Solutions to Problem Set 5 for 18.101, Fall 2007

1 Exercise 1 Solution

For the counterexample, let us consider $M = (0, +\infty)$ and let us take $V = \frac{\partial}{\partial x}$ on M. Let W be the vector field on M that is identically equal to 0. Then the given ODE has the trivial solution $W_t = 0$. We will now construct another solution to the ODE.

We first extend V to $\mathbb{R}$ by $\overline{V} = \frac{\partial}{\partial x}$, and we extend W to $\mathbb{R}$, by $\overline{W} = f\frac{\partial}{\partial x}$, where $f \in C^\infty(\mathbb{R})$ is an arbitrary function that is identically zero on M. We know that $\Phi_{t\overline{V}}(x) = x + t$ We now let W_t be the restriction to M of $(\Phi_{-t\overline{V}})_*\overline{W}$ Then, we know that $W_0 = W$. Also, by Theorem 9 from the lecture notes, we know that $\frac{\partial W_t}{\partial t} = (L_V)W_t$.

However, by construction, for $x \in M$, we have: $W_t(x) = f(x+t)\frac{\partial}{\partial x}$, which is not identically zero for all $t < 0$ if we take f not to be identically zero on $(-\infty, 0)$. Hence, the given ODE doesn't have a unique solution.

Remark: We observe that the vector field V is not complete. Namely, its trajectories starting from each point are not defined for sufficiently negative time.

2 Exercise 2 Solution

Let us fix $p \in M$, and $t_0 \in \mathbb{R}$.

We define $f(t) := (\Phi_{tV}^*\alpha)(p)$. Then $f(0) = \alpha(p)$, Also, $f : \mathbb{R} \to T_p^*M$ is smooth and

$$\frac{d}{dt}|_{t=t_0} f(t) = \frac{d}{dt}|_{t=t_0}(\Phi_{tV}^*\alpha)(p) =$$

$$= \frac{d}{ds}|_{s=0}(\Phi_{(s+t_0)V}^*\alpha)(p) =$$

$$= \frac{d}{ds}|_{s=0}((\Phi_{sV} \circ \Phi_{t_0V})^*\alpha)(p)$$

$$= \frac{d}{ds}|_{s=0}(\Phi_{t_0V}^* \circ \Phi_{sV}^*)\alpha)(p) =$$

$$\Phi_{t_0V}^*\alpha(\frac{d}{ds}|_{s=0}\Phi_{sV}^*\alpha)(p) =$$

$$= \Phi_{t_0V}^*(L_V\alpha)(p) = \{\text{since } L_V\alpha = 0\} = 0$$

Hence $\frac{d}{dt}f = 0$, so f is constant, and equal to $\alpha(p)$.

It follows that

$$\Phi_{tV}^{*}\alpha = \alpha.$$

3 Exercise 3 Solution

Let us denote by n the dimension of M. We start out by observing that M can be covered by a countable union of sets A_i which are images of $[0,1]^n$ under a diffeomorphism from an open subset of $\mathbb{R}^n$ containing $[0,1]^n$ onto a subset of M. We deduce this fact by using appropriate coordinate charts to find an open cover (not necessarily countable satisfying this property) The fact that we can extract a countable subcover follows from the Second Countability property of subspaces of Euclidean Space (i.e. that every open cover has a countable subcover. This can be proved by noting that every open set is a union of balls centered on rational points with rational radii. A countable subcover can be obtained by representing each open set as a union of these rational balls, then by choosing one set containing each rational ball involved as our countable subcover.)

Since the countable union of sets of measure zero has measure zero and since f "preserves unions" in the sense that $f(\bigcup A_i) = \bigcup f(A_i)$, it follows that we have to show that the image under f of every $f(A_i) \in \mathbb{R}^k$ has measure zero.

For a fixed index i, we look at:

$$[0,1]^n \xrightarrow{\phi_i} A_i \xrightarrow{f} \mathbb{R}^k$$

where $\phi_i : [0,1]^n \to A_i$ is the map constructed earlier.

Now, by compactness of $[0,1]^n$ A_i is also compact and ϕ_i is Lipschitz. Furthermore, by compactness of A_i, f restricted to A_i is also Lipschitz. Hence, $f \circ \phi_i : [0,1]^n \to \mathbb{R}^k$ is Lipschitz, being the composition of two Lipschitz functions. Thus if, we define $F : [0,1]^n \times \mathbb{R}^{k-n} \to \mathbb{R}^k$ by $F(x,y) := f(x)$, F then be a Lipschitz function from a subset of $\mathbb{R}^k$ to $\mathbb{R}^k$.

We know that $F([0,1]^n \times 0) = f(A_i)$.

Hence, we have reduced the claim to showing that $F([0,1]^n \times 0) \in \mathbb{R}^k$ has measure zero.

Let $s \in \mathbb{N}$ be fixed. Let us take a partition of $[0,1]^n$ by s^n cubes in $\mathbb{R}^k$ of sidelength $\frac{1}{s}$ in the canonical way. By "adding $k-n$ dimensions to each cube", we obtain a cover of $[0,1]^n$ by cubes in $\mathbb{R}^k$. We call these cubes $Q_j, j = 1, \ldots s^n$ By the Lipschitz property of F, each $F(Q_j)$ is contained in a cube of sidelength $\leq C\frac{1}{s}$, where the constant $C > 0$ depends only on the function F and the dimension k (and not on s and j).

Hence, it follows that $F([0,1]^n \times 0) \in \mathbb{R}^k$ can be covered by cubes in $\mathbb{R}^k$ whose volume adds up to:

$$\leq C^k s^{-k} s^n = C^k s^{n-k} \to 0 \text{ as } s \to \infty$$

It follows that $F([0,1]^n \times 0) \in \mathbb{R}^k$ can be covered by cubes in $\mathbb{R}^k$ of arbitrarily small volume so $F([0,1]^n \times 0) \in \mathbb{R}^k$ has measure zero. The claim now follows.

4 Exercise 4 Solution

Suppose that f is Riemann Integrable on Q. We argue that:

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$$

We will show that

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \int_Q f(x)dx$$

which we denote by I.

We first observe that both limits indeed exist because $P_{(n+1)}$ is a refinement of P_n. Also, the sequence $U(f, P_n)$ is monotonically decreasing, whereas the sequence $L(f, P_n)$ is monotonically increasing.

Let us show that:

$$\lim_{n\to\infty} U(f, P_n) = I$$

The other case is analogous.

It is true by definition that

$$I = \inf_{P\text{-partition of Q}} U(f, P) \leq \lim_{n\to\infty} U(f, P_n)$$

Let us now show that:

$$I \geq \lim_{n\to\infty} U(f, P_n)$$

Let $\epsilon > 0$ be given. By definition, we can find a partition P such that $U(f, P) \leq \frac{\epsilon}{2}$.

We observe that for n sufficiently large

$$U(f, P_n) \leq U(f, P) + \frac{C}{2^n}$$

for some constant $C > 0$ depending only on the partition P and the function.(*)

We now prove the claim (*)

Let R be a fixed rectangle in the partition P. Then:

$$\sum_{Q_i \in P_n, Q_i \subseteq R} \sup_{Q_i} f|Q_i| \leq \{\text{since each } Q_i \subseteq R\} \leq \sum_{Q_i \in P_n, Q_i \subseteq R} \sup_{R} f|Q_i|$$

$$\leq \sup_{R} f|R|$$

This first estimate deals with the dyadic cubes that are wholly contained in R.

We now have to consider the dyadic cubes that intersect R but are not wholly contained inside it. Intuitively, we expect the contribution from these cubes to the sum $U(f, P_n)$ to be small as n grows large.

Namely, we observe

$$\sum_{Q_i \in P_n, Q_i \bigcap R \neq \emptyset, Q_i \nsubseteq R} \sup_{Q_i} f|Q_i| \leq \sum_{Q_i \in P_n, Q_i \bigcap R \neq \emptyset, Q_i \nsubseteq R} \sup_{Q} f|Q_i| \leq$$

$$\leq K \sup_{Q} f| \text{ boundary of R } |\frac{1}{2^n}$$

for some constant $K > 0$ depending on R. The last observation follows from the fact that all of the Q_i we are considering in the above sum lie in an appropriate "thickening of the boundary of R by $\frac{1}{2^n}$.

Summing over all rectangles R in the partition P, we obtain the claim (*).

We now find $n \in \mathbb{N}$ sufficiently large so that $\frac{C}{2^n} < \frac{\epsilon}{2}$.

Using (*) and the construction of P, we obtain that:

$$U(f, P_n) \leq I + \epsilon$$

Such an n can be found for all $\epsilon > 0$ so, by the fact that $U(f, P_n)$ is monotonically decreasing, we obtain:

$$\lim_{n \to \infty} U(f, P_n) \leq I$$

.

Since we know from before that the opposite inequality holds, we obtain:

$$\lim_{n \to \infty} U(f, P_n) = I$$

Analogously:

$$\lim_{n\to\infty} L(f, P_n) = I$$

Hence:

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = I$$

Conversely,

Suppose that f isn't Riemann Integrable on Q. Let I_+ and I_- denote the Upper and Lower Riemann Integral of f over Q. Then

$$\forall n \in N \; L(f, P_n) \leq I_- < I_+ \leq U(f, P_n)$$

So:

$$\lim_{n\to\infty} L(f, P_n) \leq I_- < I_+ \leq \lim_{n\to\infty} U(f, P_n)$$

In particular:

$$\lim_{n\to\infty} L(f, P_n) \neq \lim_{n\to\infty} U(f, P_n)$$

5 Exercise 5 Solution:

5.1 Part a)

Let k denote the dimension of M. For the first part, we use adapted coordinates (in other words, the Canonical Immersion Theorem). Let $p \in M$ be given. Then, we can find a neighborhood U_p of p in $\mathbb{R}^n$ and a map $\phi : U_p \to \mathbb{R}^n$, which is a diffeomorphism such that:

$$\phi(U_p \cap M) = \{x_{k+1} = \ldots = x_n = 0\}$$

In other words,near p, we get local coordinates $(x_1, ..., x_k)$ on M, which we can extend to local coordinates $(x_1, ..., x_n)$ on $\mathbb{R}^n$.

Let us show how this construction allows us to extend the restriction of V to $U_p \cap M$ to a smooth vector field on U_p.

We observe that on U_p, we can write

$$V = \sum_{i=1}^{k} f_i \frac{\partial}{\partial x_i}$$

where $f_1, ..., f_k$ are smooth functions on U_p. Hence, $f_1 \circ \phi^{-1}, ..., f_k \circ \phi^{-1}$ are smooth functions from $W \times \{0\}$ to $\mathbb{R}$, where W is the open set in $\mathbb{R}^k$ such that $W \times \{0\} = \phi(U_p) \cap (\mathbb{R}^k \times \{0\})$ We can canonically extend $f_1 \circ \phi^{-1}, ..., f_k \circ \phi^{-1}$ to smooth functions on all of $\phi(U_p)$ by letting them be independent of the last $n-k$ variables. Pulling back by ϕ, it follows that we can extend $f_1, ..., f_k$ to smooth functions $\tilde{f}_1, ..., \tilde{f}_k$ on U_p.

If we let

$$\tilde{V_p} := \sum_{i=1}^{k} \tilde{f}_i \frac{\partial}{\partial x_i}$$

$\tilde{V_p}$ will be an extension of V as a vector field on all of U_p. (Namely, we are now thinking of $x_1, ..., x_k$ as coordinates on $\mathbb{R}^n$)

We now consider the collection of sets

$$\{\{U_p,\ p \in M\}, \{\mathbb{R}^n - M\}\}$$

By the previous construction and the fact that M is closed, it follows that this collection gives us an open cover of $\mathbb{R}^n$.

Let $\{\psi_p; p \in M, \psi_0\}$ be a subordinate Partition of Unity of this open cover.

It follows that then:

$$\sum_{p \in M} \psi_p = 1 \text{ on } M.\mathbf{(1)}$$

In addition to that, by construction we know:

$$(\psi_p \tilde{V_p})|_M = (\psi_p V)|_M \ \forall p \in M\mathbf{(2)}$$

Hence, if we define

$$U := \sum_{p \in M} \psi_p \tilde{V_p}$$

then U will be a smooth vector field on $\mathbb{R}^n$

By **(2)**, we have that

$$U|_M = \sum_{p \in M} (\psi_p V)|_M$$

By **(1)**, this is precisely equal to V.

Hence, U gives us the desired extension.

5.2 Part b)

Arguing as we did in the previous part of the problem, given a point $p \in M$, we can find an open set $W_p \subseteq \mathbb{R}^n$ containing p on which we can extend f to a smooth map $\tilde{(f_p)} : W_p \to \mathbb{R}^k$ (the fact that the map f is vector-valued doesn't affect the previous proof)

Let

$$U := \bigcup_{p \in M} W_p$$

Then, $U \subseteq \mathbb{R}^n$ is an open set containing M, and $\{\{W_p\},\ p \in M\}$ is an open cover of U.

We find a Partition of Unity $\{\psi_p,\ p \in M\}$ subordinate to this cover.

If we let $\tilde{f} := \sum_{p \in M} \psi_p \tilde{f}_p,\ \tilde{f}$ gives us a smooth map $\tilde{f} : U \to \mathbb{R}^k$

Since $\forall p \in M\ \ \psi_p \tilde{(f_p)}|_M = \psi_p f|_M$ and since $\sum_{p \in M} \psi_p = 1$ by properties of the partition of unity, we have that $\tilde{f}$ extends f to all of U.

If M is closed, we consider as before the open cover:

$$\{\{U_p,\ p \in M\}, \{\mathbb{R}^n - M\}\}$$

of $\mathbb{R}^n$

We look at a subordinate Partition of Unity:

$$\{\psi_p; p \in M, \psi_0\}$$

Then, we have $\sum_{p \in M} \psi_p = 1$ on M.

We argue as in part a) to deduce that

$$\tilde{f} := \sum_{p \in M} \psi_p f_p$$

is a smooth map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^k$ that extends f.

6 Exercise 6 Solution

6.1 Part a)

We first observe that, under the assumptions on f and g, $f * g$ is well-defined. Namely, for given $x \in \mathbb{R}^n$, the integrand in the integral is nonzero outside a compact set by the assumption that g is compactly supported. Since f, g are continuous, $(f * g)(x)$ is thus given as an integral of a bounded continuous function over a compact set, which is indeed well-defined.

Let us now check that $(f * g)$ is continuous.

Let us fix $x \in \mathbb{R}^n$. We consider $v \in \mathbb{R}^n$ with $|v| \leq 1$

Then

$$(f * g)(x+v) - (f * g)(x) = \int_{\mathbb{R}^n} f(s)(g(x+v-s) - g(x-s))ds$$

Since g is compactly supported and since we are considering $v \in \mathbb{R}^n$ with $|v| \leq 1$, it follows that the integrand is zero outside a compact set $K \subseteq \mathbb{R}^n$ which depends on x, but not on v.

In other words, for all $v \in \mathbb{R}^n$ with $|v| \leq 1$

$$(f * g)(x+v) - (f * g)(x) = \int_K f(s)(g(x+v-s) - g(x-s))ds$$

By using the Mean Value Theorem, the Triangle inequality the compactness of K, it follows that there exists a constant $C > 0$ such that $\forall s \in K$ and $\forall v \in \mathbb{R}^n$ with $|v| \leq 1$

$$|g(x+v-s) - g(x-s)| \leq C|v|$$

Combining the previous results, we obtain:

$\forall v \in \mathbb{R}^n$ with $|v| \leq 1$

$$|(f * g)(x+v) - f(x)| \leq C|v| \int_K |f(s)|ds$$

Since f is continuous and K is compact, $\int_K |f(s)|ds$ is finite. Consequently, $C|v| \int_K |f(s)|ds$ converges to 0 as $|v| \to 0$

Hence $(f * g)$ is continuous.

6.2 Part b)

Let $i \in \{1, ..., n\}$ be given. From the first part, we know that $f * D_i g$ makes sense.

For $h \in \mathbb{R} - \{0\}$ with $|h| \leq 1$ we look at the expression:

$$\frac{(f * g)(x + he_i) - (f * g)(x)}{h} - (f * D_i g)(x) =$$

$$\int_{\mathbb{R}^n} f(s)(\frac{g(x+he_i-s)-g(x-s)}{h} - D_i g(x-s))ds$$

We use Taylor's Theorem for the function $t \longrightarrow (f*g)(x+te_i-s)$ and the fact that g is smooth and compactly supported, it follows that there exists a constant $C>0$ and a compact subset $K \subseteq \mathbb{R}^n$ such that for all $h \in \mathbb{R}^n - \{0\}$ with $|h| \leq 1$

$$|\frac{(f*g)(x+he_i)-(f*g)(x)}{h} - (f*D_i g)(x)| \leq C|h| \int_K |f(s)|ds$$

The quantity on the right-hand side converges to 0 as $|h| \to 0$

It follows that $D_i(f*g)$ exists and equals $f*D_i g$.

Iterating this procedure, we obtain that $f*g$ is smooth.

6.3 Part c)

Suppose that f is a continuous function on $\mathbb{R}^n$.

Let $K \subseteq \mathbb{R}^n$ be compact.

Let $x \in K$ be fixed.

Then, for $t>0$, we observe by using a Change of Variable that $\int_{\mathbb{R}^n} \phi_t = \int_{\mathbb{R}^n} \phi = 1$. Hence:

$$(f*\phi_t)(x) - f(x) = \int_{\mathbb{R}^n} f(x-s)\phi_t(s)ds - f(x) =$$

$$= \{ \text{ Since} \int_{\mathbb{R}^n} \phi_t = 1\} = \int_{\mathbb{R}^n} f(x-s)\phi_t(s)ds - \int_{\mathbb{R}^n} f(x)\phi_t(s)ds =$$

$$= \int_{\mathbb{R}^n} (f(x-s)-f(x))\phi_t(s)ds = \{ \text{ By the support properties of } \phi_t\} =$$

$$= \int_{|s|\leq t} (f(x-s)-f(x))\phi_t(s)ds$$

Thus, we have:

$$|(f*\phi_t)(x) - f(x)| \leq \sup_{|y|\leq t} |f(x-y)-f(x)| \int_{\mathbb{R}^n} |\phi_t(y)|dy$$

We observe as before by rescaling that $\int_{\mathbb{R}^n} |\phi_t(s)|ds$ is independent of $t>0$

Hence, we just have to show that $\sup_{|y|\leq t} |f(x-y)-f(x)| \to 0$ uniformly in $x \in K$. **(*)**

To do this, we define $\Phi : K + \bar{B}(0,1) \to \mathbb{R}$ by:

$$\Phi(x,y) := f(x-y) - f(x)$$

Φ is continuous because f is continuous. Furthermore, $K + \bar{B}(0,1)$ is compact. Hence Φ is uniformly continuous on $K + \bar{B}(0,1)$. Using this observation, (*) immediately follows so the claim holds.

6.4 Part d)

Let f be a continuous real-valued function on M and let $\epsilon > 0$ be given.

Let $p \in M$ be given. Let n denote the dimension of M.

We find a coordinate chart (U, ϕ) centered at p (i.e. such that $\phi(p) = 0$)

Then $f \circ \phi^{-1}$ is a continuous function on $\phi(U)$, which is an open subset of $\mathbb{R}^n$.

We find a closed ball $V \subseteq \mathbb{R}^n$ centered at 0 that is contained in $\phi(U)$.

Then, the restriction of $f \circ \phi^{-1}$ to V is a continuous function defined on a closed ball of $\mathbb{R}^n$,.Let us observe that we can extend $f \circ \phi^{-1}|_V$ to a continuous function $F : \mathbb{R}^n \to \mathbb{R}$. To do this, we argue similarly as in Problem 5. Let f_V denote the given restriction of f We can find for each point $q \in V$ an open set W_q in $\mathbb{R}^n$ on which we can extend f_V to W_q. Call this extension f_V^q. Since V is closed and the W_q cover V, we get that $\{W_q, \ ; q \in V, \mathbb{R}^n - V\}$ is an open cover of $\mathbb{R}^n$. We find a subordinate partition of unity $\{\alpha_q, \ ; q \in V, \ \alpha\}$ We consider,

$$\sum_{q \in V} \alpha_q f_V^q$$

Arguing analogously as in Problem 5, we obtain that this gives us a continuous extension of f_V to all of $\mathbb{R}^n$.

By using part c), we can find a smooth function $G : \mathbb{R}^n \to \mathbb{R}$ such that $|F - G| < \epsilon$ on V.

We now find an open ball $W \subseteq \mathbb{R}^n$ which is centered at 0 and which is contained in V.

Let

$$W_p := \phi^{-1}(W), \ g_p := (G \circ \phi) \text{ restricted to } W_p$$

Then, by construction W_p is an open subset of M containing p and $g_p : W_p \to \mathbb{R}$ is smooth.

Furthermore, since $\phi(W_p) \subseteq V$, we have that on W_p $|f - g_p| = |(F \circ \phi) - (G \circ \phi)| < \epsilon$ by the fact that $|F - G| < \epsilon$ on V.

We obtain in this way an Open Cover of M:

$$\{\{W_p\},\ p \in M\}$$

We find a corresponding Partition of Unity $\{\psi_p,\ p \in M\}$
We define:

$$g := \sum_{p \in M} \psi_p g_p$$

Then $g : M \to \mathbb{R}$ is smooth.
By using the fact that $\sum_{p \in M} \psi_p = 1$ and the fact that $\psi_p \geq 0\ \forall p \in M$, we obtain:

$$|f - g| = |\sum_{p \in M} (\psi_p f) - \sum_{p \in M} (\psi_p g_p)| \leq$$

$$\leq \{ \text{ By the Triangle Inequality}\} \leq \sum_{p \in M} \psi_p |f - g_p| \leq$$

$$\leq \{ \text{ Since by construction } \psi_p |f - g_p| < \psi_p \epsilon\} < \sum_{p \in M} \psi_p \epsilon = \epsilon$$

This proves the claim.