# 1 Exercise 1 solution

## 1.1 Part (a)

We define the map $F : \mathbb{R}^4 \to \mathbb{R}^2$ by:

$$F(x_1, x_2, x_3, x_4) := (x_1^2 + x_2^2 - x_3^2 - x_4^2, x_1^2 + x_2^2 + x_3^2 + x_4^2 - 4)$$

Then, we have that the wanted set $M$ is equal to $F^{-1}\{0\}$. If we show that 0 is a regular value of $F$, it will follow that $M$ is a submanifold of $\mathbb{R}^4$ of dimension $4 - 2 = 2$.

We know that at $(x_1, x_2, x_3, x_4)$ the differential of $F$ equals

$$\begin{bmatrix} 2x_1 & 2x_2 & -2x_3 & -2x_4 \\ 2x_1 & 2x_2 & 2x_3 & 2x_4 \end{bmatrix}$$

If $(x_1, x_2, x_3, x_4) \in F^{-1}\{0\}$ then we can check that $x_1$ and $x_2$ can't simultaneously be 0. Similarly, both $x_3$ and $x_4$ can't be 0. Hence, the above matrix has full rank. Thus $M$ is indeed a 2 dimensional submanifold of $\mathbb{R}^4$.

## 1.2 Part (b)

We know that the tangent space to $M$ at $p = (1, 1, -1, -1)$ is 2 dimensional, and given by the kernel of $T_pF$. As computed above,

$$T_pF = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & -2 & -2 \end{bmatrix}$$

The kernel of this, we can see is the span of the two vectors $\{(p, (1, -1, 0, 0)), (p, (0, 0, 1, -1))\}$.

## 1.3 Part (c)

Denote the span of the vectors $\{e_2, e_3\}$ by $V$, and $\{e_1, e_4\}$ by $W$. Note that $\mathbb{R}^4 = V \oplus W$.

$$DF(p)|_W = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$$

The determinant of this is 8, so $DF(p)|_W$ is invertible. $F$ is also smooth, so we can use the implicit function theorem to tell us that that there exists some smooth function $g : V \longrightarrow W$ so in a neighborhood of $(x_2, x_3) = (1, -1)$,

$$F(g_1(x_2, x_3), x_2, x_3, g_2(x_2, x_3)) = 0$$

Then the chain rule tells us that

$$DF(p)|_W Dg(1,-1) + DF(p)|_V = 0$$

So

$$Dg(1-1) = -\begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

# 2 Exercise 2 solution

The question of whether $TX$ is a manifold is local on $TX$, since we ask that there exists some open set about each point $(x, v)$. Thus for each $x$, we may replace $X$ by an open neighborhood $U$ of $x$ in $X$, for which there exists a diffeomorphism $\phi : U \longrightarrow V$, where $V \subset \mathbb{R}^n$ is open. We now answer the equivalent question for $U$. We know that $T(\phi) : TU \longrightarrow TV \cong V \times \mathbb{R}^n$ is then a smooth map. We know also that $T(\phi^{-1})$ maps $TV$ to $TU$ and is smooth. We must check that these maps are inverses to each other. This follows from the chain rule. Thus, $TU$ is diffeomorphic to $TV \cong V \times \mathbb{R}^n$. This is an open subset of $\mathbb{R}^{2n}$. Hence, it follows that $TX$ is a $2n$-dimensional manifold.

# 3 Exercise 3 solution

## 3.1 Part (a)

Let $S_n$ denote the set of all $n \times n$ symmetric matrices. As suggested in the lecture notes, Lecture 2, Example 3, we consider the map

$$\begin{aligned} f : M_n &\rightarrow S_n \\ A &\mapsto A^T A - I, \end{aligned}$$

which is a smooth map, as it is given in coordinates by polynomial expressions. Then, $O(n) = f^{-1}(0)$. To see this note that

$$Ax \cdot Ay = x^T A^T A y$$

so $A$ preserves the dot product if and only if $A^T A = I$.

To apply Theorem 1 from the same lecture, we need to check that every element $A \in O(n)$ is a regular value for the map $f$. Since $M_n \cong \mathbb{R}^{n^2}$, we may identify the tangent space at every point $A$ naturally with $M_n$ itself. Thus, to compute $Df(A)$, we may consider paths of the form $p(t) = A + tX$, where $X$ is an arbitrary element of $M_n$.

In particular, we have that

$$\begin{aligned} f(A+tX) &= (A+tX)^T(A+tX) - Id = A^TA + t(X^TA + A^tX) + t^2X^TX - Id \\ &= t(X^TA + A^TX) + t^2X^TX. \end{aligned}$$

We apply $\frac{d}{dt}|_{t=0}$, to get $Df(A)(X) = X^TA + A^TX$. We immediately get that the above matrix lies in $S_n$. Now, given any $A \in O^n$, and $Y \in T_0S_n \cong S_n$, we let $X = AY/2$. Then $Df(A)(X) = Y/2^TA^TA + A^TAY/2 = Y$. Therefore, $Df(A)$ is surjective for all $A \in O^n$.

Using the fact that the space of all $n \times n$ symmetric matrices is a vector space of dimension $\frac{n(n+1)}{2}$, it follows that $O(n)$ is a smooth $\frac{n(n-1)}{2}$ submanifold of $\mathbb{R}^{n^2}$ of dimension $\frac{n(n+1)}{2}$. Every element of $0(n)$ has orthonormal rows, so $O(n)$ is a bounded subset of the space of all $n \times n$ matrices. (In particular, each entry is bounded by 1.) Also, since the map $A \mapsto AA^t - Id$ is continuous, the preimage of zero under this map, which is precisely $O(n)$ is closed. Consequently, $O(n)$ is indeed compact.

### 3.2 Part (b)

Identifying $T_IM_n$ with $M_n$ as above, we have

$$T_IO(n) = \ker Df(I)$$

As computed above, $Df(I)(X) = X^TI + I^TX = X^T + X$. We therefore have that

$$T_IO(n) = Skew(n) = \{A \in M_n | A^T + A = 0\}$$

## 4 Exercise 4 solution

We reduce this to the Theorem 1 of Lecture 2, as follows.

First, the condition that $X = f^{-1}(Y)$ is a manifold is local on both $X$ and $Y$, which we see as follows. $X$ is a submanifold if every point $x \in X$ has some open neighborhood V diffeomorphic to an open subset in $\mathbb{R}^s$. Thus, for a fixed $x$, we can replace $X$ by an open neighborhood $U$ of $x$, and $V$ by $U \cap V$ (and conversely, $U \cap V$ is open in $X$). Similarly, we can replace $Y$ by a neighborhood of $y = f(x)$.

By choosing a sufficiently small neighborhood of $y$, we may in fact take $Y$ to be the coordinate plane $\mathbb{R}^k \subset \mathbb{R}^m$. Consider the projection $\pi : \mathbb{R}^m \to \mathbb{R}^m/\mathbb{R}^k \cong \mathbb{R}^{m-k}$.

Then, clearly, $Y = \pi^{-1}(0)$, and thus, $f^{-1}(Y) = f^{-1}(\pi^{-1}(0)) = (\pi \circ f)^{-1}(0)$. The condition that $f$ was transversal to $Y$ at $x$ clearly implies that $x$ is a regular point of $(\pi \circ f)$, and we thus by Theorem 1 of the notes, we are done.