Solutions to Problem Set 3 for 18.101, Fall 2007

1 Exercise 1 solution

We write $f = f(u, v)$ and $g = g(u, v) = (g_1(u, v), g_2(u, v))$. Then $f \circ g : \mathbb{R}^2 \to \mathbb{R}$ is given by: $(f \circ g)(u, v) = f(g_1(u, v), g_2(u, v))$. By using the Chain Rule, it follows that:

$$D_2(f\circ g)(u,v) = (D_1 f)(g_1(u,v), g_2(u,v))\cdot(D_2 g_1)(u,v)+(D_2 f)(g_1(u,v), g_2(u,v))\cdot(D_2 g_2)(u,v)$$

Applying the Chain Rule again, we obtain:

$$D_1 D_2(f \circ g)(u,v) =$$

$$\begin{gathered}
(D_1^2 f)(g_1(u,v), g_2(u,v)) \cdot (D_1 g_1)(u,v) \cdot (D_2 g_1)(u,v) \\
+(D_2 D_1 f)(g_1(u,v), g_2(u,v)) \cdot (D_1 g_2)(u,v) \cdot (D_2 g_1)(u,v) \\
+(D_1 f)(g_1(u,v), g_2(u,v)) \cdot (D_1 D_2 g_1)(u,v) \\
+(D_1 D_2 f)(g_1(u,v), g_2(u,v)) \cdot (D_1 g_1)(u,v) \cdot (D_2 g_2)(u,v) \\
+(D_2^2 f)(g_1(u,v), g_2(u,v)) \cdot (D_1 g_2)(u,v) \cdot (D_2 g_2)(u,v) \\
+(D_2 f)(g_1(u,v), g_2(u,v)) \cdot (D_1 D_2 g_2)(u,v)
\end{gathered}$$

From the preceding formula, we may conclude that, in order to find $D_1 D_2(f \circ g)$ at a point $(u, v) \in \mathbb{R}^2$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a known function, it is not enough to know the values of the first partial derivatives of the components of $g = (g_1, g_2)$ at (u, v). As the above formula shows, we need to know $(D_1 D_2 g_1)(u, v)$ and $(D_1 D_2 g_2)(u, v)$ in order to find the quantity $D_1 D_2(f \circ g)(u, v)$.

2 Exercise 2 solution

Throughout the rest of the problem set, we use the notation: $V(f) := df(V)$, whenever V is a vector field and f is a function.

2.1 Part a)

We work in a coordinate chart $(U, x_1, ..., x_n)$ on M. Suppose that $f, g : M \to \mathbb{R}$ are smooth. We know that on U:

$$d(fg) = \sum_{i=1}^{n} \frac{\partial(fg)}{\partial x_i} dx_i =$$

$$= \{\text{by the Product Rule}\} =$$

$$= \sum_{i=1}^{n} f \frac{\partial g}{\partial x_i} dx_i + \sum_{i=1}^{n} g \frac{\partial f}{\partial x_i} dx_i =$$

$$= fdg + gdf$$

Since this is a local property, the claim follows.

2.2 Part b)

Suppose now that M, N are manifolds and $f : M \to N$ and $g : N \to \mathbb{R}$ are smooth. Let v be a smooth vector field on M. We then have that:

$$f^* dg(v) = dg \circ Tf(v) = Tf(v)(g) = v(g \circ f) = d(g \circ f)(v)$$

Hence indeed:

$$f^* dg = d(g \circ f)$$

2.3 Part c)

Let M, f be as before. Let α be a 1-form on N, and let $\lambda : N \to \mathbb{R}$ be smooth. Suppose $p \in M$, and $X \in T_p M$ are given. Then we have:

$$(f^*(\lambda\alpha))_p(X) = (\lambda\alpha)_{f(p)}(T_p f(X)) =$$

$$= \lambda(f(p)) \cdot \alpha_{f(p)}(T_p f(X)) = ((\lambda \circ f)(p)) \cdot (f^*(\alpha))_p(X)$$

Thus we indeed get:

$$f^*(\lambda\alpha) = (\lambda \circ f) f^*(\alpha)$$

2.4 Part d)

With f, α as in part c), let us suppose that v is a vector field on M. Then, for all $p \in M$, we have:

$$((f^*\alpha(v))(p) = \alpha_{f(p)}(T_p f(v(p))) =$$

$$= (\alpha(f_* v))(f(p)) = ((\alpha(f_* v) \circ f)(p)$$

It follows that:

$$\alpha(f_* v) \circ f = f^* \alpha(v)$$

2.5 Part e)

Let $f : M \to N$ be a diffeomorphism. Suppose that v is a vector field on M and that $\lambda : M \to \mathbb{R}$ is smooth. Let $p \in N$ be given. Since f is a diffeomorphism, we can find $q \in M$ such that $f(p) = q$.

$$(f_*(\lambda v))(p) = (f_*(\lambda v)(f(q)) = T_q f((\lambda v)(q)) =$$

$$= \lambda(q) T_q f(v(q)) = \lambda(q)(f_* v)(p) =$$

$$= ((\lambda \circ f^{-1}) f_* v)(p)$$

Consequently, we get:

$$f_*(\lambda v) = (\lambda \circ f^{-1}) f_* v$$

3 Exercise 3 solution

Suppose that $f : M \to \mathbb{R}$ is as in the problem. By compactness of M, f achieves its minimum and maximum. Using the fact that M is not the empty set or a single point, it follows that we can find distinct points p_1, p_2 in M which are either a local maximum or local minimum for f. We want to argue that $(df)_{p_i} = 0$ for $i = 1, 2$. Let $i \in \{1, 2\}$. Then, we can find a coordinate system $(U, x_1, ..., x_n)$ around p_i. We know that $\frac{\partial f}{\partial x_j}(p_i) = 0$ for all $j = 1, ..n$. Hence, we have:

$$(df)_{p_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p_i) \cdot (dx_j)_{p_i} = 0$$

The claim now follows.

4 Exercise 4 solution

In this problem, we again use the notation:

$$V(f) := df(V)$$

when V is a vector field and f is a smooth function. Hence, for a smooth function f and smooth vector fields V and W, we define:

$$[V, W](f) := V(W(f)) - W(V(f))$$

4.1 Part a)

Let V, W be vector fields on M.It suffices to check that $[V, W]$ locally defines a vector field. The fact that it is unique is immediate since we know how it acts on functions. Suppose that $(U, x_1, ..., x_n)$ is a coordinate system on M. In this coordinate system, we write $V = \sum_{i=1}^n V_i \frac{\partial}{\partial x_i}, W = \sum_{j=1}^n W_j \frac{\partial}{\partial x_j}$, for some smooth functions $V_i, W_j; i, j = 1, ..., n$ on U. Let $f : M \to \mathbb{R}$ be smooth. We know that then on U

$$[V, W](f) = V(W(f)) - W(V(f)) =$$

$$= \sum_{i=1}^n V_i \frac{\partial}{\partial x_i} (\sum_{j=1}^n W_j \frac{\partial f}{\partial x_j}) - \sum_{j=1}^n W_j \frac{\partial}{\partial x_j} (\sum_{i=1}^n V_i \frac{\partial f}{\partial x_i}) =$$

$$= \sum_{i,j=1}^n (V_i \frac{\partial W_j}{\partial x_i} \frac{\partial f}{\partial x_j} + V_i W_j \frac{\partial^2 f}{\partial x_i \partial x_j})$$

$$- \sum_{i,j=1}^n (W_j \frac{\partial V_i}{\partial x_j} \frac{\partial f}{\partial x_i} + W_j V_i \frac{\partial^2 f}{\partial x_j \partial x_i}) =$$

$$= \sum_{j=1}^n (\sum_{i=1}^n (V_i \frac{\partial W_j}{\partial x_i} - W_i \frac{\partial V_j}{\partial x_i}) \frac{\partial f}{\partial x_j})$$

Hence, on U, we have:

$$[V, W] = \sum_{j=1}^n (\sum_{i=1}^n (V_i \frac{\partial W_j}{\partial x_i} - W_i \frac{\partial V_j}{\partial x_i}) \frac{\partial}{\partial x_j})$$

It follows that $[V, W]$ is indeed a well-defined vector field.

Remark: The reason why we subtract the other term is because we want to cancel the term which involves "second derivatives" so that we get a vector field.

4.2 Part b)

We begin by observing the following fact about pushforwards:

$(*)$Let $F : M \to N$ be a diffeomorphism between two manifolds. Let X, Y be smooth vector fields on M and N respectively. Then $Y = F_*X$ if and only if $X(f \circ F) = Y(f) \circ F$ for all $f : N \to \mathbb{R}$ which are smooth.

Proof of $(*)$

Let $p \in M$ be given. Suppose $f : N \to \mathbb{R}$ is smooth. Then $((F_*X)(f))(F(p)) = (X(F \circ f))(p)$ We know fromt the previous equality that $Y = F_*X$ if and only if for all f and p as above we have $Y(f)(F(p)) = X(F \circ f)(p)$, which is equivalent to: $(Y(f) \circ F)(p) = X(F \circ f)(p)$. From here $(*)$ immediately follows.

We now use $(*)$ to prove the claim.

Suppose that v, w are as in the problem. Suppose that $g : M \to \mathbb{R}$ is smooth. Then, we have:

$$v(w(g \circ f)) = \{\text{by } (*)\} = v(f_*w(g) \circ f) =$$
$$= \{\text{by } (*)\} = f_*v(f_*w(g)) \circ f$$

Analogously, we have:

$$w(v(g \circ f)) = f_*w(f_*v(g)) \circ f$$

Subtracting the last two equalities, we get, for all smooth functions $g : M \to \mathbb{R}$:

$$[v, w](g \circ f) = [f_*v, f_*w](g)$$

Using $(*)$ (the converse direction now), it follows that:

$$f_*[v, w] = [f_*v, f_*w]$$

This proves the claim.

4.3 Part c)

Let u, v, w be smooth vector fields.

Suppose that $f : m \to \mathbb{R}$ is smooth. We know that then:

$$[u,[v,w]](f) = u([v,w](f)) - [v,w](u(f)) =$$
$$= u(v(w(f))) - u(w(v(f))) - v(w(u(f))) + w(v(u(f)))$$

By symmetry:

$$[v,[w,u]](f) = v(w(u(f))) - v(u(w(f))) - w(u(v(f))) + u(w(v(f)))$$

$$[w,[u,v]](f) = w(u(v(f))) - w(v(u(f))) - u(v(w(f))) + v(u(w(f)))$$

Summing the last three equalities, we obtain the Jacobi Identity. Hence indeed:

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$