The Lie Derivative of Densities

Definition 1. *Let v be a C^1 vector field on M, and σ a C^1 density on M. Denote by Φ_{tv} the flow induced by v on M. The Lie derivative of σ with respect to v is a density $L_v\sigma$ on M which satisfies*

$$(L_v\sigma)(p) = \frac{d}{dt}|_{t=0}\Phi_{tv}^*\sigma(p)$$

Note that $\Phi_{tv}^*\sigma(p)$ is inside the vector space $|T_pM|$, so it makes sense to try to differentiate this with respect to t. We shall prove that this derivative exists by calculating it explicitly.

We shall use the following lemma

Lemma 1. *Suppose that $X(t)$ and $A(t)$ are a family of $n \times n$ matrices so that*

$$\frac{d}{dt}X(t) = A(t)X(t)$$

Then the determinant of $X(t)$ satisfies

$$\frac{d}{dt}(\det X(t)) = Tr(A(t)) \det X(t)$$

where the trace, $Tr(A)$ means the sum of the diagonal entries of the matrix A (which is the sum of the eigenvalues).

We shall leave the proof of this as an exercise, but mention the following hint:

The chain rule tells us that the derivative of $\det X(t)$ depends only on the derivative of $X(t)$ and the derivative of det.

$$\frac{d}{dt}X(t) = \frac{d}{ds}|_{s=0}(I + sA(t))X(t) \text{ for each } t$$

Therefore the lemma will follow if you can prove that

$$\frac{d}{ds}|_{s=0} \det(I + sA(t)) = Tr(A(t))$$

A slick proof could use $e^{sA(t)}$ instead of $(I + sA(t))$, and use the definition of det and Tr in terms of eigenvalues, and the knowledge of how eigenvalues of e^{sA} compare to those of A.

Lemma 2. *If v is a C^1 vector field on an open set $U \subset \mathbb{R}^n$ given by*

$$v = v_1\frac{\partial}{\partial x_1}, \ldots, v_n\frac{\partial}{\partial x_n}$$

the Lie derivative of σ_{Leb} is given by

$$L_v\sigma_{\text{Leb}} = \sum \frac{\partial v_i}{\partial x_i}\sigma_{Leb}$$

Proof. First, using Lemma 1 we get

$$D\sigma_{Leb}(p)(e_1, \ldots, e_n)(u^1, \ldots, u^n) = \sum_i dx_i(u^i) = \sum_i u_i^i$$

$$\begin{aligned} L_v \sigma_{Leb} &= \frac{d}{dt}|_{t=0} \Phi_{tv}^* \sigma_{\text{Leb}}(p) \\ &= \frac{d}{dt}|_{t=0} \sigma_{\text{Leb}}(\Phi_{tv}(p))(D\Phi_{tv}(p)e_1, \ldots, D\Phi_{tv}e_n)\sigma_{Leb}(p) \\ &= \left(\sum_i dx_i \left(\frac{d}{dt}|_{t=0} D\Phi_{tv}(e_i) \right) \right) \sigma_{Leb}(p) \\ &= \left(\sum_i dx_i \left(-L_v \frac{\partial}{\partial x_i} \right) \right) \sigma_{Leb}(p) \\ &= \sum \frac{\partial v_i}{\partial x_i} \sigma_{Leb} \end{aligned}$$

□

To compute $L_v \sigma$ in coordinates, we can use the above calculation and the following lemma, the proof of which is an exercise:

Lemma 3. *If ϕ is a C^1 function, σ a C^1 density, and v a C^1 vectorfield on a manifold M,*

$$L_v(\phi\sigma) = (L_v\phi)\sigma + \phi(L_v\sigma)$$

Using the above two lemmas, we can observe

Lemma 4.

$$L_{v_1+v_2}\sigma = L_{v_1}\sigma + L_{v_2\sigma}$$

You should think of $L_v\ \sigma$ as how you see σ change when you flow yourself using the vector field v. If you think of yourself as a set U on which σ can be integrated, this suggests the following theorem:

Theorem 5. *Suppose that $U \subset M$ is an open subset of M so that its closure $\bar{U} \subset M$ is compact. Let v be a complete C^1 vector field, and σ a C^1 density, then*

$$\frac{d}{dt}|_{t=0} \left(\int_{\Phi_{tv}(U)} \sigma \right) = \int_U L_v \sigma$$

where $\Phi_{tv}(U)$ indicates the image of U under the flow of v for time t.

Proof. Note that the change of variables theorem tells us that

$$\int_{\Phi_{tv}(U)} \sigma = \int_U \Phi_{tv}^* \sigma$$

We need to justify moving $\frac{d}{dt}$ under the integral sign. Define the density

$$\sigma_t := \Phi_{tv}^* \sigma$$

As we are more familiar with the properties of functions then densities, fix a smooth embedding $M \subset \mathbb{R}^N$ so we can use the smooth positive density σ_{vol} coming from the restriction of the Euclidean metric on $\mathbb{R}^N$, and define the function $\varphi : M \times \mathbb{R} \longrightarrow \mathbb{R}$ so that

$$\varphi(p,t)\sigma_{vol}(p) = \sigma_t(p)$$

The function φ is C^1, so $\frac{\partial \varphi}{\partial t}(x,t)$ converges uniformly to $\frac{\partial \varphi}{\partial t}(x,0)$ as $t \to 0$ on the compact set $\bar{U}$. Therefore, using the mean value theorem, we get that for all $\epsilon > 0$, there exists some $\delta > 0$ so that for $|t| < \delta$ and $x \in \bar{U}$,

$$\left| \frac{\varphi(x,t) - \varphi(x,0)}{t} - \frac{\partial \varphi}{\partial t}(x,0) \right| < \epsilon$$

so

$$\begin{aligned} \left| \frac{\int_U \sigma_t - \int_U \sigma_0}{t} - \int_U L_v \sigma \right| &\leq \int_U \left| \frac{\sigma_t - \sigma_0}{t} - L_v \sigma \right| \\ &= \int_U \left| \frac{\varphi(x,t) - \varphi(x,0)}{t} - \frac{\partial \varphi}{\partial t}(x,0) \right| \sigma_{vol} \\ &\leq \int_U \epsilon \sigma_{vol} = \epsilon \int_U \sigma_{vol} \end{aligned}$$

As ϵ was arbitrary,

$$\lim_{t \to 0} \frac{\int_U \Phi_{tv}^* \sigma - \int_U \sigma}{t} = \int_U L_v \sigma$$

so

$$\frac{d}{dt}|_{t=0} \left(\int_{\Phi_{tv}(U)} \sigma \right) = \int_U L_v \sigma$$

□

Exercises

1. Prove Lemma 1

2. Prove Lemma 3

3. Prove Lemma 4

4. In standard coordinates on $\mathbb{R}^3$, let

$$v := x_1^2 \frac{\partial}{\partial x_1} - x_1 x_2 \frac{\partial}{\partial x_2} - x_1 x_3 \frac{\partial}{\partial x_3}$$

   suppose that U is a bounded open set in $\mathbb{R}^3$, and Φ_{tv} is defined on U. Prove that

$$\int_U \sigma_{\text{Leb}} = \int_{\Phi_{tv}(U)} \sigma_{Leb}$$

   Prove also that in this case, the naive change of variables theorem holds:

$$\int_U f\sigma_{\text{Leb}} = \int_{\Phi_{tv}(U)} (f \circ \Phi_{tv})\sigma_{Leb}$$

5. Suppose that σ is a compactly supported C^1 density on M. Prove that if v is any C^1 vector field on M,

$$\int_M L_v \sigma = 0$$