1 Exercise 1 solution

1.1 Part (a)

We define the map $F : \mathbb{R}^4 \to \mathbb{R}^2$ by:

$$F(x_1, x_2, x_3, x_4) := (x_1^2 + x_2^2 - x_3^2 - x_4^2, x_1^2 + x_2^2 + x_3^2 + x_4^2 - 4)$$

Then, we have that the wanted set $M$ is equal to $F^{-1}\{0\}$. If we show that 0 is a regular value of $F$, it will follow that $M$ is a submanifold of $\mathbb{R}^4$ of dimension $4 - 2 = 2$.

We know that at $(x_1, x_2, x_3, x_4)$ the differential of $F$ equals

$$\begin{bmatrix}
2x_1 & 2x_2 & -2x_3 & -2x_4 \\
2x_1 & 2x_2 & 2x_3 & 2x_4
\end{bmatrix}$$

If $(x_1, x_2, x_3, x_4) \in F^{-1}\{0\}$ then we can check that $x_1$ and $x_2$ can’t simultaneously be 0. Similarly, both $x_3$ and $x_4$ can’t be 0. Hence, the above matrix has full rank. Thus $M$ is indeed a 2 dimensional submanifold of $\mathbb{R}^4$.

1.2 Part (b)

We know that the tangent space to $M$ at $p = (1, 1, -1, -1)$ is 2 dimensional, and given by the kernel of $T_pF$. As computed above,

$$T_pF = \begin{bmatrix}
2 & 2 & 2 & 2 \\
2 & 2 & -2 & -2
\end{bmatrix}$$

The kernel of this, we can see is the span of the two vectors $\{(p, (1, -1, 0, 0)), (p, (0, 0, 1, -1))\}$.

1.3 Part (c)

Denote the span of the vectors $\{e_2, e_3\}$ by $V$, and $\{e_1, e_4\}$ by $W$. Note that $\mathbb{R}^4 = V \oplus W$.

$$DF(p)|_W = \begin{bmatrix}
2 & 2 \\
2 & -2
\end{bmatrix}$$

The determinant of this is 8, so $DF(p)|_W$ is invertible. $F$ is also smooth, so we can use the implicit function theorem to tell us that that there exists some smooth function $g : V \to W$ so in a neighborhood of $(x_2, x_3) = (1, -1),

F(g_1(x_2, x_3), x_2, x_3, g_2(x_2, x_3)) = 0$
Then the chain rule tells us that
\[ DF(p)|_W Dg(1, -1) + DF(p)|_V = 0 \]

So
\[ Dg(1 - 1) = - \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \]

2 Exercise 2 solution

The question of whether \( TX \) is a manifold is local on \( TX \), since we ask that there exists some open set about each point \((x, v)\). Thus for each \( x \), we may replace \( X \) by an open neighborhood \( U \) of \( x \) in \( X \), for which there exists a diffeomorphism \( \phi : U \rightarrow V \), where \( V \subseteq \mathbb{R}^n \) is open. We now answer the equivalent question for \( U \).

We know that \( T(\phi) : TU \rightarrow TV \cong V \times \mathbb{R}^n \) is then a smooth map. We know also that \( T(\phi^{-1}) \) maps \( TV \) to \( TU \) and is smooth. We must check that these maps are inverses to each other. This follows from the chain rule. Thus, \( TU \) is diffeomorphic to \( TV \cong V \times \mathbb{R}^n \). This is an open subset of \( \mathbb{R}^{2n} \). Hence, it follows that \( TX \) is a 2n-dimensional manifold.

3 Exercise 3 solution

3.1 Part (a)

Let \( S_n \) denote the set of all \( n \times n \) symmetric matrices. As suggested in the lecture notes, Lecture 2, Example 3, we consider the map
\[
 f : M_n \rightarrow S_n \\
 A \mapsto A^T A - I,
\]

which is a smooth map, as it is given in coordinates by polynomial expressions. Then, \( O(n) = f^{-1}(0) \). To see this note that
\[
 Ax \cdot Ay = x^T A^T Ay
\]
so \( A \) preserves the dot product if and only if \( A^T A = I \).

To apply Theorem 1 from the same lecture, we need to check that every element \( A \in O(n) \) is a regular value for the map \( f \). Since \( M_n \cong \mathbb{R}^{n^2} \), we may identify the
tangent space at every point $A$ naturally with $M_n$ itself. Thus, to compute $Df(A)$, we may consider paths of the form $p(t) = A + tX$, where $X$ is an arbitrary element of $M_n$. In particular, we have that

\[ f(A + tX) = (A + tX)^T(A + tX) - I_d = A^T A + t(X^T A + A^T X) + t^2 X^T X - I_d \]

We apply $\frac{d}{dt}|_{t=0}$, to get $Df(A)(X) = X^T A + A^T X$. We immediately get that the above matrix lies in $S_n$. Now, given any $A \in O(n)$, and $Y \in T_0 S_n \cong S_n$, we let $X = AY/2$. Then $Df(A)(X) = Y/2^T A^T A + A^T AY/2 = Y$. Therefore, $Df(A)$ is surjective for all $A \in O(n)$.

### 3.2 Part (b)

Identifying $T_I M_n$ with $M_n$ as above, we have

\[ T_I O(n) = \ker Df(I) \]

As computed above, $Df(I)(X) = X^T I + I^T X = X^T + X$. We therefore have that

\[ T_I O(n) = Skew(n) = \{ A \in M_n | A^T + A = 0 \} \]

### 4 Exercise 4 solution

We reduce this to the Theorem 1 of Lecture 2, as follows.

First, the condition that $X = f^{-1}(Y)$ is a manifold is local on both $X$ and $Y$, which we see as follows. $X$ is a submanifold if every point $x \in X$ has some open neighborhood $V$ diffeomorphic to an open subset in $\mathbb{R}^s$. Thus, for a fixed $x$, we can replace $X$ by an open neighborhood $U$ of $x$, and $V$ by $U \cap V$ (and conversely, $U \cap V$ is open in $X$). Similarly, we can replace $Y$ by a neighborhood of $y = f(x)$.

By choosing a sufficiently small neighborhood of $y$, we may in fact take $Y$ to be the coordinate plane $\mathbb{R}^k \subset \mathbb{R}^m$. Consider the projection $\pi : \mathbb{R}^m \to \mathbb{R}^m/\mathbb{R}^k \cong \mathbb{R}^{m-k}$. 

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Then, clearly, $Y = \pi^{-1}(0)$, and thus, $f^{-1}(Y) = f^{-1}(\pi^{-1}(0)) = (\pi \circ f)^{-1}(0)$. The condition that $f$ was transversal to $Y$ at $x$ clearly implies that $x$ is a regular point of $(\pi \circ f)$, and we thus by Theorem 1 of the notes, we are done.