The Lie Derivative of Densities

Definition 1. Let $v$ be a $C^1$ vector field on $M$, and $\sigma$ a $C^1$ density on $M$. Denote by $\Phi_{tv}$ the flow induced by $v$ on $M$. The Lie derivative of $\sigma$ with respect to $v$ is a density $L_v\sigma$ on $M$ which satisfies

$$(L_v\sigma)(p) = \frac{d}{dt}|_{t=0}\Phi_{tv}^\ast \sigma(p)$$

Note that $\Phi_{tv}^\ast \sigma(p)$ is inside the vector space $|T_pM|$, so it makes sense to try to differentiate this with respect to $t$. We shall prove that this derivative exists by calculating it explicitly.

We shall use the following lemma

Lemma 1. Suppose that $X(t)$ and $A(t)$ are a family of $n \times n$ matrices so that

$$\frac{d}{dt}X(t) = A(t)X(t)$$

Then the determinant of $X(t)$ satisfies

$$\frac{d}{dt}(\det X(t)) = Tr(A(t))\det X(t)$$

where the trace, $Tr(A)$ means the sum of the diagonal entries of the matrix $A$ (which is the sum of the eigenvalues).

We shall leave the proof of this as an exercise, but mention the following hint:

The chain rule tells us that the derivative of $\det X(t)$ depends only on the derivative of $X(t)$ and the derivative of det.

$$\frac{d}{dt}X(t) = \frac{d}{ds}|_{s=0}(I + sA(t))X(t)$$

for each $t$

Therefore the lemma will follow if you can prove that

$$\frac{d}{ds}|_{s=0}\det(I + sA(t)) = Tr(A(t))$$

A slick proof could use $e^{sA(t)}$ instead of $(I + sA(t))$, and use the definition of det and $Tr$ in terms of eigenvalues, and the knowledge of how eigenvalues of $e^{sA}$ compare to those of $A$.

Lemma 2. If $v$ is a $C^1$ vector field on an open set $U \subset \mathbb{R}^n$ given by

$$v = v_1 \frac{\partial}{\partial x_1}, \ldots, v_n \frac{\partial}{\partial x_n}$$

the Lie derivative of $\sigma_{Leb}$ is given by

$$L_v\sigma_{Leb} = \sum \frac{\partial v_i}{\partial x_i} \sigma_{Leb}$$
Proof. First, using Lemma 1 we get

\[ D\sigma_{Leb}(p)(e_1, \ldots , e_n)(u^1, \ldots , u^n) = \sum dx_i (u^i) = \sum u_i^i \]

\[ L_v \sigma_{Leb} = \frac{d}{dt} \Big|_{t=0} \Phi_{tv}^* \sigma_{Leb}(p) \]
\[ = \frac{d}{dt} \Big|_{t=0} \sigma_{Leb}(\Phi_{tv}(p)) (D\Phi_{tv}(p)e_1, \ldots , D\Phi_{tv}e_n) \sigma_{Leb}(p) \]
\[ = \left( \sum dx_i \left( \frac{d}{dt} \Big|_{t=0} D\Phi_{tv}(e_i) \right) \right) \sigma_{Leb}(p) \]
\[ = \left( \sum dx_i \left( -Lv \frac{\partial}{\partial x_i} \right) \right) \sigma_{Leb}(p) \]
\[ = \sum \frac{\partial v_i}{\partial x_i} \sigma_{Leb} \]

To compute $L_v \sigma$ in coordinates, we can use the above calculation and the following lemma, the proof of which is an exercise:

**Lemma 3.** If $\phi$ is a $C^1$ function, $\sigma$ a $C^1$ density, and $v$ a $C^1$ vectorfield on a manifold $M$,

\[ L_v(\phi \sigma) = (L_v \phi) \sigma + \phi(L_v \sigma) \]

Using the above two lemmas, we can observe

**Lemma 4.**

\[ L_{v_1+v_2} \sigma = L_{v_1} \sigma + L_{v_2} \sigma \]

You should think of $L_v \sigma$ as how you see $\sigma$ change when you flow yourself using the vector field $v$. If you think of yourself as a set $U$ on which $\sigma$ can be integrated, this suggests the following theorem:

**Theorem 5.** Suppose that $U \subset M$ is an open subset of $M$ so that its closure $\bar{U} \subset M$ is compact. Let $v$ be a complete $C^1$ vector field, and $\sigma$ a $C^1$ density, then

\[ \frac{d}{dt} \Big|_{t=0} \left( \int_{\Phi_{tv}(U)} \sigma \right) = \int_{U} L_v \sigma \]

where $\Phi_{tv}(U)$ indicates the image of $U$ under the flow of $v$ for time $t$. 

2
Proof. Note that the change of variables theorem tells us that
\[ \int_{\Phi tv(U)} \sigma = \int_U \Phi_{tv}^* \sigma \]
We need to justify moving \( \frac{d}{dt} \) under the integral sign. Define the density
\[ \sigma_t := \Phi_{tv}^* \sigma \]
As we are more familiar with the properties of functions then densities, fix a smooth embedding \( M \subset \mathbb{R}^N \) so we can use the smooth positive density \( \sigma_{vol} \) coming from the restriction of the Euclidean metric on \( \mathbb{R}^N \), and define the function \( \varphi : M \times \mathbb{R} \rightarrow \mathbb{R} \) so that
\[ \varphi(p, t)\sigma_{vol}(p) = \sigma_t(p) \]
The function \( \varphi \) is \( C^1 \), so \( \frac{\partial \varphi}{\partial t}(x, t) \) converges uniformly to \( \frac{\partial \varphi}{\partial t}(x, 0) \) as \( t \to 0 \) on the compact set \( \bar{U} \). Therefore, using the mean value theorem, we get that for all \( \epsilon > 0 \), there exists some \( \delta > 0 \) so that for \( |t| < \delta \) and \( x \in \bar{U} \),
\[ \left| \frac{\varphi(x, t) - \varphi(x, 0)}{t} - \frac{\partial \varphi}{\partial t}(x, 0) \right| < \epsilon \]
so
\[ \left| \int_U \frac{\sigma_t - \sigma_0}{t} - \int_U L_v \sigma \right| \leq \int_U \left| \frac{\sigma_t - \sigma_0}{t} - L_v \sigma \right| = \int_U \left| \frac{\varphi(x, t) - \varphi(x, 0)}{t} - \frac{\partial \varphi}{\partial t}(x, 0) \right| \sigma_{vol} \]
\[ \leq \int_U \epsilon \sigma_{vol} = \epsilon \int_U \sigma_{vol} \]
As \( \epsilon \) was arbitrary,
\[ \lim_{t \to 0} \int_U \frac{\Phi_{tv}^* \sigma - \sigma_0}{t} = \int_U L_v \sigma \]
so
\[ \frac{d}{dt} \Big|_{t=0} \left( \int_{\Phi tv(U)} \sigma \right) = \int_U L_v \sigma \]

Exercises
1. Prove Lemma 1
2. Prove Lemma 3
3. Prove Lemma 4

4. In standard coordinates on $\mathbb{R}^3$, let

$$v := x_1^2 \frac{\partial}{\partial x_1} - x_1 x_2 \frac{\partial}{\partial x_2} - x_1 x_3 \frac{\partial}{\partial x_3}$$

suppose that $U$ is a bounded open set in $\mathbb{R}^3$, and $\Phi_{tv}$ is defined on $U$. Prove that

$$\int_U \sigma_{Leb} = \int_{\Phi_{tv}(U)} \sigma_{Leb}$$

Prove also that in this case, the naive change of variables theorem holds:

$$\int_U f \sigma_{Leb} = \int_{\Phi_{tv}(U)} (f \circ \Phi_{tv}) \sigma_{Leb}$$

5. Suppose that $\sigma$ is a compactly supported $C^1$ density on $M$. Prove that if $v$ is any $C^1$ vector field on $M$,

$$\int_M L_v \sigma = 0$$