1 Exercise 1 Solution

Let us first observe the following fact **(*)**

If $Y \subseteq \mathbb{R}^k$ is bounded, then $\overline{\int}\chi_Y = 0$ implies that Y has measure zero. Furthermore, if Y has measure zero, then $\underline{\int}\chi_Y = 0$.

Proof of **(*)**

Suppose first that $Y \subseteq \mathbb{R}^k$ is bounded and $\overline{\int}\chi_Y = 0$. Let Q be a rectangle containing Y. By definition of $\overline{\int}\chi_Y$, it follows that for all $\epsilon > 0$, we can find a partition of Q such that $U(\chi_Y, P) \leq \epsilon$, where we are now considering χ_Y to be a function on Q. This, in turn, gives us a finite collection of rectangles covering Y whose volume adds up to less than ϵ. Such a collection can be found for all $\epsilon > 0$ so, in particular, Y has measure zero.

Suppose now that Y has measure zero. Let Q be a rectangle in $\mathbb{R}^k$ containing Y as before. Then, for all partitions P of Q, we have $L(\chi_Y, P) = 0$ since the fact that Y has measure zero implies that no rectangle of P having non-zero measure is wholly contained in Y, so every term in the sum defining $L(\chi_Y, P)$ is zero. This holds for all partitions P of Q hence $\underline{\int}\chi_y = 0$.

This proves **(*)**

We now prove the claim:

We first fix some notation. Let us suppose that $X \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is bounded and rectifiable. From the fact that X is bounded, it follows that we can find rectangles $Q_1 \subseteq \mathbb{R}^n$ and $Q_2 \subseteq \mathbb{R}^m$ such that $X \subseteq Q_1 \times Q_2$. Let $Q := Q_1 \times Q_2$. Then Q is a rectangle in $\mathbb{R}^n \times \mathbb{R}^m$ that contains X. Let us denote by f the function χ_Y restricted to Q.

Finally,we denote by:

$$A := \{p \in \mathbb{R}^n \ \pi(X \cap (\{p\} \times \mathbb{R}^m)) \subseteq \mathbb{R}^m \text{ doesn't have measure zero}\}$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is the canonical projection map $\pi(x, y) = y$. In other words, A is the subset of all $p \in \mathbb{R}^n$ whose "horizontal slice in $\mathbb{R}^m$" doesn't have measure zero in $\mathbb{R}^m$

1)Suppose that X has measure zero.

Then

$$0 = \int_Q f(x, y) = \{\text{Using } \textbf{Fubini's Theorem}\} =$$

$$= \int_{Q_1} \overline{\int}_{Q_2} f(x, y)$$

We know by Fubini's Theorem that the function

$$x \mapsto \overline{\int}_{Q_2} f(x,y)$$

is an integrable function on Q_1. By construction, this function is non-negative and from the preceding calculations it follows that its integral over Q_1 equals zero. By applying **Theorem 11.3.b)** on Page 96 of the textbook "Analysis on Manifolds" by James Munkres, we deduce that $\overline{\int}_{Q_2} f(x,y)$ vanishes except on a set of measure zero.

On the other hand, by using **(*)** and arguing contrapositively together with the definition of A, it follows that for all $x \in A$ $\overline{\int}_{Q_2} f(x,y) > 0$

Combining the previous results, it follows that A has measure zero.

2)Suppose that A has measure zero.

We know that then, by **(*)** for all $x \in Q_1 - A$ we have $\underline{\int}_{Q_2} f(x,y) = 0$ $(\diamond)$

Also, by Fubini's Theorem $x \mapsto \underline{\int}_{Q_2} f(x,y)$ is integrable on Q_1 $(\diamond\diamond)$

Hence, we have:

$$\int_Q f(x,y) = \int_{Q_1} \underline{\int}_{Q_2} f(x,y) =$$

$$= \underline{\int}_{Q_1} \underline{\int}_{Q_2} f(x,y) = 0$$

by using $(\diamond)$ and **(*)** together with the fact that A has measure zero.

We deduce that

$$\int_Q f(x,y) = \int \chi_X = 0$$

Since X is rectifiable, we know that $\overline{\int} \chi_X = \int \chi_X$. Hence $\overline{\int} \chi_X = 0$

By **(*)**, we deduce that X has measure zero.

Conclusion:) X has measure zero if and only if A has measure zero. $\square$

2 Exercise 2 Solution

Solution 1:

We first consider the cases $n = 1$ and $n = 2$ separately.

For n=1:

$$\lambda_1 = \int_{-1}^{1} dx = 2$$

For n=2: By polar coordinates:

$$\lambda_2 = \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = \pi$$

Suppose now that $n \geq 3$
We recall the fact from Calculus that

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

From this observation, it follows that:

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \int_{\mathbb{R}^n} e^{-x_1^2 - \cdots - x_n^2}\ dx_1 \cdots dx_n =$$

$$= (\int_{-\infty}^{+\infty} e^{-x_1^2}\ dx_1) \cdots (\int_{-\infty}^{+\infty} e^{-x_n^2}\ dx_n) = \pi^{\frac{n}{2}} \textbf{(*)}$$

On the other hand, by using n-dimensional polar coordinates, we have:

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \int_0^{+\infty} \int_{S^{n-1}} e^{-r^2} r^{n-1}\ d\sigma\, dr$$

where $d\sigma$ denotes the surface measure on S^{n-1}.
Let

$$\omega_n := \int_{S^{n-1}} d\sigma = \text{Surface Measure of } S^{n-1} \subseteq \mathbb{R}^n$$

By using **(*)**, it follows that

$$\omega_n \cdot \int_0^{+\infty} e^{-r^2} r^{n-1}\ dr = \pi^{\frac{n}{2}}$$

In other words,

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\int_0^{+\infty} e^{-r^2} r^{n-1}\ dr}$$

Using n dimensional polar coordinates again, we deduce that:

$$\lambda_n := \mathrm{vol}(\overline{B(0,1)} \subseteq \mathbb{R}^n) = \int_0^1 \int_{S^{n-1}} r^{n-1}\ d\sigma\, dr = \frac{1}{n} \cdot \omega_n$$

Hence, in order to calculate λ_n, we need to calculate ω_n.

To calculate ω_n explicitly, we observe:

$$\int_0^{+\infty} e^{-r^2} r^{n-1} dr = \{\text{Change variables } s = r^2\} =$$

$$= \int_0^{+\infty} e^{-s}\, s^{\frac{n-1}{2}}\, \frac{1}{2}\, s^{-\frac{1}{2}}\, ds =$$

$$= \frac{1}{2} \int_0^{+\infty} e^{-s}\, s^{\frac{n}{2}-1}\, ds$$

Now:

$$\int_0^{+\infty} e^{-s}\, s^{\frac{n}{2}-1}\, ds =$$

$$= \{\text{Integrating by parts by setting } u = s^{\frac{n}{2}-1}\ , dv = e^{-s},$$

$$\text{and noting that } u \text{ vanishes at zero, since } n \geq 3\} =$$

$$= (\frac{n}{2} - 1) \int_0^{+\infty} e^{-s}\, s^{\frac{n}{2}-2}\, ds =$$

$$= (\frac{n}{2} - 1) \int_0^{+\infty} e^{-s}\, s^{\frac{n-2}{2}-1}\, ds \text{ (**)}$$

We can also define ω_1, ω_2 by $\omega_j := j \cdot \lambda_j$, $j = 1, 2$ and we get:

$$\omega_j = \frac{2\pi^{\frac{j}{2}}}{\int_0^{+\infty} e^{-s}\, s^{\frac{j}{2}-1}}\, ds$$

since the whole previous calculation up to the integration by parts step still works for these j

Combining (**) with this extended definition of the ω_j-s, we obtain the recursion that for all $n \geq 3$

$$\omega_n = \frac{\pi}{\frac{n}{2} - 1} \omega_{n-2} = \frac{2\pi}{n-2}\, \omega_{n-2}\ (\diamond)$$

This is the main recursive step.

By $\diamond$, we obtain by induction on k that for all $k \in \mathbb{N}_0$

$$\omega_{2k+1} = \frac{2^k \, \pi^k}{(2k-1)!!} \, \omega_1 = \frac{2^k \, \pi^k}{(2k-1)!!} \cdot 2 =$$
$$= \frac{2^{k+1} \, \pi^k}{(2k-1)!!}$$

Here $(2k-1)!!$ denotes $(2k-1) \cdot (2k-3) \cdots 1$ if $k \geq 1$ and $1!! = 0$ by convention.
It follows that for all $k \in \mathbb{N}_0$

$$\lambda_{2k+1} = \frac{1}{2k+1} \omega_{2k+1} = \frac{1}{2k+1} \, \frac{2^{k+1} \, \pi^k}{(2k-1)!!}$$

(This gives us the volume of the unit ball for odd dimensions)
Similarly, for $k \in \mathbb{N}$ we obtain:

$$\omega_{2k} = \frac{\pi^{k-1}}{(k-1)!} \, \omega_2 = \frac{\pi^{k-1}}{(k-1)!} \cdot 2\pi =$$
$$= \frac{2 \, \pi^k}{(k-1)!}$$

It follows that for all $k \in \mathbb{N}$

$$\lambda_{2k} = \frac{1}{2k} \omega_{2k} = \frac{1}{2k} \, \frac{2\pi^k}{(k-1)!} = \frac{\pi^k}{k!}$$

(This gives us the volume of the unit ball for even dimensions)
Conclusion: For all $k \in \mathbb{N}_0$ we have:

$$\lambda_{2k+1} = \frac{1}{2k+1} \, \frac{2^{k+1} \, \pi^k}{(2k-1)!!}$$

and for all $k \in \mathbb{N}$ we have:

$$\lambda_{2k} = \frac{\pi^k}{k!}. \quad \Box$$

Alternative Solution:
Let us denote by $\lambda_{n,a}$ the volume of the ball of radius a in $\mathbb{R}^n$. Then, by definition $\lambda_{n,1} = \lambda_n$. By using the change of variables $x \mapsto ay$, we obtain:

$$\lambda_{n,a} = a^n \lambda_n$$

As before,

$$\lambda_1 = 2,\ \lambda_2 = \pi$$

Applying Fubini's Theorem, we obtain:

$$\lambda_n = \int_{x_1^2+\cdots+x_n^2\leq 1} 1dx_1\cdots dx_n =$$

$$= \int_{x_1^2+x_2^2\leq 1} (\int_{x_3^2+\cdots+x_n^2\leq 1} 1dx_3\cdots dx_n)dx_1dx_2 =$$

$$= \int_{x_1^2+x_2^2\leq 1} \lambda_{n-2,1-x_1^2-x_2^2}\ dx_1dx_2 =$$

$$= \int_{x_1^2+x_2^2\leq 1} \lambda_{n-2} \cdot (1-x_2^2-x_2^2)^{\frac{n-2}{2}} dx_1dx_2 =$$

$$= \{\text{By using Polar Coordinates in two dimensions}\} =$$

$$= \lambda_{n-2} \cdot \int_0^1 \int_0^{2\pi} (1-r^2)^{\frac{n-2}{2}}\ drd\theta =$$

$$= -\lambda_{n-2} \cdot 2\pi \cdot \frac{1}{n} \cdot (1-r^2)^{\frac{n}{2}}|_{r=0}^{r=1} =$$

$$= \frac{2\pi}{n}\lambda_{n-2}$$

In this way, we obtain the same recursion as earlier. From here, we analogously deduce what λ_n is, considering separately the cases when n is odd and when n is even.

3 Exercise 3 Solution

We will use **Theorem** 15.2 on Page 123 of the textbook "Analysis on Manifolds" by James Munkres.

With notation as in the Theorem, we take

$$C_k := \{\frac{1}{k} \leq \|x\| \leq 1 - \frac{1}{k}\}$$

Then (C_k) is a sequence of compact rectifiable subsets of A whose union is A such that $C_k \subseteq \text{int}(C_{k+1})$ for all k.

We must show that $\int_{C_k} \|x\|^{-(n-1)}$ is uniformly bounded in k.

By using n-dimensional polar coordinates, we have that for all k

$$\int_{C_k} \|x\|^{-(n-1)} \leq \int_{\frac{1}{k}}^{1} \int_{S^{n-1}} r^{-(n-1)} r^{n-1} d\sigma dr$$

where $d\sigma$ denotes the surface measure on S^{n-1}
The above expression equals:

$$\mathrm{vol}(S^{n-1}) \int_{\frac{1}{k}}^{1} dr$$

This expression is less than or equal to $\mathrm{vol}(S^{n-1})$.
Hence, the claim now follows by applying Theorem 15.2.$\square$

4 Exercise 4 Solution

4.1 Part a)

Let $\sigma_0 \in |T_e G|$ be an arbitrary positive density on $T_e G$.
We define for $h \in G$:

$$\sigma(h) := (T_h l_{h^{-1}})^*(\sigma_0) \in |T_h G|$$

Hence, σ gives us a density on G.
For all $h \in G$, $v_1, \ldots, v_n \in T_h G$ a basis of $T_h G$ we have:

$$\sigma(h)(v_1, \ldots, v_n) = (T_h l_{h^{-1}})^*(\sigma_0)(v_1, \ldots, v_n) =$$
$$= \sigma_0(T_{l_h^{-1}} v_1, \ldots, T_{l_h^{-1}} v_n) > 0$$

since by construction $\sigma_0 \in |T_e G|$ is positive.
So, σ gives us a positive density on G.
Let us check that σ is left-invariant.
To do this, suppose that $g, h \in G$ and $v_1, \ldots, v_n \in T_h G$ are given. Then:

$$(l_g^* \sigma)(h)(v_1, \ldots, v_n) =$$
$$= \sigma(l_g(h))(T_h l_g v_1, \ldots, T_h l_g v_n) =$$
$$= \sigma(gh)(T_h l_g v_1, \ldots, T_h l_g v_n) =$$
$$= ((T_{gh} l_{(gh)^{-1}})^* \sigma_0)(T_h l_g v_1, \ldots, T_h l_g v_n) =$$

$$= ((T_{gh}(l_{h^{-1}} \circ l_{g^{-1}}))^* \sigma_0)(T_h l_g v_1, \dots, T_h l_g v_n) =$$
$$= ((T_h l_{h^{-1}} \circ T_{gh} l_{g^{-1}})^* \sigma_0)(T_h l_g v_1, \dots, T_h l_g v_n) =$$
$$= T_{gh} l^*_{g^{-1}} \circ T_h l^*_{h^{-1}} \circ \sigma_0 (T_h l_g v_1, \dots, T_h l_g v_n) =$$
$$= (T_h l^*_{h^{-1}} \sigma_0)(T_{gh} l_{g^{-1}} T_h l_g v_1, \dots, T_{gh} l_{g^{-1}} T_h l_g v_n) =$$
$$= \{\text{By the Chain Rule}\} = (T_h l^*_{h^{-1}} \sigma_0)(v_1, \dots, v_n) =$$
$$= \{\text{By construction of } \sigma\} = \sigma(h)(v_1, \dots, v_n)$$

It follows from the above calculation that $l_g^* \sigma = \sigma$ for all $g \in G$ so σ is left-invariant.

We now check that σ is smooth.

To do this, suppose that $\psi : U \to M$ is a coordinate chart with coordinates $(x_1, \dots, x_n)$. Then, for $p \in U$:

$$(\psi^* \sigma)(p)((\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_n})_p) =$$
$$= \sigma(\psi(p))(T_p \psi(\frac{\partial}{\partial x_1})_p, \dots, T_p \psi(\frac{\partial}{\partial x_n})_p) =$$
$$= ((T_{\psi(p)} l_{(\psi(p))^{-1}})^* \sigma_0)(T_p \psi(\frac{\partial}{\partial x_1})_p, \dots, T_p \psi(\frac{\partial}{\partial x_n})_p) =$$
$$= \sigma_0(T_{\psi(p)} l_{(\psi(p))^{-1}} T_p \psi(\frac{\partial}{\partial x_1})_p, \dots, T_{\psi(p)} l_{(\psi(p))^{-1}} T_p \psi(\frac{\partial}{\partial x_n})_p)$$

We observe that this expression depends smoothly on p. Namely, for $i = 1, \dots, n$ the vector

$$T_{\psi(p)} l_{(\psi(p))^{-1}} T_p \psi(\frac{\partial}{\partial x_i})_p \in T_e G$$

depends smoothly on p. Namely, arguing similarly as in Homework 4 Problem 4, we know that $T_{\psi(p)} l_{(\psi(p))^{-1}} T_p \psi(\frac{\partial}{\partial x_i})_p$ depends smoothly on $\psi(p)^{-1}$ on $T_p \psi(\frac{\partial}{\partial x_i})_p$, both of which depend smoothly on p.

Hence, we conclude that: σ **is a positive smooth left-invariant density on** G

Since σ is smooth and G is compact, it follows that σ is integrable on G. Hence $\int_g \sigma$ is a finite real number. Furthermore, since σ is positive, it follows that $\int_G \sigma$ is positive, hence nonzero. In particular, $C := \int_G \sigma \in (0, +\infty)$.

Let

$$\sigma_G := \frac{1}{C} \sigma$$

Then: σ_G **is a left-invariant density on** G **that satisfies** $\int_G \sigma_G = 1$. This shows existence.

Let us now check uniqueness.

Suppose that $\overline{\sigma_G}$ is a left-invariant density on G whose integral over G exists and equals 1.

Then, since $|T_eG|$ is one-dimensional and since $\sigma_G(e) = \sigma_0 \neq 0$ we can find $\alpha \in \mathbb{R}$ such that $\overline{\sigma_G} = \alpha\sigma_0$

By left-invariance of $\overline{\sigma_G}$, for all $h \in G$ and for all $v_1, \ldots, v_n$ in T_hG

$$\overline{\sigma_G}(h)(v_1, \ldots, v_n) = ((l_{h^{-1}})^*\overline{\sigma_G})(h)(v_1, \ldots, v_n) =$$

$$= \overline{\sigma_G}(e)(T_hl_{h^{-1}}v_1, \ldots, T_hl_{h^{-1}}v_n) =$$

$$= \alpha\sigma_0(T_hl_{h^{-1}}v_1, \ldots, T_hl_{h^{-1}}v_n) =$$

$$= \alpha\sigma_G(e)(T_hl_{h^{-1}}v_1, \ldots, T_hl_{h^{-1}v_n}) =$$

$$= \{\text{By an analogous calculation, using the left-invariance of } \sigma_G\} =$$

$$= \alpha\sigma_G(h)(v_1, \ldots, v_n)$$

We deduce that $\overline{\sigma_G} = \alpha\sigma_G$.

Since

$$\int_G \overline{\sigma_G} = \int_G \sigma_G = 1$$

it follows that $\alpha = 1$ so indeed

$$\overline{\sigma_G} = \sigma_G$$

which proves uniqueness.

4.2 Part b)

Let $g \in G$ be fixed.

We consider $r_g^*\sigma_G$, which is a density on G.

We know that for all $h \in G$

$$l_h^*(r_g^*\sigma_G) =$$

$$= (r_g \circ l_h)^*\sigma_G =$$

$$= \{\text{since } r_g \circ l_h = l_h \circ r_g, \text{ i.e left and right multiplication commute}\} =$$

$$= (l_h \circ r_g)^*\sigma_G = r_g^*(l_h^*\sigma_G) =$$

$$= r_g^*\sigma_G$$

Consequently, r_g^* is left-invariant.

Since r_g is a diffeomorphism, we also know by Theorem 3 from the handout on Integrating Densities that:

$$\int_G r_g^*\sigma_G = \int_G \sigma_G = 1$$

By using the uniqueness part of a), it follows that

$$r_g^*\sigma_G = \sigma_G$$

This holds for all $g \in G$.

We may thus conclude that σ_G is right-invariant.

4.3 Part c)

Let μ be an arbitrary positive density on M (which exists by patching together locally defined Lebesgue Densities via a Partition of Unity).

We define:

$$\sigma := \int_G (\phi_g^*\mu)\sigma_G$$

where σ_G is the density defined in a).

More precisely:

Given $p \in M$ and $v_1, \ldots, v_n$ vectors in T_pM

$$\sigma(p)(v_1, \ldots, v_n) := \int_G (\phi_g^*\mu)(p)(v_1, \ldots, v_n)\sigma_G$$

This quantity is well-defined because,arguing as earlier $g \mapsto (\phi_g^*\mu)(p)(v_1, \ldots, v_n)$ is smooth and G is compact.

In this way, we indeed obtain a density on G because if $A : T_pM \to T_pM$ is linear then:

$$\sigma(p)(Av_1, \ldots, Av_n) =$$

$$= \int_G (\phi_g^*\mu)(p)(Av_1, \ldots, Av_n)\sigma_G =$$

$$= \{\text{Since } \phi_g^*\mu \text{ is a density for all } g \in G\} =$$

$$= \int_G |\text{det}A|\phi_g^*\mu(p)(v_1, \ldots, v_n)\sigma_G =$$

$$= |\text{det}A| \int_G \phi_g^*\mu(p)(v_1, \ldots, v_n)\sigma_G =$$

$$= |\text{det}A|\sigma(p)(v_1, \ldots, v_n)$$

So σ is indeed a density on M.

We observe that σ is a positive density on M.

Namely, if $\{v_1, \ldots, v_n\}$ is a basis for T_pM then for all $g \in G$

$$(\phi_g^*\mu)(p)(v_1, \ldots, v_n) = \mu(\phi_g(p))(T_p\phi_g v_1, \ldots, T_p\phi_g v_n) > 0$$

since μ was chosen to be a positive density on M and since ϕ_g is a diffeomorphism of M so $\{T_p\phi_g v_1, \ldots, T_p\phi_g v_n\}$ is a basis of $T_{\phi_g(p)}M$.

Hence σ is a positive density on M.

We now check the invariance of σ under the action of G. In order to do this, we want to "put the pullback under the integral sign"

As before, we fix $p \in M$ and $v_1, \ldots, v_n \in T_pM$

Then, for all $h \in G$

$$(\phi_h^*\sigma)(p)(v_1, \ldots, v_n) =$$

$$= \sigma(\phi_h(p))(T_p\phi_h v_1, \ldots, T_p\phi_h v_n) =$$

$$\int_G (\phi_g^*\mu)(\phi_h(p))(T_p\phi_h v_1, \ldots, T_p\phi_h v_n)\sigma_G =$$

$$= \{\text{By definition of the pullback of a density}\} =$$

$$= \int_G (\phi_h^*(\phi_g^*\mu))(p)(v_1, \ldots, v_n)\sigma_G =$$

$$= ((\phi_g \circ \phi_h)^*\mu)(p)(v_1, \ldots, v_n)\sigma_G =$$

$$= \{\text{By the definition of Group Action: } \phi_g \circ \phi_h = \phi_{gh}\} =$$

$$= \int_G (\phi_{gh}^*\mu)(p)(v_1, \ldots, v_n)\sigma_G$$

Now, let us define $f : G \to \mathbb{R}$ by

$$f(g) := (\phi_g^*\mu)(p)(v_1, \ldots, v_n)$$

Then, the latter integral equals:

$$\int_G f(gh)\sigma_G = \int_G (f \circ r_h)(g)\sigma_G =$$

$$= \int_G (r_h^* f)(g)\sigma_G =$$

$$= \{\text{Using the right-invariance of } \sigma_G \text{ from part b)}\} =$$

$$= \int_G r_h^* f \cdot r^* \sigma_G = \int_G r_h^*(f\sigma_G) =$$

$$= \{\text{By using Theorem 3 from the handout on Integrating Densities}\} =$$

$$= \int_G f\sigma_G =$$

$$= \{\text{By the definition of } f\} =$$

$$= \int_G (\phi_g^* \mu)(p)(v_1, \dots, v_n)\sigma_G =$$

$$= \sigma(p)(v_1, \dots, v_n)$$

Hence:

$$(\phi_h^* \sigma)(p)(v_1, \dots, v_n) = \sigma(p)(v_1, \dots, v_n)$$

This holds for all $h \in G$, $p \in M$ and for all $v_1, \dots, v_n \in T_pM$

Conclusion: σ is a positive density on M that is preserved by the action of G. $\square$