Solutions to Problem Set 4 for 18.101, Fall 2007

1 Exercise 1 Solution

Suppose that v is a vector field on $\mathbb{R}^N$ which is tangent to a closed submanifold M of $\mathbb{R}^N$. Let $\gamma : (a,b) \to M$ be an integral curve of v which intersects M, i.e. such that $\gamma(\theta) \in M$ for some $\theta \in (a,b)$. We are supposing that (a,b) is the maximal interval of existence of γ.

We define $A := \{t \in (a,b) \text{ such that } \gamma(t) \in M\}$. Let us argue that $A = (a,b)$.

We know that $\theta \in A$ so **A is nonempty. (1)**

Furthermore, we observe that A is open in (a,b). Suppose that $t_0 \in A$. Then, we find a coordinate neighborhood $(U, x_1, \ldots, x_n)$ of $\gamma(t_0)$ in M. We can then find a diffeomorphism $\phi : U \to V$, where $V \subseteq \mathbb{R}^N$ is open. Then $(\phi)_*(v)$ is a C^1 vector field on V. Hence, this vector field is locally Lipschitz on V. By using the Existence Part of the Existence and Uniqueness Theorem for ODEs, we can find $\epsilon > 0$ and $\sigma : (t_0 - \epsilon, t_0 + \epsilon) \to M$ such that:

$$\sigma'(t) = (\phi_* v)(\sigma(t)), \forall t \in (t_0 - \epsilon, t_0 + \epsilon)$$
$$\sigma(t_0) = \phi \circ \gamma(t_0)$$

However, we also know that, since γ is an integral curve of v that $\phi \circ \gamma$ is an integral curve of $\phi_* v$, namely:

$$\phi \circ \gamma'(t) = (\phi_* v)(\sigma(t)), \forall t \ \in (t_0 - \epsilon, t_0 + \epsilon)$$

and

$$\phi \circ \gamma(t_0) = \sigma(t_0)$$

By using the Uniqueness Part of the Existence and Uniqueness Theorem for ODEs, it follows that

$$\phi \circ \gamma(t) = \sigma(t) \ , \forall t \in (t_0 - \epsilon, t_0 + \epsilon)$$

Since $\sigma(t) \in V \ , \forall t \in (t_0 - \epsilon, t_0 + \epsilon)$, it follows that $\gamma(t) \in \phi^{-1}(V) = U \subseteq M \ , \forall t \in (t_0 - \epsilon, t_0 + \epsilon)$

Hence $(t_0 - \epsilon, t_0 + \epsilon) \subseteq A$. Such an $\epsilon > 0$ can be found for all $t_0 \in A$, thus **A is open in (a,b). (2)**

Finally, we observe that A is closed in (a,b). To see this, suppose that (t_n) is a sequence in A such that $t_n \to t$, for some $t \in (a,b)$. By continuity of γ, we know that $\gamma(t_n) \to \gamma(t)$. By construction of (t_n), we know that $\gamma(t_n) \in M$ $,\forall n$. Using the last two observations together with the assumption that M is closed, it follows that $\gamma(t) \in M$. By definition of A, this implies that $t \in A$. Hence **A is closed in (a,b). (3)**

Using **(1),(2),(3)** together with the connectedness of (a, b), it follows that $A = (a, b)$.

Consequently, γ is entirely contained in M, as was claimed.

2 Exercise 2 Solution

2.1 Part a)

We know that:

$$dH(v_H) = \sum_i (\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i}) = 0$$

Hence H is a conserved quantity of v_H.

2.2 Part b)

Suppose that F is smooth. Then:

$$L_{v_H} F = \sum_i (\frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i})$$

$$L_{v_F} H = \sum_i (\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i})$$

From here, it immediately follows that

$$L_{v_H} F = -L_{v_F} H$$

2.3 Part c)

Suppose that F is a conserved quantity of v_H. This means that, $L_{v_H}(F) = 0$. The latter is equivalent to:

$$\sum_i (\frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i}) = 0 \ (*)$$

We know that:

$$v_H = \sum_i (\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i})$$

$$v_F = \sum_j (\frac{\partial F}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial}{\partial q_j})$$

We then calculate that:

$$[v_H, v_F] =$$

$$= \sum_j ((\sum_i (\frac{\partial H}{\partial q_i}\frac{\partial^2 F}{\partial p_i \partial q_j} - \frac{\partial H}{\partial p_i}\frac{\partial^2 F}{\partial q_i \partial q_j} - \frac{\partial F}{\partial q_i}\frac{\partial^2 H}{\partial p_i \partial q_j} + \frac{\partial F}{\partial p_i}\frac{\partial^2 F}{\partial q_i \partial q_j}))\frac{\partial}{\partial p_j} +$$

$$+(\sum_i (-\frac{\partial H}{\partial q_i}\frac{\partial^2 H}{\partial p_i \partial p_j} + \frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial q_i \partial p_j} + \frac{\partial F}{\partial q_i}\frac{\partial^2 H}{\partial p_i \partial p_j} - \frac{\partial F}{\partial p_i}\frac{\partial^2 H}{\partial q_i \partial p_j}))\frac{\partial}{\partial q_j})$$

We now want to argue that for all j:

$$\sum_i (\frac{\partial H}{\partial q_i}\frac{\partial^2 F}{\partial p_i \partial q_j} - \frac{\partial H}{\partial p_i}\frac{\partial^2 F}{\partial q_i \partial q_j} - \frac{\partial F}{\partial q_i}\frac{\partial^2 H}{\partial p_i \partial q_j} + \frac{\partial F}{\partial p_i}\frac{\partial^2 F}{\partial q_i \partial q_j}) = 0 \text{ (A)}$$

$$\sum_i (-\frac{\partial H}{\partial q_i}\frac{\partial^2 H}{\partial p_i \partial p_j} + \frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial q_i \partial p_j} + \frac{\partial F}{\partial q_i}\frac{\partial^2 H}{\partial p_i \partial p_j} - \frac{\partial F}{\partial p_i}\frac{\partial^2 H}{\partial q_i \partial p_j} = 0 \text{ (B)}$$

However, we observe that **(A)** follows from differentiating equality $(*)$ with respect to q_j and that **(B)** follows from differentiating $(*)$ with respect to p_j.

Hence $[v_H, v_F] = 0$ so v_H and v_F commute.

3 Exercise 3 Solution

Let M be a compact manifold and led v be a smooth vector field on M. Since M is compact, the flow Φ_{tv} of v is defined for all times t.

3.1 Part a)

Let h be a smooth function on M such that $(L_v)^N h = 0$ for N sufficiently large. We argue that for all times $t \in \mathbb{R}$

$$e^{tL_v h} := \sum_{n=0}^{\infty} \frac{t^n}{n!}(L_v)^n h = (\Phi_{tv})^* h$$

We will use the Existence and Uniqueness Theorem for ODEs. We fix $p\ in M$. We let $F(t) := e^{tL_v h}(p)$ and we let $G(t) := (\Phi_{tv})^* h(p)$.The assumption that $(L_v)^N h = 0$ for N sufficiently large implies that $F(t)$ is well-defined, that it is a smooth function in t and that we can differentiate in t term by term. Hence $F'(t) = (\sum_{n=0}^{\infty} \frac{t^n}{n!}(L_v)^{n+1} h)(p) = L_v(\sum_{n=0}^{\infty} \frac{t^n}{n!}(L_v)^n h)(p) = v(F(t))$ We know that: $G'(t) = L_v((\Phi_{tv})^* h)(p) = v(G(t))$ by Theorem 9 from the handout on the Flows of Vector Fields on Manifolds. Since F(0)=G(0)=h(p), the claim indeed follows from the Existence and Uniqueness Theorem for ODEs.

3.2 Part b)

Suppose now that w is a smooth vector field on M such that $(L_v)^N w = 0$ for all N sufficiently large.

Let us define $w_t := \sum_{n=0}^{\infty} \frac{t^n}{n!}(L_v)^n w$. Using the fact that the a priori infinite series terminates. Hence, arguing as in part a) we can differentiate term by term to obtain:

$$\frac{\partial}{\partial t} w_t = L_v w_t$$

Using Theorem 9 from the handout on the Flows of Vector Fields on Manifolds, we get that

$$w_t = \Phi_{tv}^* w.$$

Hence, we get:

$$e^{tL_v w} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(L_v)^n w = \Phi_{tv}^* w$$

as was claimed.

4 Exercise 4 solution

4.1 Part a)

Let $g \in G$ be given. We define $\phi_g : G \to G \times G$ to be $\phi_g(h) := (g, h)$. We have that then:

$$G \stackrel{\phi_g}{\to} G \times G \stackrel{m}{\to} G$$

and $m \circ \phi_g = l_g$. Hence, we get:

$$TG \stackrel{T\phi_g}{\to} TG \times TG \stackrel{Tm}{\to} TG$$

which by the Chain Rule and the previous observation is:

$$TG \stackrel{Tl_g}{\to} TG.$$

We observe that for $g \in G$ we have: $T\phi_g(I, v_0) = ((g, 0), (I, v_0))$ Hence, combining the previous facts, it follows that: given $v_0 \in T_I G$, we have:

$$(I, v_0) \stackrel{T_I\phi_g}{\to} ((g, 0), (I, v_0)) \stackrel{T_{(g,I)}m}{\to} (g, T_I l_g(v_0))$$

Now, the map

$$g \in G \mapsto ((g, 0), (I, v_0))$$

is smooth in g. Furthermore $Tm : TG \to TG$ is smooth because m is smooth. Hence the map:

$$g \in G \mapsto (g, T_I l_g(v_0))$$

is a smooth map. Hence, if we define v by $v(g) := T_I l_g(v_0)$, then $v(g) \in T_I G$ by construction and by the previous observation $v(g)$ will depend smoothly on g. Hence, v defines a smooth vector field on G.

4.2 Part b)

Suppose that v is a vector field as constructed from $v_0 \in T_I M$ as in part a). Suppose that $g, h \in G$ are given. We have:

$$((l_h)_*)(g) = (T_{h^{-1}g} l_h)(v(h^{-1}g)) = T_{h^{-1}g} l_h \circ T_I l_{h^{-1}g}(v_0) =$$

$$= \{ \text{ by the Chain Rule } \} = T_I(l_h \circ l_{h^{-1}g})(v_0) = \{ \text{ by the construction of } v \} = v(g)$$

Hence, v is left-invariant.

Conversely, suppose that v is a left-invariant vector field on G We then have that $\forall g \in G$:

$$v(g) = \{ \text{ by left-invariance } \} = ((l_g)_* v)(g) = T_I l_g(v(I))$$

So, v is obtained as in part a) if we set $v_0 := v(I)$.

Remark: Combining parts a) and b) it follows that every left invariant vector field is smooth.

4.3 Part c)

Suppose that v, w are left-invariant vector fields on G. We have that then for all $h \in G$:

$$(l_h)_*[v, w] = \{ \text{ by using the result of Problem } 4b) \text{ from last week's homework } \} =$$

$$= [(l_h)_* v, (l_h)_* w] = \{ \text{ by left invariance of } v \text{ and } w \} = [v, w]$$

4.4 Part d)

Suppose that v is a left-invariant vector field on G. We argue that Φ_{tv} is defined for all time.

Suppose that $g \in G$ is given. We define $\gamma_g(t) := \Phi_{tv}(p)$. We want to show that γ_g is defined for all time. We show that it is defined for all positive times. The claim for all negative times follows by symmetry. We suppose that γ_p is defined on $[0, a)$ for some $a > 0$. Let us now show that γ_p can be extended beyond time a.

Let $h := \gamma_g(\frac{a}{2})$.

We define $\sigma : [0, a) \to G$ by:

$$\sigma(t) := (l_{hg^{-1}} \circ \gamma_g)(t)$$

Then, we know:

$$\sigma(0) = (l_{hg^{-1}} \circ \gamma_g)(0) = l_{hg^{-1}}(g) = h$$

and

$$\sigma'(t) = T_g l_{hg^{-1}}(\gamma_g^{'}(t)) = \{ \text{ by the fact that } \gamma_g \text{ is an integral curve of } v \} = T_g l_{hg^{-1}}(v(\gamma_g(t)))$$

$$= ((l_{hg^{-1}})_* v)((l_{hg^{-1}} \circ \gamma_g)(t))) = \{ \text{ by left-invariance of } v \} =$$

$$= v((l_{hg^{-1}} \circ l_g)(t)) = v(\sigma(t))$$

We know that:

$$\gamma_g(\frac{a}{2}) = h \text{ and } \frac{d}{dt}(\gamma_g(t + \frac{a}{2})) = v(\gamma_g(t + \frac{a}{2}))$$

By the Existence and Uniqueness Theorem for ODEs, it follows that

$$\sigma(t) = \gamma_g(t + \frac{a}{2}) \ \forall t \in [\frac{a}{2}, a)$$

Hence, we can extend γ_g to $[0, \frac{3a}{2})$ by letting it equal $\sigma(t - \frac{a}{2})$ on $[a, \frac{3a}{2})$.

It follows that the flow Φ_{tv} is defined for all time.

Suppose v is a left-invariant vector field on G as above and suppose that $g \in G$ is given.

We now argue that:

$$\Phi_v(g) = m(g, \Phi_v(I))$$

We let $\gamma(t) := \gamma_I(t)$. By the previous observation, γ is defined for all $t \in \mathbb{R}$. Let $\sigma(t) := m(g, \gamma(t)) = g \cdot \gamma(t)$. The earlier calculation shows that σ is an integral curve of v with $\sigma(0) = g$. Hence we get $\sigma(t) = \Phi_{tv}(g)$.

We deduce that $\forall t \in \mathbb{R}$:

$$\Phi_{tv}(g) = m(g, \gamma(t)) = m(g, \Phi_{tv}(I))$$

We set $t = 1$ to deduce that:

$$\Phi_v(g) = m(g, \Phi_v(I))$$

Hence, flowing by a left-invariant vector field corresponds to multiplication on the right.