsequence $\mathbf{x}^{n_i}(t)$ $(i = 1, 2, \ldots)$ *which is uniformly convergent in the interval.*

Applying this result to the sequence $\mathbf{x}^n(t)$, we see that it must contain a uniformly convergent subsequence $\mathbf{x}^{n_i}(t)$, converging to a continuous function $\mathbf{x}^\infty(t)$ as $n_i \to \infty$.

It is now easy to verify that this limit function $\mathbf{x}^\infty(t)$ satisfies the integral equation (27). Indeed, (28) can be written in the form

$$\mathbf{x}^{n_i}(t) = \mathbf{c} + \int_0^t \mathbf{X}(\mathbf{x}^{n_i}(s), s)\, ds - \int_{t - T_1/n_i}^{t} \mathbf{X}(\mathbf{x}^{n_i}(s), s)\, ds. \tag{29}$$

As $n_i \to \infty$, $\int_0^t \mathbf{X}(\mathbf{x}^{n_i}(s), s)\, ds \to \int_0^t \mathbf{X}(\mathbf{x}^\infty(s), s)\, ds$ because $\mathbf{X}(\mathbf{x}, t)$ is uniformly continuous; and the last term of (29) tends to zero, because, by the inequality (3),

$$\left|\int_{t - T_1/n_i}^{t} \mathbf{X}\, ds\right| \leq \int_{t - T_1/n_i}^{t} M\, ds = M\,\frac{T_1}{n_i} \to 0.$$

Therefore, taking limits on both sides of (29) as $n_i \to \infty$, we find that $\mathbf{x}^\infty$ satisfies the integral equation (27), q.e.d.

EXERCISES F

1. Let $F(x, y)$ be continuous for $|x - a| \leqq T$, $|y - c| \leqq K$. Show that the set of all solutions of $y' = F(x, y)$, satisfying the same initial condition $f(a) = c$, is equicontinuous.
2. Show that, if $\mathbf{X}(\mathbf{x}, t)$ is continuous and satisfies a Lipschitz condition for $a \leqq t \leqq b$, every solution of DE (2) satisfying $\mathbf{x}(a) = \mathbf{c}$ is bounded for $a \leqq t \leqq b$. Show that the corresponding result is not true for open intervals $a < t < b$.
3. Let $xX(x, y) + y^3 Y(x, y) = 0$, where X and Y are of class $\mathscr{C}^1$. Show that the system $x' = X(x, y)$, $y' = Y(x, y)$ has infinite escape time. [HINT: Show that $2x^2 + y^4$ is an integral of the system.]
4. Let the function $\mathbf{X}(\mathbf{x}, t)$ be defined for $0 \leqq t < \infty$ and for all $\mathbf{x}$, and let $|\mathbf{X}(\mathbf{x}, t) - \mathbf{X}(\mathbf{y}, t)| \leqq L(t)\,|\mathbf{x} - \mathbf{y}|$, where $\int_0^\infty L(t)\, dt < \infty$. Show that the DE $d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}, t)$ has a solution on $0 \leqq t < +\infty$ for every initial condition $\mathbf{x}(a) = \mathbf{c}$. Show that, if one solution is bounded, then all are.
5. Let $\mathbf{X}(\mathbf{x}, t, s)$ be of class $\mathscr{C}^1$ for $|\mathbf{x} - \mathbf{c}| \leqq K$, $|t - a| \leqq T$, $|s - s_0| \leqq S$. Let $\mathbf{x}(t, s)$ be the solution of $\mathbf{x}' = \mathbf{X}(\mathbf{x}, t, s)$ satisfying $\mathbf{x}(a) = \mathbf{c}$. Show that $\mathbf{x}$ is a differentiable function of s.

6*. Under the assumptions of Exercise 5, suppose that $\mathbf{X}(\mathbf{x}, t, s)$ is of class $\mathscr{C}^n$. Show that $\mathbf{x}(t, s)$ has n continuous partial derivatives relative to s.

7*. Show that if there are two distinct solutions f and g of $y' = F(x, y)$ satisfying the same initial condition $c = f(a) = g(a)$ (F continuous in $|x - a| \leqq T$, $|y - c| \leqq K$), there are infinitely many of them.

8*. Show that there is a *maximal* and a *minimal solution* $f_M(x)$ and $f_m(x)$ of the DE in Exercise 7, such that $f_m(x) \leqq f(x) \leqq f_M(x)$ for any other solution f such that $f(a) = f_M(a) = f_m(a)$. [HINT: See Chapter 1, Exercise E4.]

9*. Let $F(x, y)$ and $G(x, y)$ be continuous for $a \leqq x \leqq T$, $|y - c| \leqq K$, and $F(x, y) \leqq$

$G(x, y)$. Let f be a solution of $y' = F(x, y)$, and let g be the maximal solution of $y' = G(x, y)$. Show that, if $f(a) \leqq g(a)$, then $f(x) \leqq g(x)$ for $x > a$.

ADDITIONAL EXERCISES

1*. Let $dx/dt = X(x, y, t)$ and $dy/dt = Y(x, y, t)$, where

$$(x - x')[X(x, y, t) - X(x', y', t)] + (y - y')[Y(x, y, t) - Y(x', y', t)]$$

is everywhere negative or zero. Show that, for $t > 0$, the above system has at most one solution satisfying a given initial condition at $t = 0$.

In Exercises 2 through 4, f'_+ means the right-derivative; prove the implication specified. You may assume the existence of f'_+ and g'_+ freely.

2. If $f'_+(x) \leqq g'_+(x)$, then $f(x) - f(y) \leqq g(x) - g(y)$ for $x \geqq y$.
3. If $|f'_+(x)| \leqq K|f(x)|$ then $|f(x)| \leqq |f(a)|e^{K|x-a|}$ for $x \geqq a$.
4. If $|f'_+(x)| \leqq K|f(x)| + \epsilon$, then $|f(x)| \leqq |f(a)|e^{K|x-a|} + (\epsilon/K)(e^{K|x-a|} - 1)$.
5. Let $dz_i/dt = Q_i(z_1, \ldots, z_n)$, where the Q_i are quadratic polynomials. Show that, for any initial condition, the nth Picard approximation to the solution is a polynomial in t of degree at most $2^n - 1$.
6. (a) Prove that, if there is a normal kth-order ordinary DE satisfied by two functions u and v and if $n > k$, there is a normal nth-order DE satisfied by both functions. State your differentiability assumptions.
   (b) Prove that, if the given kth-order DE is linear, then the nth-order DE can also be chosen to be linear.
   (c) Prove that there is no fourth-order normal DE $u^{\text{iv}} = F(u, u', u'', u''', t)$ satisfied by both $u = t^4$ and $v = t^6$ for all real t.
   (d) Prove that $u = t^6$ satisfies no normal linear homogeneous DE of degree six or less with continuous coefficients.
7. Show that, if $X_1, \ldots, X_n$ satisfy Lipschitz conditions on a compact domain, so does any polynomial function of the X_i.
8. Show that, if $X(t) = \|x_{ij}(t)\|$ is a matrix whose columns are solutions of the homogeneous linear system $X' = A(t)X$, then $\det X(t) = (\det X(a)) \exp \int_a^t \sum a_{kk}(s)ds$.
9. A matrix $X(t)$ is a *fundamental matrix* for $a \leqq t \leqq a + T$ of a homogeneous linear system $X' = A(t)X$ if its columns are solutions of the system and $\det (X(t)) \neq 0$. Show that, if the columns of X are solutions of the system and if $\det X(a) \neq 0$, then X is a fundamental matrix.
10*. Show that, if $X(t)$ is a fundamental matrix of the reduced linear system, the function $x(t) = X(t) \int_a^t X^{-1}(s)b(s)\,ds$ is the solution of the inhomogeneous system such that $x(a) = 0$ (X^{-1} is the matrix inverse of X).
11*. (Nagumo). Let X be continuous for $|t - a| \leqq T$, $|\mathbf{x} - \mathbf{c}| \leqq K$, and $|t - a|\,|X(\mathbf{x}, t) - X(\mathbf{y}, t)| \leqq |\mathbf{x} - \mathbf{y}|$. Show that the solution of (2) satisfying $\mathbf{x}(a) = \mathbf{c}$ is unique. [HINT: Use the equivalent integral equation, and consider the maximum of $|\mathbf{x}(t) - \mathbf{y}(t)|/(t - a)$ for $t \geqq a$].
12. Let X be continuous for $|\mathbf{x} - \mathbf{c}| \leqq K$, $|t| \leqq T$, and let

$$2(\mathbf{x} - \mathbf{c}) \cdot \mathbf{X}(\mathbf{x}, t) \leqq \psi(|\mathbf{x} - \mathbf{c}|^2, t),$$

where $\psi(u, t)$ is positive and increasing in u for fixed t, ($u \geqq 0$. Let $\phi \in \mathscr{C}^1$ satisfy $\phi'(t) > \psi(\phi(t), t)$, and $\phi(0) = 0$. Show that if $\mathbf{x}$ is a solution of (2) such that $\mathbf{x}(0) = c$, then $|\mathbf{x}(t) - c| \leqq \sqrt{\phi(t)}$. [HINT: Show that $\sigma'(t) \leqq \psi(\sigma(t), t)$, where $\sigma(t) = |x(t) - c|^2$.]

13*. Assume that $\mathbf{X}$ is continuou

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where $\psi(u, t)$ is nonnegative, con
$|t - a| \leqq T$ and $u \geqq 0$. Let $\phi(t)$ be
and $\phi(0) = 0$. Let $\mathbf{x}(t)$ be a soluti

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14*. Let X be continuous for $|t$ -
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Show that $\sigma'(t) \leqq \psi(\sigma(t), t)$, where

13*. Assume that $\mathbf{X}$ is continuous for $|t - a| \leqq T$, $|\mathbf{x} - \mathbf{c}| \leqq K$, and that

$$|\mathbf{X}(\mathbf{x}, t)| \leqq \psi(|\mathbf{x} - \mathbf{c}|, |t - a|),$$

where $\psi(u, t)$ is nonnegative, continuous, increasing in u for fixed t, defined for $|t - a| \leqq T$ and $u \geqq 0$. Let $\phi(t)$ be of class $\mathscr{C}^1$ and $\phi'(t) > \psi(\phi(t), t)$ for $|t - a| \leqq T$, and $\phi(0) = 0$. Let $\mathbf{x}(t)$ be a solution of (2) such that $\mathbf{x}(a) = \mathbf{c}$. Show that

$$|\mathbf{x}(t) - \mathbf{c}| \leqq \phi(|t - a|).$$

14*. Let X be continuous for $|t - a| \leqq T$, let $\phi(u)$ be continuous and increasing for $u \geqq 0$, and let $\lim_{N \to \infty} \int_1^N du/\phi(u) = \infty$. Show that, if $|\mathbf{X}(\mathbf{x}, t)| \leqq \phi(|\mathbf{x}|)$, all solutions of (2) are bounded on $|t - a| \leqq T$. [HINT: Use Ex. 13]